Regular Cell Complexes in Total Positivity

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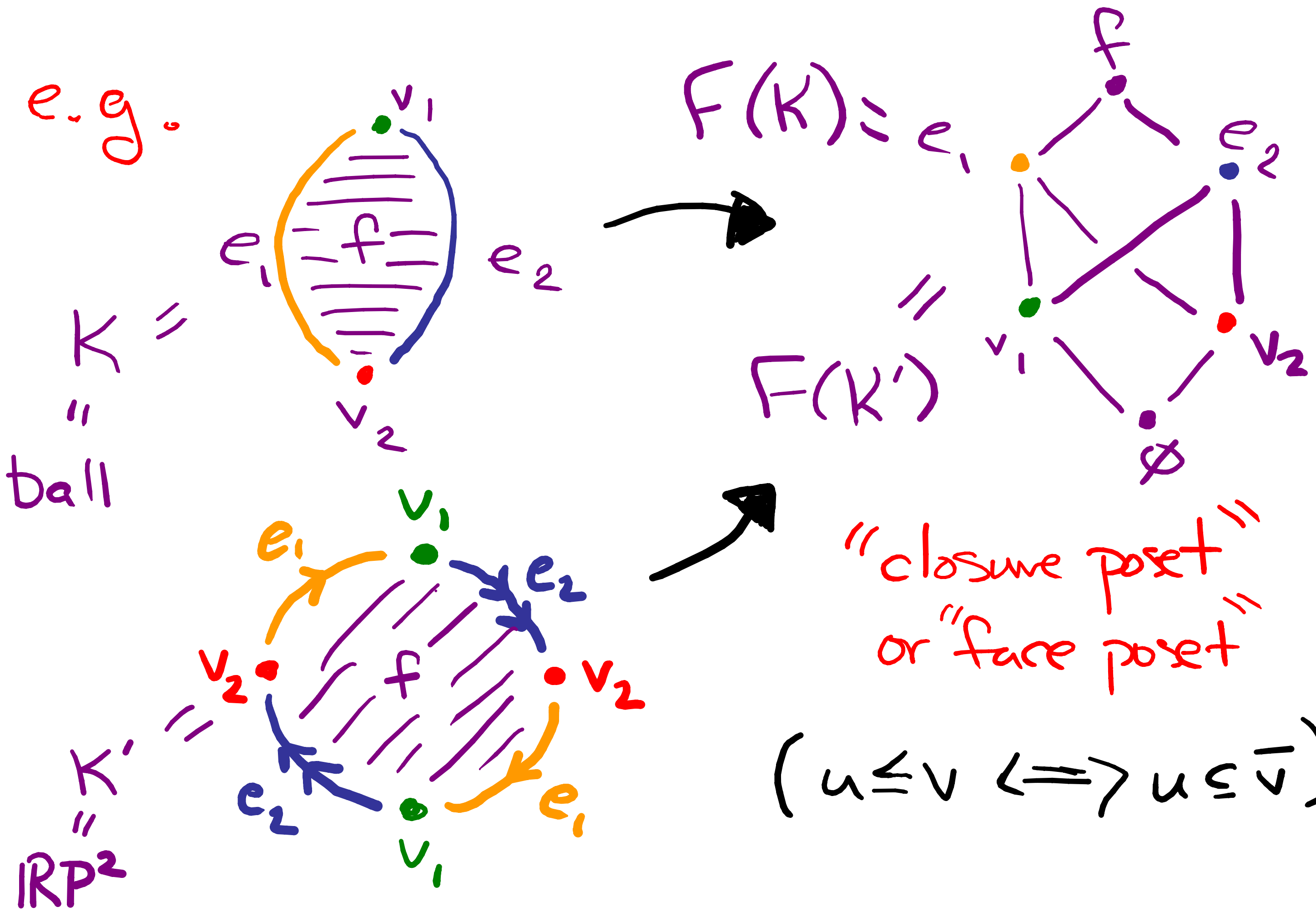
(Paper with these results by same title, to appear in Inventiones Mathematicae, 60 pages)

(See http://www4.ncsu.edu/~plhersh for slides, including appendix with more details)

Topological Aspects of Total Positivity

- Lusztig initiated study of Totally nonnegative, real part of (matrix) Schubert varieties
  (i.e. part with minors all nonnegative in spaces of matrices or of flags)
- Topology (homeomorphism type) is conjecturally/provably trivial.
- Puts restrictions on possible relations among (exponentiated) Chevalley generators.
- Reveals structure in canonical bases via tropicalization; a motivation for cluster algebras
- Main Result of Talk: Proof of Fomin-Shapiro Conjecture via new tools exploiting interplay of combinatorial data & topological data

Background: CW Complexes and their Closure Posets



"closure poset"
or "face poset"

$(u \le v \iff u \subseteq \bar{v})$

CW complexes: comprised of pieces called cells each homeomorphic to an open ball

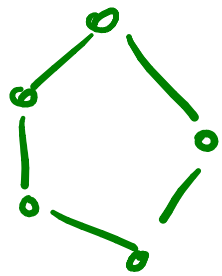
- higher dimensional cells glued to unions of lower dimensional ones by attaching maps.

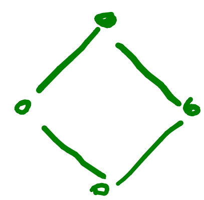
Regular CW Complexes

- A CW complex is regular if the attaching map for each cell is a homeomorphism (hence injective).
  e.g. all simplicial complexes & polytopes
- K regular $\Rightarrow K \cong \Delta(F(K) \setminus \{\hat{0}\}) = \text{sd}K$
  (Δ: poset "order complex" or "nerve")
- seemingly encompasses several spaces of interest from combinatorial rep'n theory, real algebraic geometry, total positivity theory, electrical networks, ...

Note: Our focus is on spaces themselves with posets/combinatorics of faces as auxiliary tool & interaction of rel'ns on level of points and of cells

Recall: Poset is graded if $u \le v$ implies minimal paths u to v all same length

e.g.  is not graded

A graded poset is thin if each rank 2 interval has 4 elements 

Defn (Björner): A finite, graded poset P is CW poset if

- P has unique min'l elt. $\hat{0}$
- P has additional element(s)
- $x \neq \hat{0} \Rightarrow \Delta(\hat{0},x) \cong S^{rk\,x-2}$

Thm (Björner): P is CW poset <=> there exists regular CW complex K with $P = F(K)$.

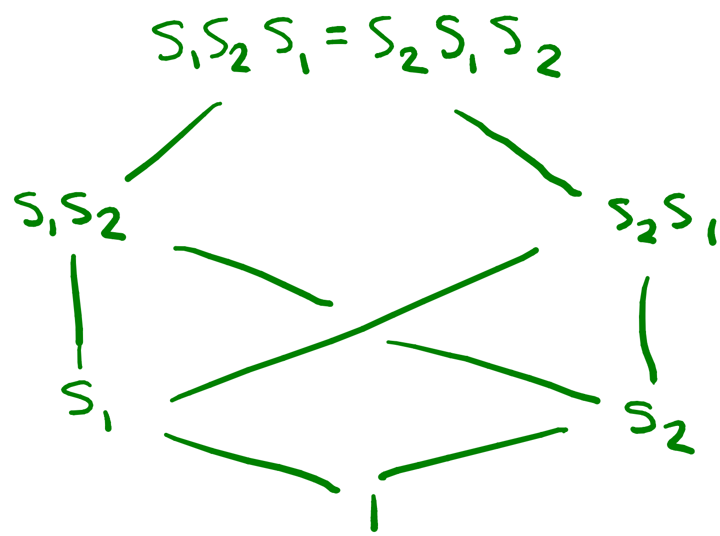
Some Examples of CW Posets

- Shellable & thin (Danaraj-Klee)
- Bruhat order (Björner & Wachs)
- Closure poset for double Bruhat decomp. of totally nonneg. part of flag variety (Williams)
- Closure poset of triangulation of double suspension of Poincare homology 3-sphere with "big cell" glued in (due to work of Cannon & Edwards)

The **Bruhat order** is partial order on Coxeter group W with $u \leq v \iff$ there exists **reduced expressions** (i.e. products of minimal # adjacent transpositions) $r(u)$ and $r(v)$ with $r(u)$ subexpression of $r(v)$.

e.g. $W = S_3$ with generators

$s_1 = (1,2)$
$s_2 = (2,3)$



- Poset of closure relations for Schubert cell decomposition of Schubert variety (a highly non-regular CW complex)

Question (Bernstein): Find regular CW complexes naturally arising from rep'n theory which are homeomorphic to closed balls and have the Bruhat intervals as closure posets.

Conjectural Solution (Fomin-Shapiro):

The Bruhat stratification of $\mathrm{lk}(\mathrm{id})$ in totally nonnegative, real part of unipotent radical in semisimple, simply connected algebraic group is regular CW complex homeomorphic to closed ball.

Theorem (H.): Fomin-Shapiro Conjecture indeed holds.

Special Case of Type A:

Space of Totally nonnegative upper triangular matrices with 1's on diagonal & entries just above diagonal summing to fixed, positive constant, stratified by which minors are positive and which are 0.

The Totally Nonnegative Part of a Space of Matrices

- $x_i(t) = I_n + tE_{i,i+1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$ ← row i, column $i+1$

  $x_i(t) = \exp(te_i)$ (general type); $I_n + tE_{i,i+1}$ (type A)

- $f_{(i_1,\ldots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$

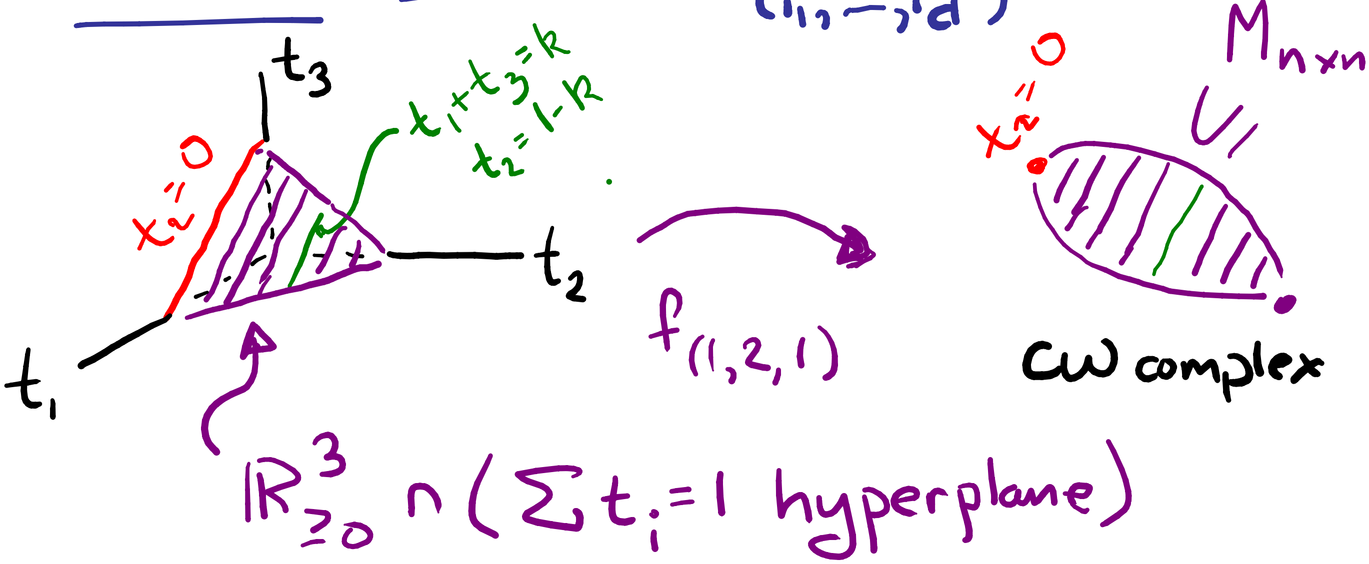
$$(t_1,\ldots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1,\dots,i_d)}$



$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$\rightsquigarrow$ $t_2 = 0$

$x_1(t_1) \circ x_1(t_3)$

$$f_{(1,2,1)}(t_1,0,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

Non-injectivity: results from "nil-moves" $x_i(u)x_i(v) = x_i(u+v)$ & "long braid moves" in 0-Hecke algebra

0-Hecke Algebra Captures which Simplex Faces have Same Image under $f_{(i_1,\ldots,i_d)}$

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$ "nil-move"

↓ suppress parameters

$x_i^2 = x_i$ (0-Hecke alg. rel'n, up to sign)

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1t_2}{t_1+t_3}\right)$

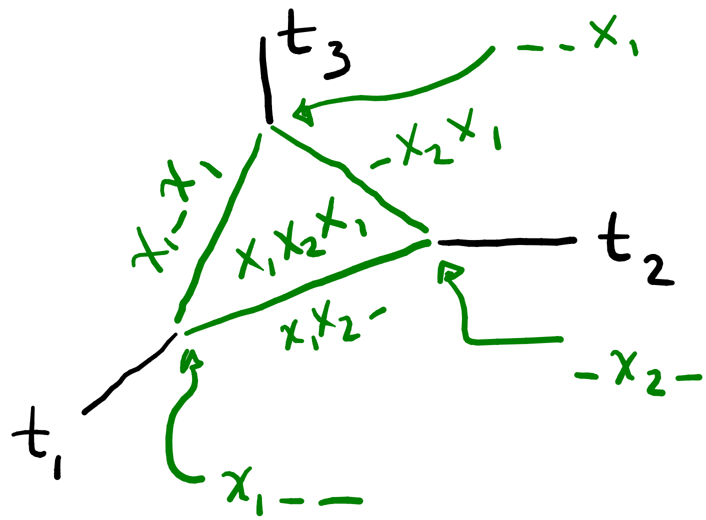
↓ (type A)

"long braid move" with enrichment from parameters

$x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1}$

(similar relation holds outside type A)

Indexing Faces of Preimage by Words in 0-Hecke Algebra



Key Observation About $f_{(i_1,\ldots,i_d)}$:

$$im(F_1) = im(F_2) \iff x(F_1) = x(F_2)$$

equal as 0-Hecke algebra elements (i.e. equal Demazure product)

Thm (Lusztig): If $(i_1,\ldots,i_d)$ is reduced word, then $f_{(i_1,\ldots,i_d)}$ acts homeomorphically on $\mathbb{R}^d_{>0}$.

Upshot: $f_{(i_1,\ldots,i_d)}$ restricts to homeomorphism on each face given by reduced subword

Properties of Change-of-Coordinates Map Given by Braid Moves

e.g. $(t_1, t_2, t_3) \mapsto \left(\frac{t_2 t_3}{t_1+t_3}, t_1+t_3, \frac{t_1 t_2}{t_1+t_3}\right)$

in type A

- Tropicalizes to change-of-basis map for Lusztig's canonical bases:

$$(a,b,c) \mapsto (b+c-\min(a,c), \min(a,c), a+b-\min(a,c))$$

- A motivation for development of cluster algebras (& mutation)

Suggested exercise: verify map is involution

Proof Strategy for Fomin-Shapiro Conjecture (& for Images of Maps from Polytopes)

Set-up: Continuous, surjective fn

$$f : P \to Y$$

from convex polytope P s.t. f maps $\text{int}(P)$ homeomorphically to $\text{int}(Y)$.

Step 1: Perform "collapses" on ∂P preserving regularity and homeomorphism type - via continuous, surjective collapsing functions $P \to P$ yielding $P/\sim$ with fewer cells s.t. $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$

(collapse away all "non-reduced" cells)

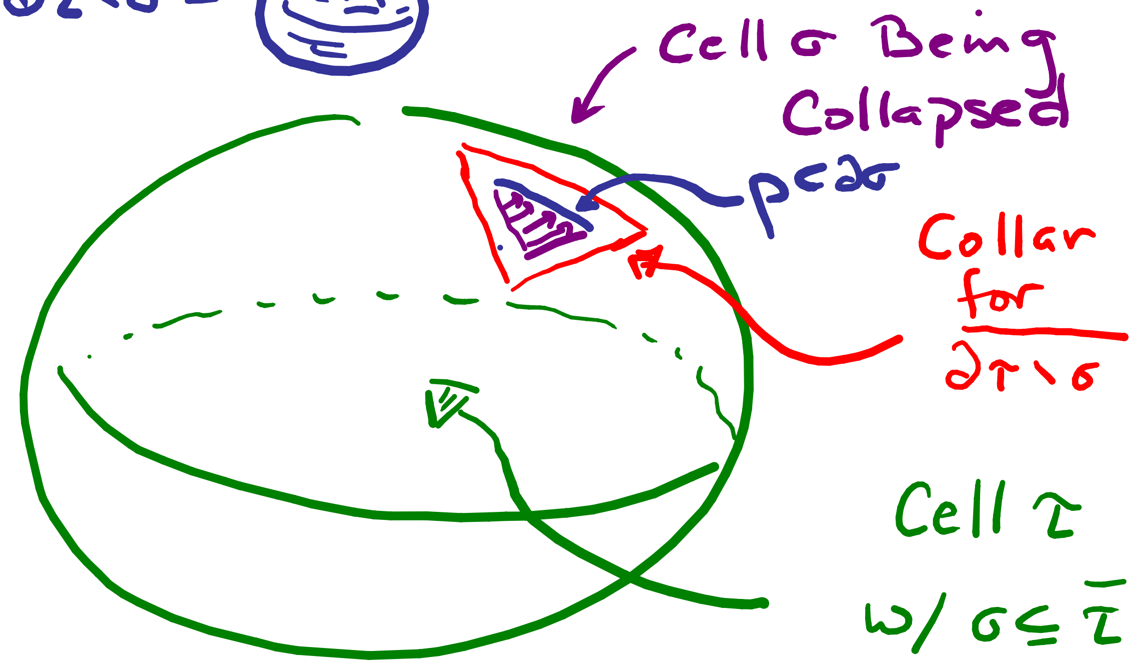
Step 2: Prove $\bar{f} : P/\sim \to Y$ is homeomorphism by new regularity criterion

Collapsing Cell σ onto Cell $\rho \le \bar{\sigma}$ within $\partial \tau$

Thm (M. Brown; Connelly): Any topological manifold with boundary ∂M has a collar, i.e. a nbh'd homeomorphic to $\partial M \times [0,1]$ &

Fact: Our collapses preserve this for:

$\overline{\partial \tau \setminus \sigma}$ = 

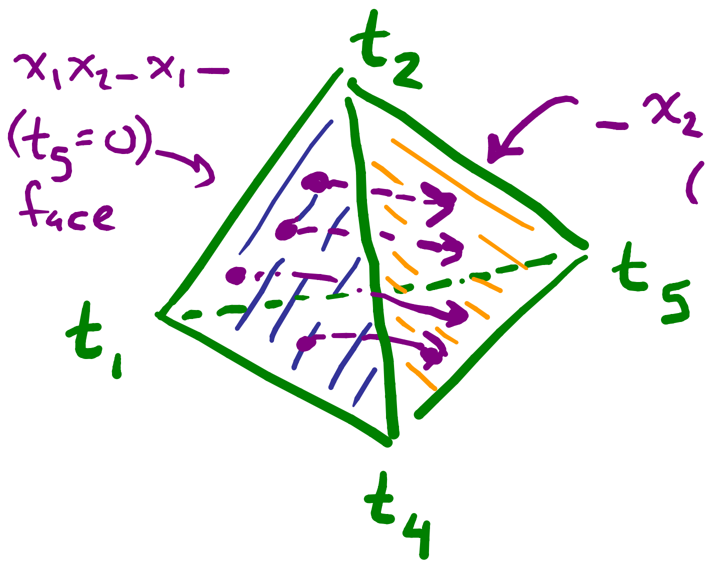


& $\dim \tau = \dim \sigma + 1$

Plan: Collapse $\bar{\sigma}$ onto $\bar{\rho} \le \partial \sigma$, stretching collar for $\overline{\partial \tau \setminus \sigma}$ to cover $\bar{\sigma} \setminus \bar{\rho}$, using polytope face convexity to get continuity

Collapsing "non-reduced" Face Across Curves (each in Single Fiber)

e.g. $f_{(1,2,3,1,2)}$ face with $t_3 = 0$



- Collapse across curves identifying $t_1t_2t_4$ & $t_2t_4t_5$ faces

$$(t_1, t_2, 0, t_4, t_5) \longmapsto x_1(t_1)x_2(t_2)x_1(t_4)x_2(t_5)$$

$$= x_2(t_1')x_1(t_2')x_2(t_4')x_2(t_5)$$

only for $t_1 + t_4 > 0$

$$= x_2(t_1')x_1(t_2')x_2(t_4' + t_5)$$

for $t_1' = \dfrac{t_2t_4}{t_1+t_4}$ $\quad t_2' = t_1 + t_4 \quad t_4' = \dfrac{t_1t_2}{t_1+t_4}$

curves: $t_1' = k_1$ & $t_2' = k_2$ & $t_4' + t_5 = k_3$

"Deletion Pairs": Transferring Coxeter Group Properties to 0-Hecke Algebra

In a non-reduced expression $s_{i_1} \cdots s_{i_d}$, let $\{s_{i_r}, s_{i_t}\}$ be **deletion pair** if $s_{i_r} \cdots s_{i_{t-1}}$ and $s_{i_{r+1}} \cdots s_{i_t}$ are reduced expressions (for same $w \in W$) while $s_{i_r} \cdots s_{i_t}$ is nonreduced.

e.g. $x_1 x_4 x_5 x_1$ or $x_1 x_2 x_1 x_2$

Key Coxeter Group Property: Any two reduced expressions for same $w \in W$ are connected by series of braid moves - ensures nonreduced expressions admit modified nil-moves.

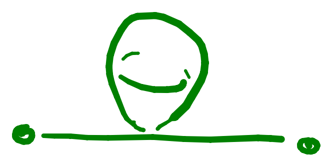
Greedy Collapsing Order: based on (1) leftmost deletion pair, (2) minimize $t-r$, (3) maximize cell dimension - amenable to induction on length & to well-defined changes-of-coord's on closed cells

New Regularity Criterion:

Prop'n (H.) Let K be a finite CW complex w/ characteristic maps $\{f_\alpha\}$. Suppose

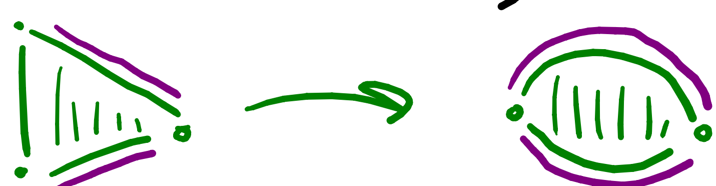
(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example:



(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example:



Then $F(K)$ is graded by cell dimension.

Remark: Next theorem "spreads around" injectivity requirement

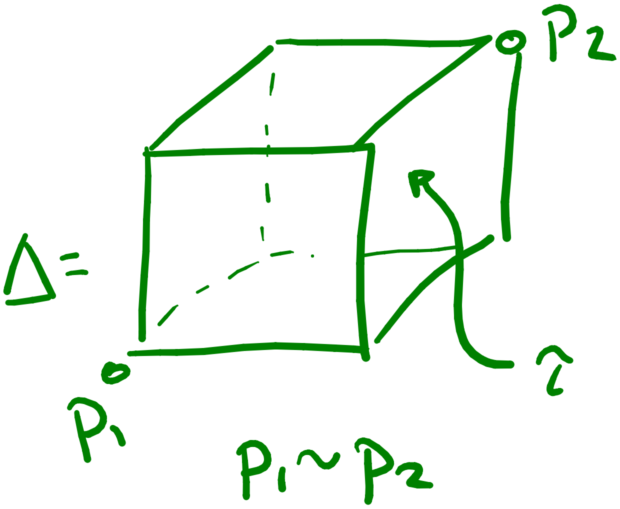
Thm (H.) Let K be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then K is regular w.r.t. $\{f_\alpha\}$ $\Longleftrightarrow$

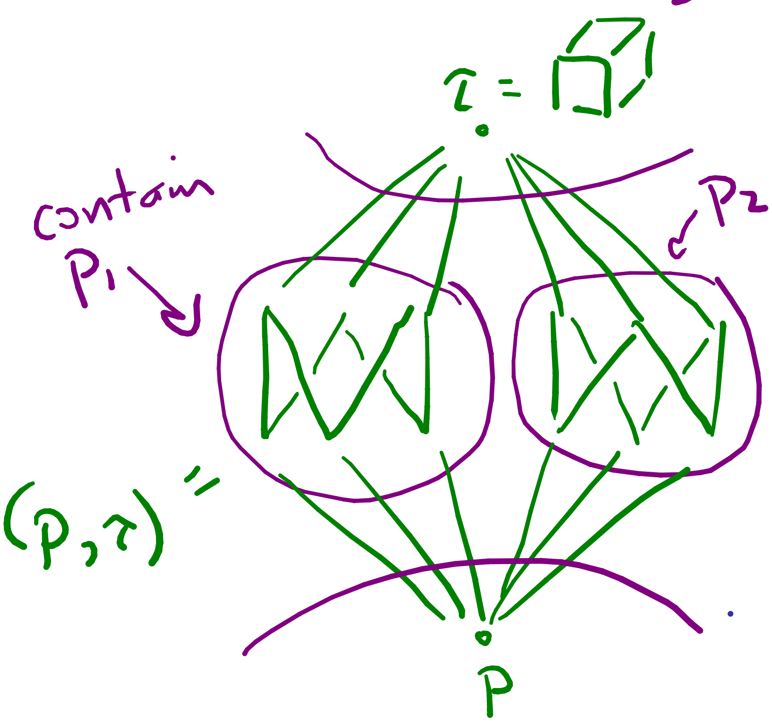
(1) K meets requirements of prop'n for $F(K)$ to be graded by cell dim.

(2) $F(K)$ is thin and each open interval (u,v) for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

Non-Example



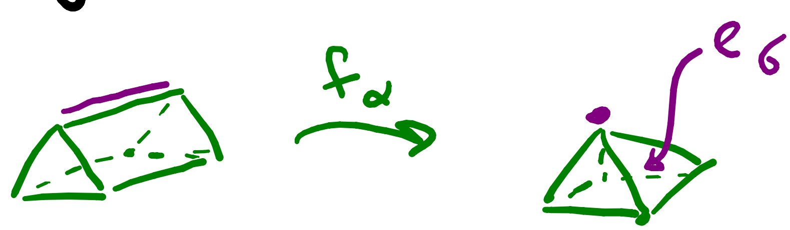


(3) For each α, the restriction of f_α to preimages of codim. one cells in $\overline{e_\alpha}$ is injective.
(topological condition)

Non-Example:

(4) $\forall\, e_\sigma \subseteq \overline{e_\alpha}$, f_σ factors as continuous inclusion $i: B^{\dim \sigma} \to B^{\dim \alpha}$ followed by f_α.

Non-Example:
(due to David Speyer)



Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one

Example Where Injectivity is (Much) Easier in Codimension One

By exchange axiom for Coxeter groups

$s_1 s_2 s_1 s_3$

↙ ↙ ↘ ↘

$s_1 s_2 s_1 -$	$s_1 s_2 - s_3$	~~$s_1 - s_1 s_3$~~	$- s_2 s_1 s_3$
$=$	$=$	~~$=$~~	$=$
3214	2341	~~1243~~	3142

various ways to delete a letter obtaining reduced expression gives distinct Coxeter group elements

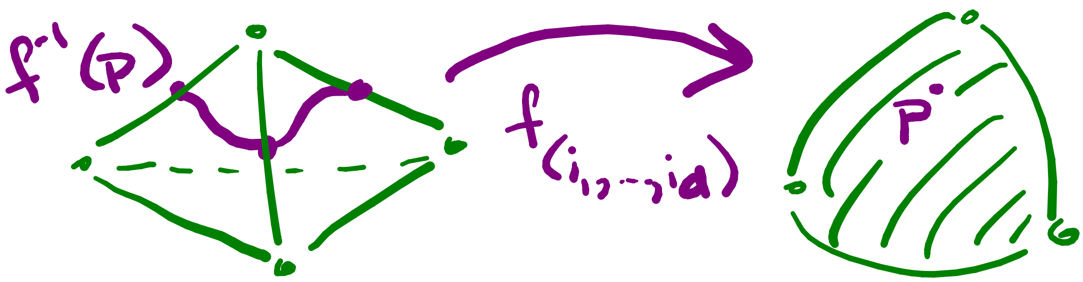
This Injectivity Idea Fails in Higher Codim:

e.g. $s_1 s_2 s_1 s_3$

↙ ↘

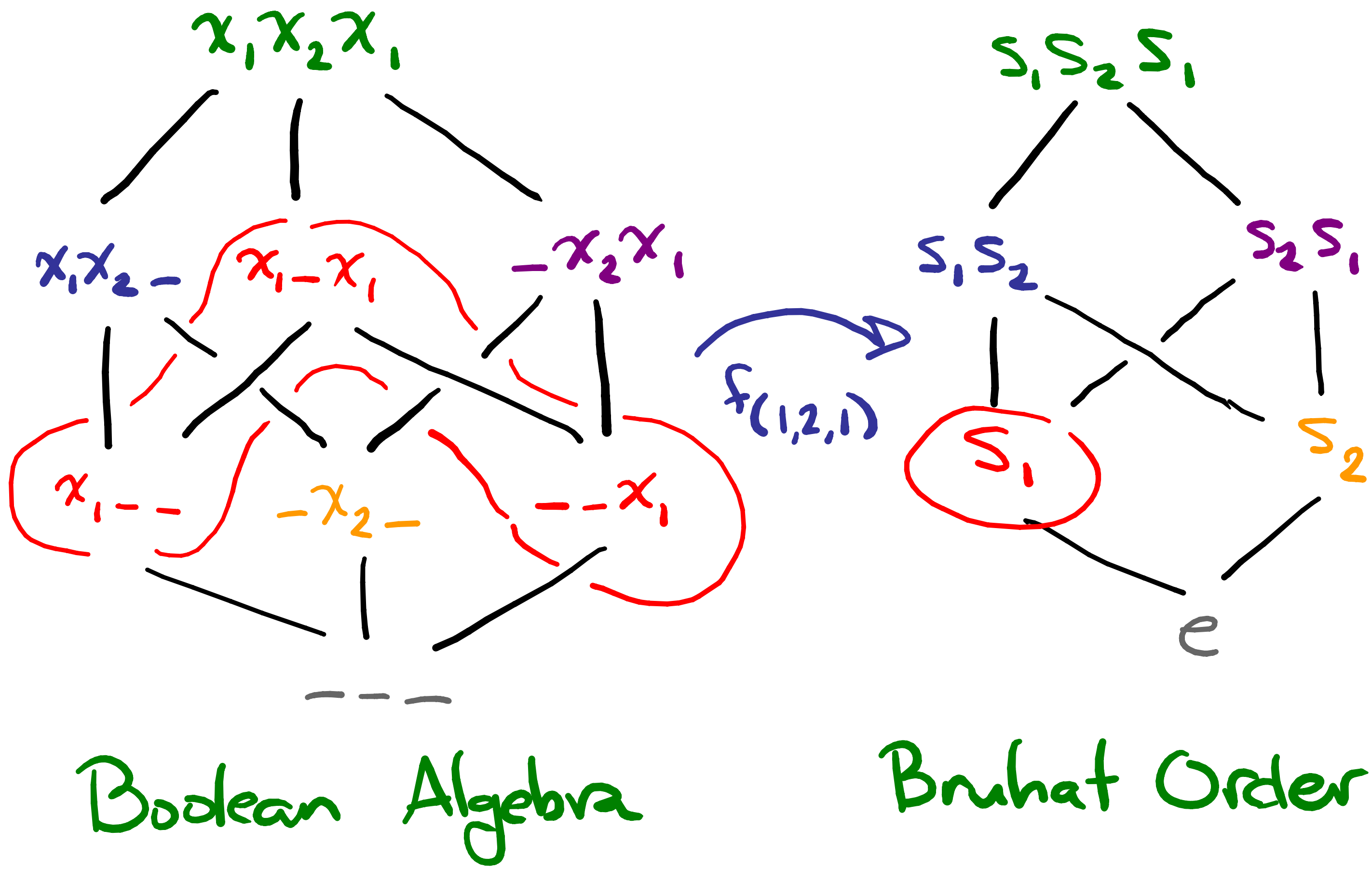
$s_1 - - -$ $\qquad$ $- - s_1 -$

Conjecture (Davis-H.-Miller): $f^{-1}_{(i_1,\ldots,i_d)}(p)$ for each $p \in Y_w^\circ$ is a regular CW complex homeomorphic to a ball with closure poset dual to face poset for interior of subword complex $\Delta((i_1,\ldots,i_d),w)$.



Remark: Subword complexes, discussed next, first arose in work of Knutson & Miller on matrix Schubert varieties as Stanley-Reisner complexes of initial ideals of coordinate rings

A Poset Map (on Face Posets) induced by $f_{(i_1,\dots,i_d)}$ and an implicit definition of subword complexes



- Fibers $f_{\sqsupseteq}^{-1}(u)$ are dual to face posets of subword complexes
- Subword complex = unique such regular CW complex (and is a Δ-complex/simplicial set)

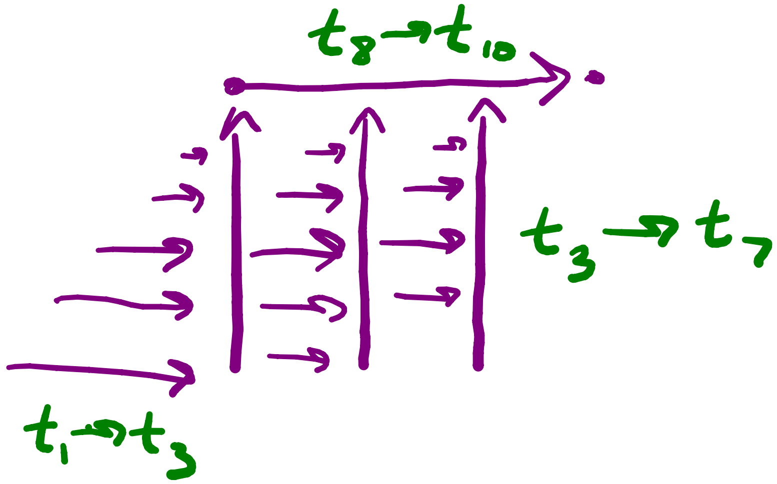
"Flow" on a Fiber from Collapsing Process to Base Point of Fiber

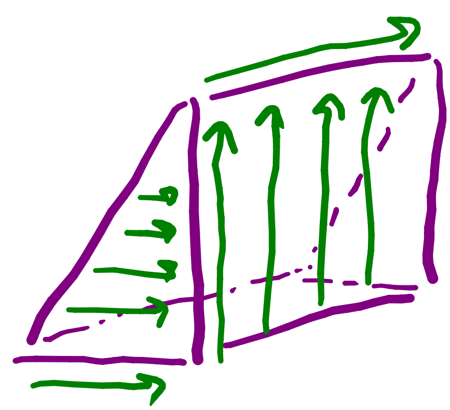
e.g. $x_1 - x_1 x_2 x_1 - x_2 x_3 - x_3$

$$x_1(t_1)\, x_1(t_3)\, x_2(t_4)\, x_1(t_5)\, x_2(t_7)\, x_3(t_8)\, x_3(t_{10})$$

$x_1(t_1+t_3)$ $\qquad$ $x_3(t_8+t_{10})$

$$x_2(t_1')\, x_1(t_2')\, x_2(t_3'+t_7)$$





"Topologist Approach" to Fomin-Shapiro Conjecture (joint work in progress with Jim Davis & Ezra Miller)

Combining Top'l Results: Let $g: B \to Z$ be continuous surjection from ball B to Hausdorff space Z whose restriction to $int(B)$ is an embedding. Suppose also:

(1) $g(\partial B) \cong \partial B = S^n$;

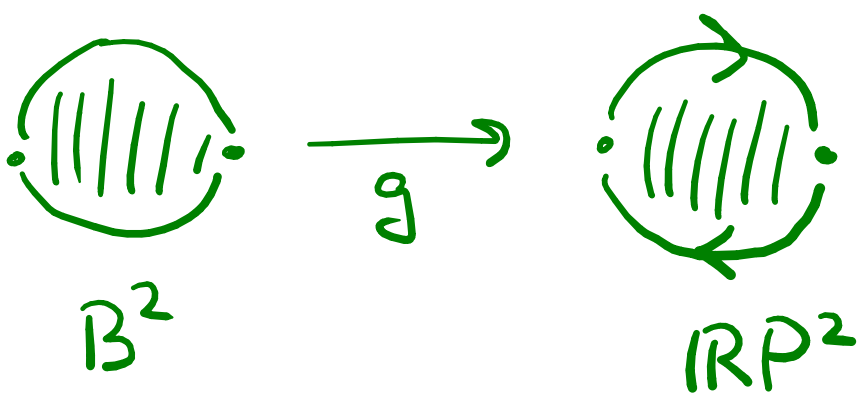
(2) $g(\partial B) \cap g(int(B)) = \emptyset$;

(3) $g^{-1}(p)$ is contractible $\forall p \in g(\partial B)$.

Then $Z \cong B$.

Example: Collapsing maps discussed earlier which collapse cells across families of closed curves – may regard collapsing process as factoring $f_{(i_1, \ldots, i_d)}$ into composition of (especially simple) such maps.

Non-example:



(1) ✓ $g(S^1) = S^1$

(2) ✓

(3) × $g^{-1}(p)$ = pair of antipodal points $\subset S^1$ $\therefore$ disconnected

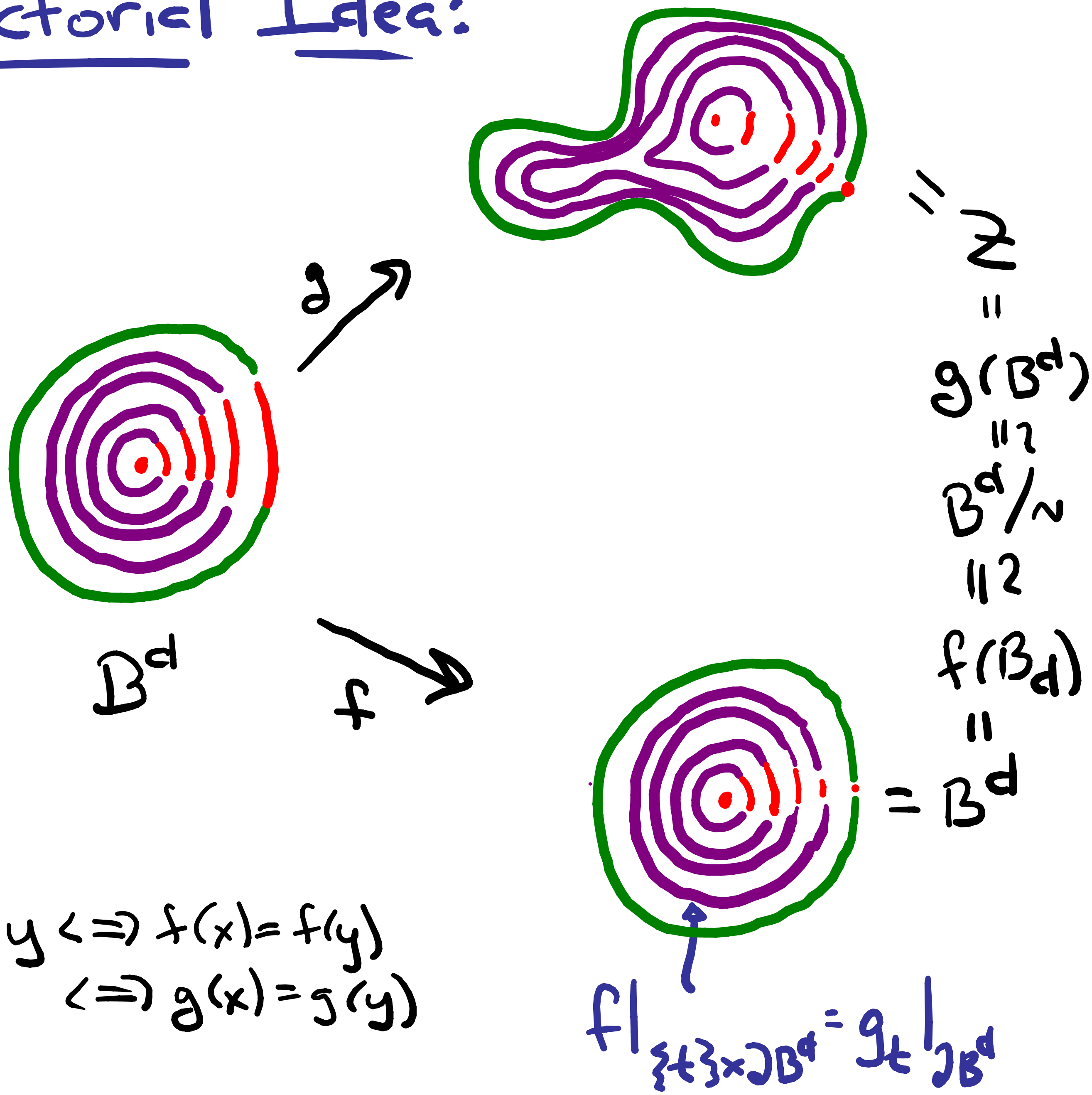
Ingredients in this Relationship Between Fibers & Image of "Nice" Map:

- CE-approx. theorem: $g : \partial B \to \partial B$ as above may be approximated by homeomorphisms
  - Kirby-Siebenmann: dim ≥ 5
  - Quinn: dim 4
  - Armentrout + Poincaré Conjecture: dim 3
- Local Contractibility of Homeo(S^n, S^n): two homeomorphisms "close enough" to each other may be connected by path of homeomorphisms

Idea: $B \cong$ metric ball $= \{\vec{0}\} \cup (0,1] \times \partial B$.

Use path of homeomorphisms converging to $g|_{\partial B}$ to construct $f : B \to B$ with $f^{-1}(p) = g^{-1}(p)$ $\forall p \in B$ and $f|_{\partial B} = g|_{\partial B}$, so $g(B) = B/\sim = f(B) \cong B$.

Pictorial Idea:



$$x \sim y \iff f(x) = f(y)$$
$$\iff g(x) = g(y)$$

Remark: May interpret proof of F-S Conjecture as (1) factoring $f_{(i_1,\dots,i_d)}$ as series of such collapses with explicit paths of homeomorphisms & requisite properties verifiable combinatorially + (2) regularity criterion

Checking Sphericity for $f_{(i_1,\dots,i_d)}(\partial\Delta^{d-1})$

1. Stratification has Bruhat intervals as closure posets.

2. By induction on dimension, cell closures in $f_{(i_1,\dots,i_d)}(\partial B)$ are balls.

3. Hence $f_{(i_1,\dots,i_d)}(\partial B)$ is regular CW complex, which we denote K.

4. Hence, $K \cong \Delta(F(K)^{\setminus\{\hat{0},\hat{1}\}}) \cong$ sphere since $F(K)$ is Bruhat order, hence thin and shellable, thus a sphere.

Upshot: Davis-H-Miller Conjecture would also imply Fomin-Shapiro Conjecture.

Further Questions

1. Analogous map, theory of "reduced expressions" & topol. results for totally nonnegative part of: Grassmannian? loop group? flag variety?
   (partial results of Postnikov, Rietsch, Williams, Speyer, Marsh, ...)
2. Explanation why subword complexes arising in distinct, but related settings? More general notion of subword complexes?

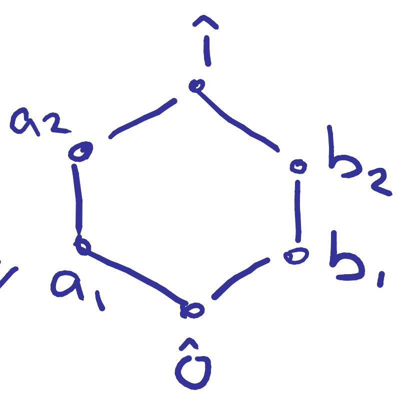
Thank you!

Appendix: Some Further Details

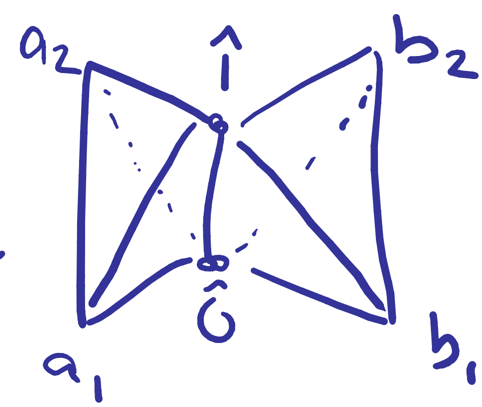
Def'n: The **order complex** (or **nerve**) of a poset P is the simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-chains $v_0 < \cdots < v_i$ in P

e.g.

$P \cong$



$\Delta(P) \cong$



Remark: Studied extensively e.g. in group theory (posets of subgroups), combinatorics (Möbius fn & group actions on posets), commutative algebra (small resolutions), etc.

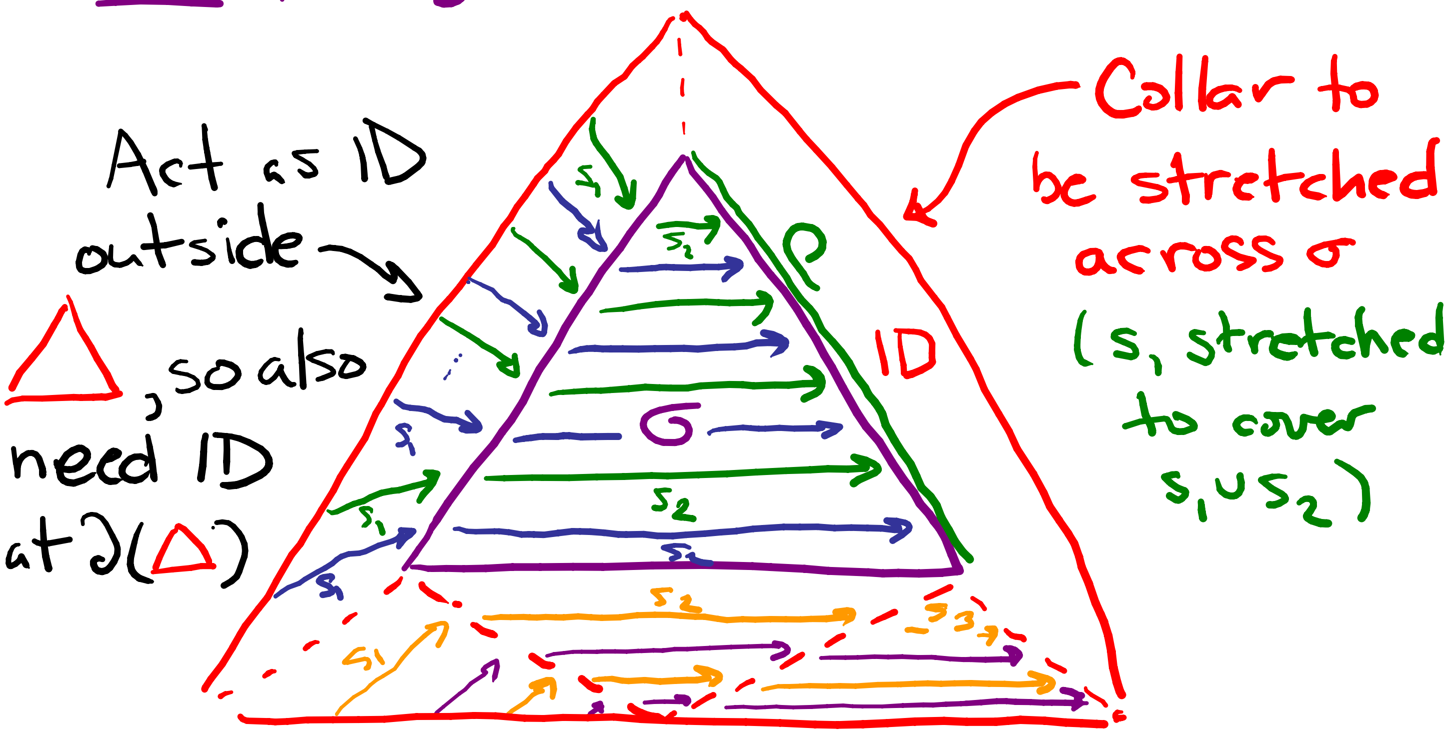
Connection to Schubert Varieties & their Schubert Cell Decompositions

- Y_w° = image of $f_{(i_1,\ldots,i_d)}: \mathbb{R}_{>0}^d \to M_{n\times n}$

- $Y_w = \overline{Y_w^\circ}$ = image from $\mathbb{R}_{\geq 0}^d$
  = totally nonnegative part of $\overline{B^- w B^-} \cap \begin{pmatrix}\text{unipotent} \\ \text{radical of } B\end{pmatrix}$

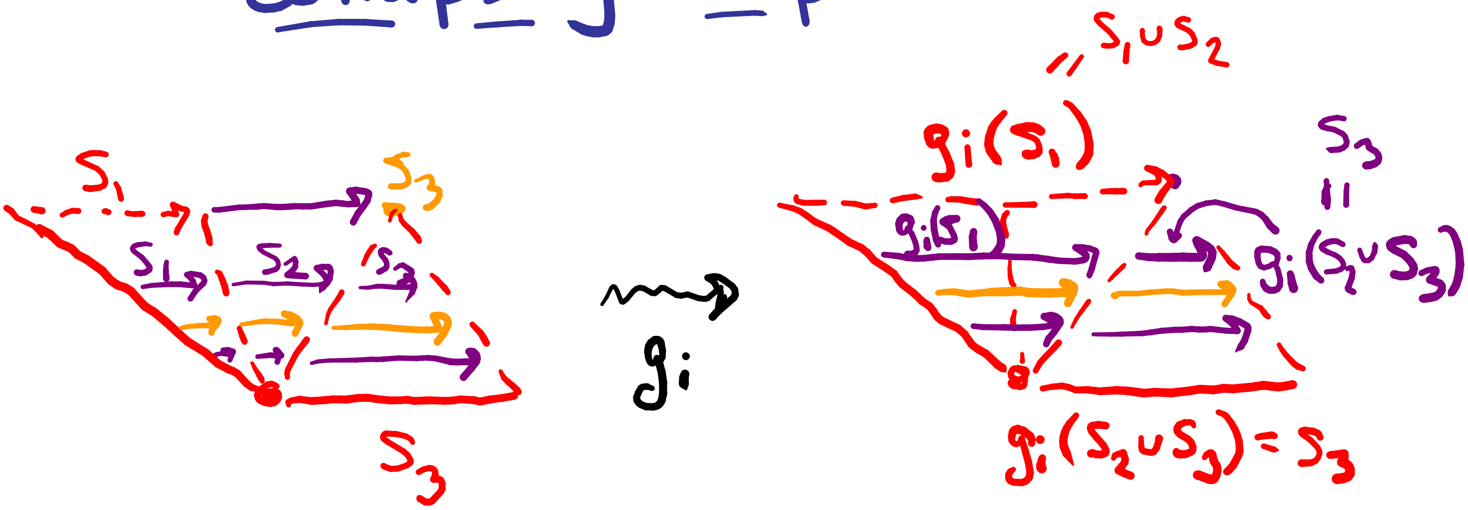
- Y_{w_0} = totally nonnegative part of space of upper triangular matrices w/ 1's on diagonal
  (old result of Whitney-type A)

Collapsing a Cell $\bar{\sigma}$ onto a Cell $\bar{\rho}$



- Map segments s_2 in $\bar{\sigma}$ onto endpoint in $\bar{\rho}$, stretch extension $s_1 \subseteq$ collar to cover $s_1 \cup s_2$, act as ID on $\bar{\rho} \times [0,1] \subseteq$ collar.
- For $c \subseteq \partial\sigma$, collapsing map on $c \times [0,1]$ will stretch s_1 to cover $s_1 \cup s_2$ & shorten $s_2 \cup s_3$ to cover s_3, as depicted next.

"Close-up" of bottom part of collapsing map



Key Observations:

(1) This type of collapse makes sense more generally, relying on existence of continuous fn $\ln : \bar{\sigma} \to \mathbb{R}$ sending point to "length" of segment in $\bar{\sigma}$ containing it.

(2) These collapses are explicitly approximable by homeomorphisms:

Approximate

by

(Mainly Combinatorial) Requirements Enabling Collapses Across Curves

There is a series of earlier face collapses

$$K_0 \xrightarrow{g_1} K_1 \xrightarrow{g_2} K_2 \to \cdots \to K_i$$

$K_0 = \Delta$

$K_1 \cong \Delta/\sim_1$ (new cell structure), K_1 " Δ (as g_1 is surjection onto Δ)

$K_2 \cong \Delta/\sim_2$

with closed cell of K_i covered by images of parallel line segments in K_0 with family G_i of "parallel-like" curves satisfying:

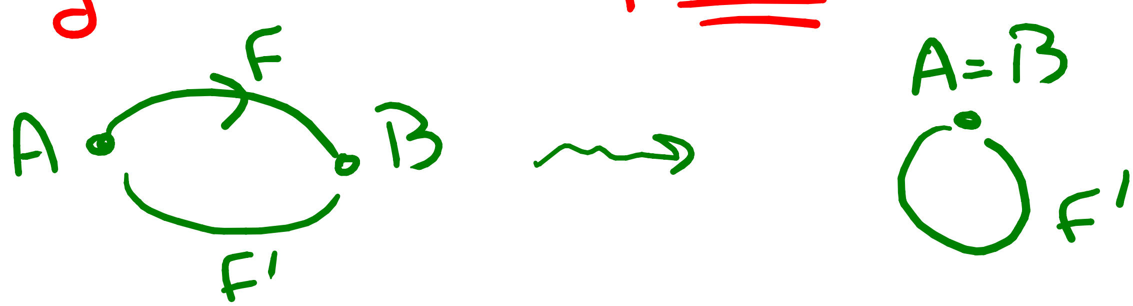
- **Distinct endpoints condition (DE):** the two ends of a nontrivial curve live in cells not yet identified

- **Distinct initial points condition (DIP):** distinct curves have distinct starting points (so collapse well-defined)

Condition to ensure collapses preserve regularity (suggested by David Speyer)

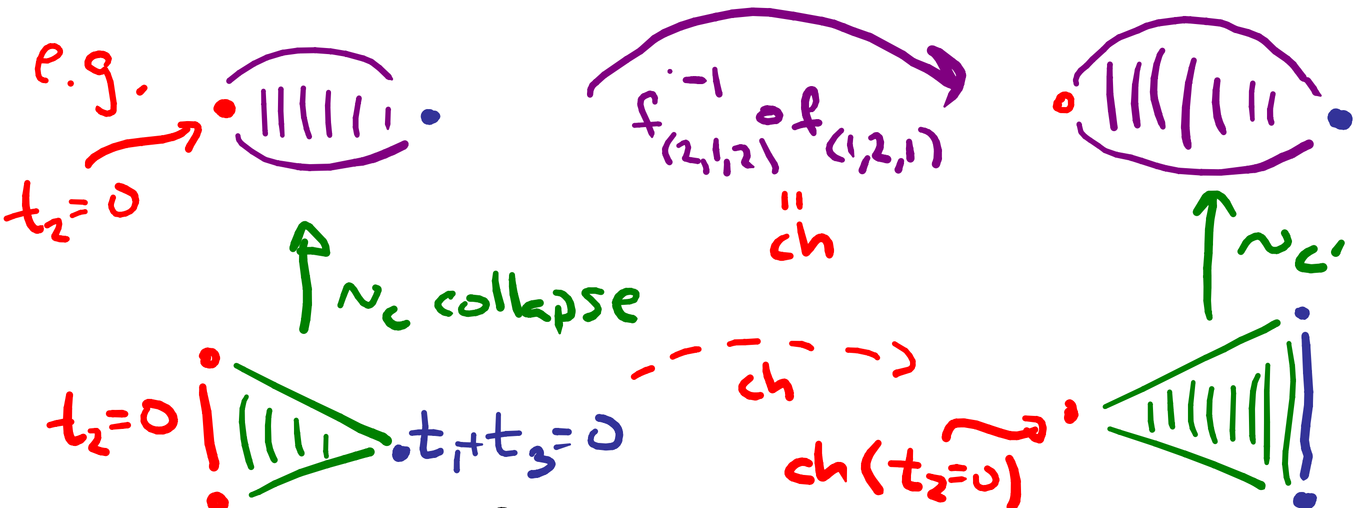
Least Upper Bd Condition: If cells A & B are IDed via face collapse of F, then all least upper bounds for A & B just prior to collapse must also be collapsed in this step.

e.g. Want to prevent



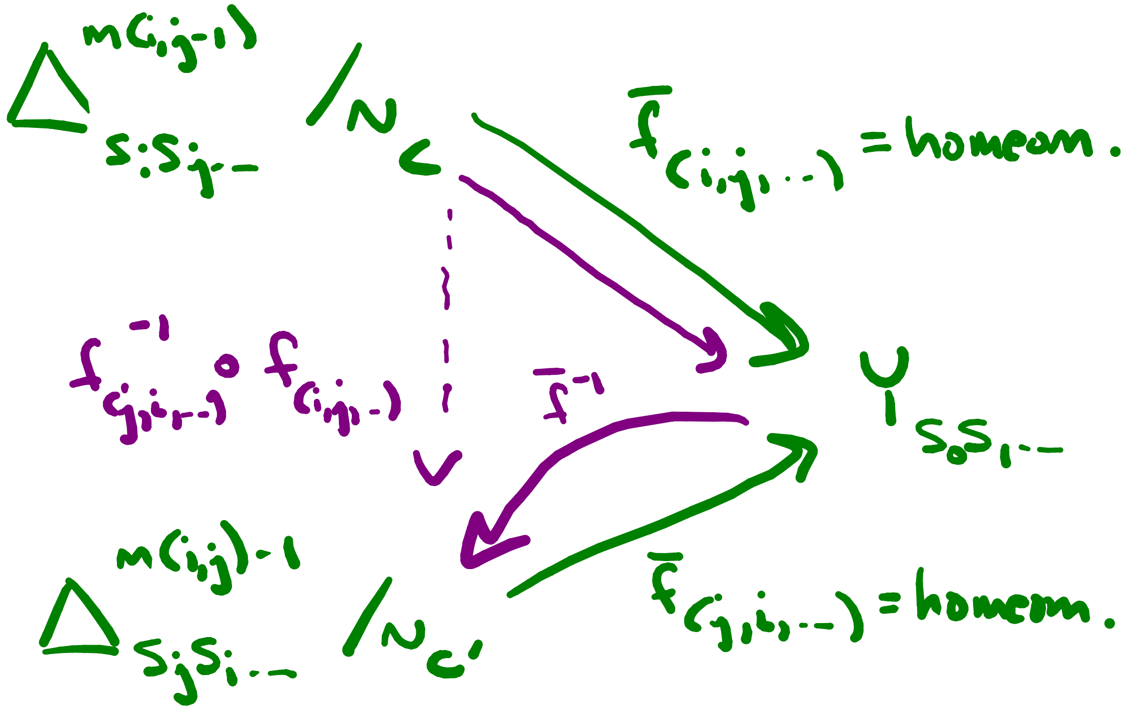
Note: Conditions checkable with combinatorics of reduced/nonreduced words of 0-Hecke algebra!

Long Braid Move as Change of Coord's
Homeomorphism on Closed Cell to Be Collapsed

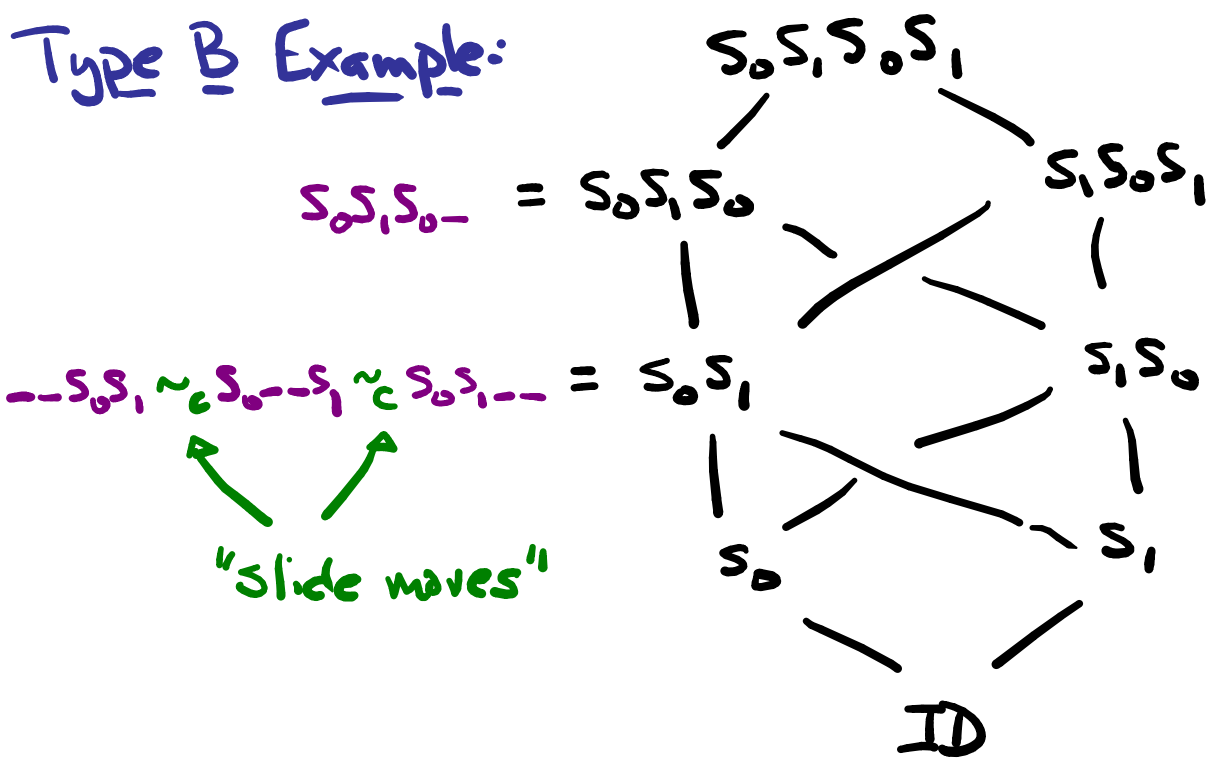


Key Lemma: Consider reduced expressions $s_i s_j s_i \ldots$ and $s_j s_i s_j \ldots$ of length $m(i,j)$ and equivalence relations $\sim_c$ and $\sim_{c'}$ on $\Delta^{m(i,j)-1}_{s_i s_j \ldots}$ and $\Delta^{m(i,j)-1}_{s_j s_i \ldots}$ given by identifications based on commutation and "slide moves". Then $\Delta^{m(i,j)-1}_{s_i s_j \ldots}/\sim_c \cong \Delta^{m(i,j)-1}_{s_j s_i \ldots}$ via the homeomorphism $f^{-1}_{(j,i,\ldots)} \circ f_{(i,j,\ldots)}$.

Idea: Subwords of $(i,j,\ldots)$ and $(j,i,\ldots)$ do not admit any long braid moves, so:

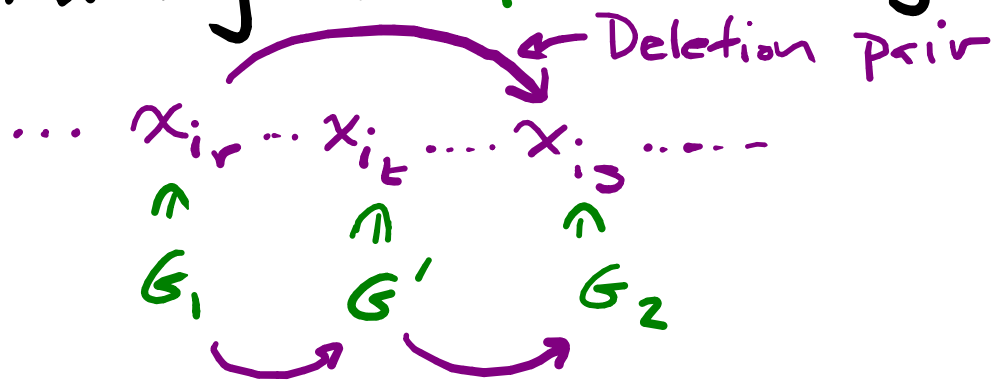


Type B Example:



Verifying DE (Distinct Endpoints Condition) with Combinatorics (Gives Flavor of Many Lemmas)

Suppose collapse of F uses curves starting in G_1 and ending in G_2



If G_1 were already identified earlier with G_2 then there exists G' with earlier steps identifying G_1 with G' and G' with G_2. But the former would have also identified $G_1 \cup \{x_{i_s}\} = F$ with the cell $G' \cup \{x_{i_s}\} = F'$ which was already collapsed in step identifying G' with $G_2 \Rightarrow\Leftarrow$

Subword Complexes (introduced by Knutson & Miller)

$Q :=$ (not necessarily reduced) expression

$w :=$ Coxeter group element

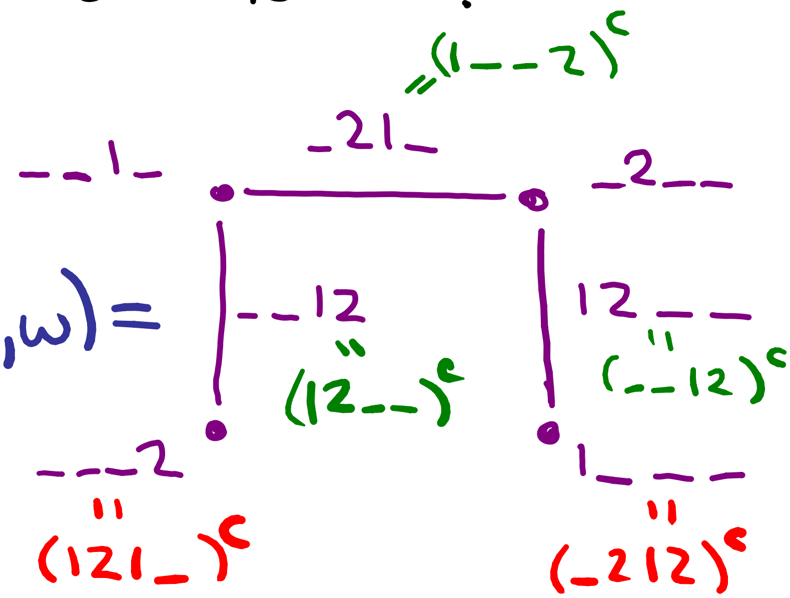
Facets of $\Delta(Q,w)$ are the subwords of Q whose complements are reduced words for w.

e.g.

$Q = (1,2,1,2)$

$w = s_1 s_2$

$\Delta(Q,w) =$



Thm (Knutson-Miller): $\Delta(Q,w)$ is vertex decomposable (hence shellable) ball or sphere.

More Generally? "Fibers" of Parametrization Maps for Nonneg flag variety, loop groups, etc.?

Homotopy Type of Bruhat Intervals:
New Proof by Quillen Fibre Lemma

Thm (Armstrong-H.): The poset map $f_{(i_1 \ldots i_d)}$ yields short proof of:

$$\Delta_{Bruhat}(u,v) \simeq S^{rk\,v - rk\,u - 2} \quad \text{for all } u \leq v$$

Idea: • fibers $f_{\geq}^{-1}(u) = \{x \in B_n \mid f(x) \geq u\}$ are dual to face posets of subword complexes – proven to be balls by Allen Knutson & Ezra Miller.

Subword complexes previously arose as:

Stanley-Reisner complex for Gröbner degeneration of matrix Schubert variety ideal (Knutson and Miller)