Combinatorics & Topology of Totally Positive Spaces

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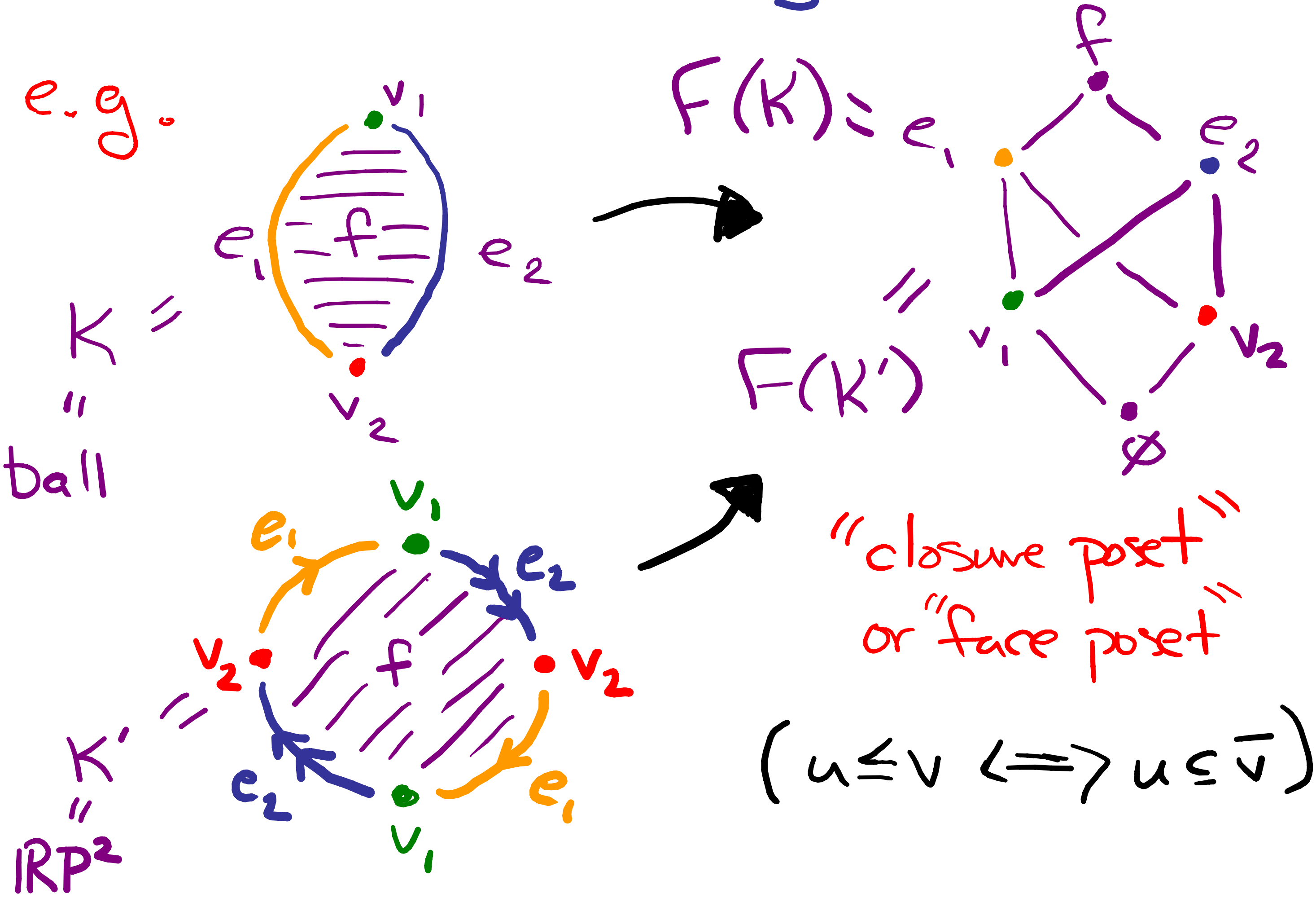
- Slides available at:
  http://www4.ncsu.edu/~plhersh/Brown-Colloq.pdf
- see "Regular cell complexes in total positivity", Inventiones Mathematicae, 197 (2014), 57-114 for details.

Topological Aspects of Total Positivity

- Lusztig initiated study of Totally nonnegative, real part of (matrix) Schubert variettes, flag varieties,...
  (i.e. part with minors all nonnegative in spaces of matrices or of flags)
- Conjecturally/provably homeomorphic to closed balls (after deconing)
- Proving this:
  - puts restrictions on relations among (exponentiated) Chevalley generators.
  - reveals structure in canonical bases; a motivation for cluster algebras.
- **Main Result of Talk:** Proof of Fomin-Shapiro Conjecture via new tools exploiting interplay of combinatorial data & topological data.

CW Complexes & their Face Posets

(i.e. Partially Ordered Sets)

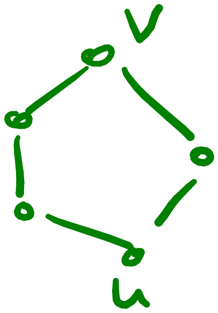


"closure poset" or "face poset"

$(u \le v \iff u \le \bar{v})$

Recall: A **CW complex**: cells $e_\alpha \cong \mathbb{R}^{d(\alpha)}$, characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \to \bigcup_{e_\beta \le \bar{e}_\alpha} e_\beta$ & attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

- A poset is **graded** if $u \le v$ in P implies all minimal paths upward from u to v have same length (i.e. #edges)

e.g. 

is not graded

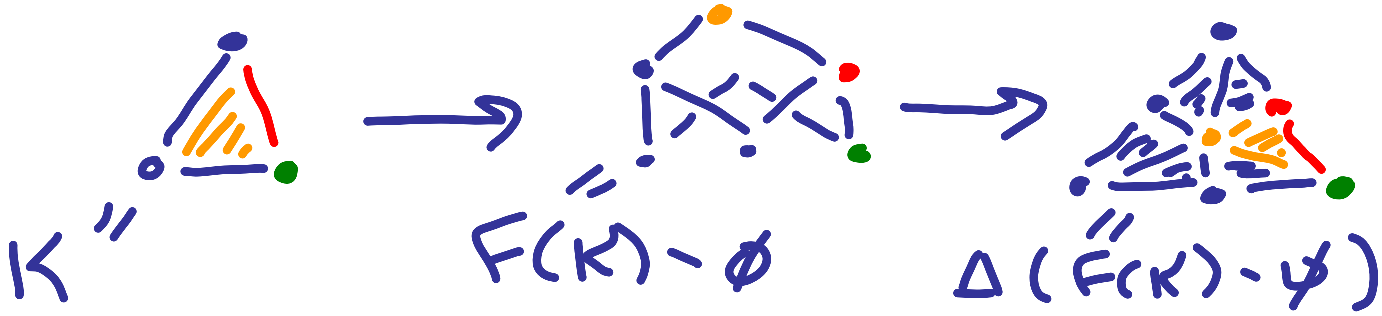
- A graded poset is **thin** if each rank 2 interval has exactly 4 elements.

Recall: A CW complex is **regular** if the attaching map f_α for each cell e_α is a homeomorphism, i.e. cell closures are closed balls.

Recall: 

- K regular $\Rightarrow K \cong \underbrace{\Delta}\left(F(K) \setminus \emptyset\}\right) = \mathrm{sd}K$

nerve or order complex, i.e. simplicial complex whose faces are chains $u_1 < \cdots < u_i$.



Defn (Björner): A finite, graded poset P is CW poset if

- P has unique min'l elt. $\hat{0}$
- P has additional element(s)
- $x \neq \hat{0} \Rightarrow \Delta(\hat{0}, x) \cong S^{\mathrm{rank}(x)-2}$

Thm (Björner): P is CW poset $\Longleftrightarrow$ there exists regular CW complex K with $P = F(K)$.

Some Examples of CW Posets

- Bruhat order (Björner & Wachs) of finite Coxeter group
- Closure poset for double Bruhat decomp. of totally nonneg. part of flag variety (Williams)
- thin, "shellable" posets (Danaraj & Klee)
- Closure poset of double suspension of triangulation of homology sphere with "big cell" glued in (due to work of J. Cannon & R. Edwards)

A Goal of Mine: Use combinatorics of $F(K)$ + limited topological info (codim. one cell incidences) to understand K.

Finite Coxeter Groups & Finite Reflection Groups: Quick Review

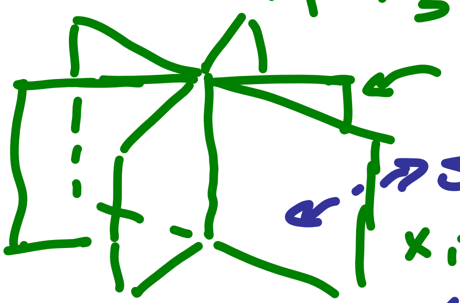
Coxeter group: generators $\{s_i \mid i \in S\}$

& rel'ns $s_i^2 = e$ & $(s_i s_j)^{m(i,j)} = e$

e.g. $$\underset{\substack{\shortparallel \\ \text{symmetric} \\ \text{group}}}{S_3} = \langle \underset{\substack{\shortparallel \\ (1,2)}}{s_1}, \underset{\substack{\shortparallel \\ (2,3)}}{s_2} \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle$$ (& other Weyl gps)

geometrically:

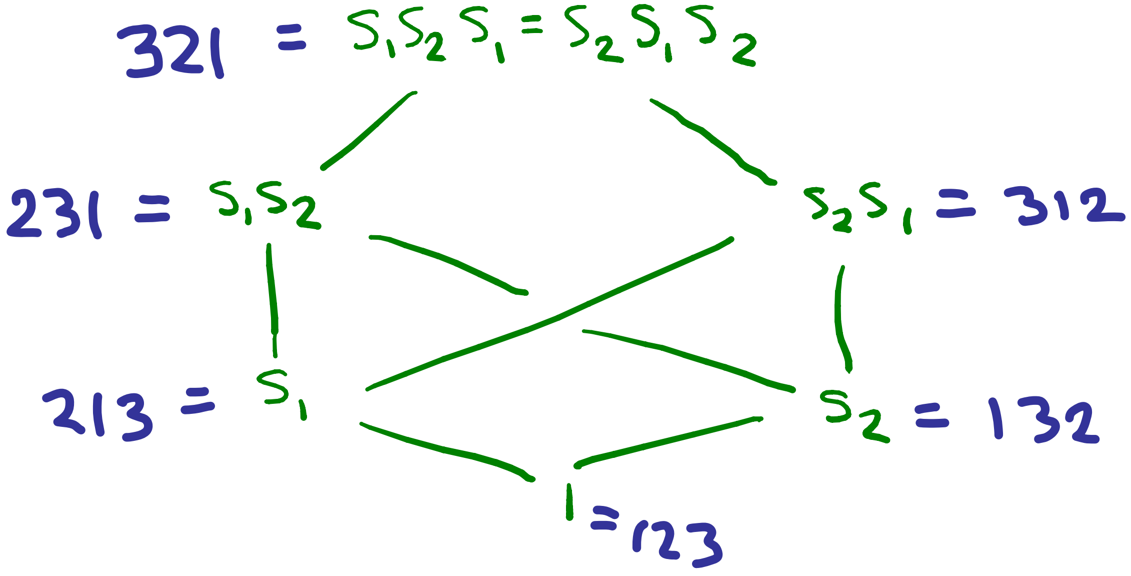
(as group of reflections)



A **reduced expression** for $w \in W$ is an expression of minimal "length" e.g. $s_1 s_2$ but not $s_1 s_2 s_1 s_2$ (which equals $s_2 s_1$)

The **Bruhat order** is partial order on elements of Coxeter group W with $u \leq v$ $\Leftrightarrow$ there exist reduced expressions $r(u)$ and $r(v)$ for u & v, respectively, with $r(u)$ subexpression of $r(v)$.

e.g. $W = S_3$ with generators $s_1 = (1,2)$, $s_2 = (2,3)$



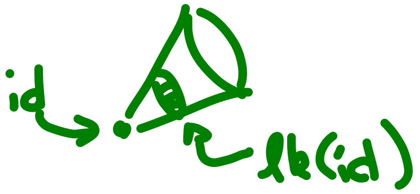
- **reduced word** $(i_1, \dots, i_d)$ for $s_{i_1} s_{i_2} \cdots s_{i_d}$
- Closure poset for Schubert cell decompositions of flag varieties G/B (all cells have even real dim)

Qn (Bernstein & Björner): Find regular CW complexes naturally arising from rep'n theory which are homeomorphic to closed balls and have the (lower) Bruhat intervals as closure posets.

Conjectural Solution (Fomin & Shapiro):

The Bruhat stratification of lk(id) in totally nonneg. real part of unipotent radical of Borel in semisimple, simply connected algebraic group defined and split over $\mathbb{R}$.

i.e. a "slice"



within:

$$Y_w = \left[\overline{B^- w B^-} \cap \begin{pmatrix}\text{unipotent}\\ \text{subgp of } B\end{pmatrix}\right]_{\geq 0}$$

lower triangular opposite Borel B^-

permutation w

upper triang. w/ 1's on diagonal

totally nonneg. part

Theorem (H.): Fomin-Shapiro Conjecture indeed holds.

Special Case (Running Example for Talk): Space of totally nonnegative upper triangular matrices with 1's on diagonal & entries just above diagonal summing to fixed, positive constant, stratified by which minors are positive and which are 0.

Concrete Realization: products of certain elementary matrices, by results of Whitney & Lusztig.

The Totally Nonnegative Part of a Space of Matrices

column $i+1$

- $x_i(t) = I_n + tE_{i,i+1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & t & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$ ← row i

$\shortparallel$

$\exp(te_i)$

↑ (general finite type, expon'd Chevalley generator)

↑ (type A)

- $f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \underline{\in} \mathbb{R}^{n^2}$

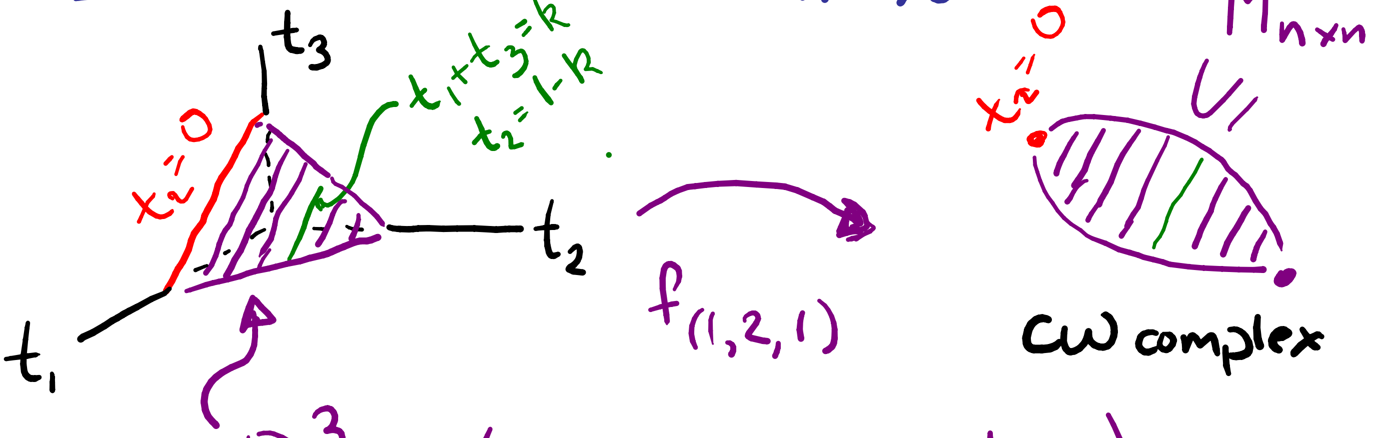
$$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1,\ldots,i_d)}$



$\mathbb{R}^3_{\geq 0} \cap (\sum t_i = 1$ hyperplane$)$

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow t_2 = 0$$

$$x_1(t_1)\cdot x_1(t_3)$$

$$f_{(1,2,1)}(t_1,0,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

Non-injectivity: results from "modified nil-moves" $x_i(u)x_i(v) \to x_i(u+v)$ directly & after "long braid moves" in 0-Hecke algebra.

1st Key Idea to Proof of F-S Conjecture

0-Hecke Algebra Captures which Simplex Faces have Same Image under $f_{(i_1,\dots,i_d)}$

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

"nil-move"

$\rightsquigarrow$ suppress parameters

$x_i^2 = x_i$ (0-Hecke alg. reln, up to sign)

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1t_2}{t_1+t_3}\right)$

$\rightsquigarrow$ (type A) assuming $t_1+t_3>0$

$x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1}$

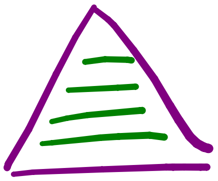
(similar relation holds outside type A)

"long braid move" with enrichment from parameters

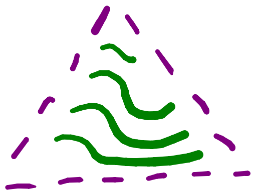
Fibers as Curves:

$x_i^2 \to x_i$

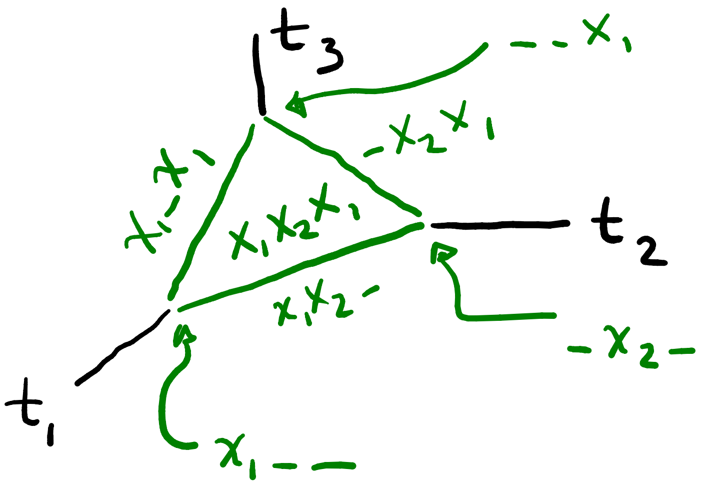
$t_1+t_2=k$



OR after a braid move:



Indexing Faces of Preimage by Words in 0-Hecke Algebra



Key Observation About $f_{(i_1,\dots,i_d)}$:

$$\mathrm{im}(f_1)=\mathrm{im}(f_2) \Leftrightarrow \underbrace{x(f_1)=x(f_2)}_{\text{equal as 0-Hecke algebra elements}}$$

Thm (Lusztig): If $(i_1,\dots,i_d)$ is reduced, then $f_{(i_1,\dots,i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

Observation: "non-reduced" subwords give redundant faces covered by curves, each in single fiber of $f_{(i_1,\dots,i_d)}$

Properties of Change-of-Coordinates Map Given by Braid Moves

e.g. $(t_1, t_2, t_3) \mapsto \left(\frac{t_2 t_3}{t_1+t_3}, t_1+t_3, \frac{t_1 t_2}{t_1+t_3}\right)$

in type A

- Tropicalizes to change-of-basis map for Lusztig's canonical bases:

$$(a,b,c) \mapsto (b+c-\min(a,c), \min(a,c), a+b-\min(a,c))$$

- A motivation for development of cluster algebras (& mutation)

Exercise: check this is an involution.

Proof Strategy (Phrased for Possible Future Applic's too)

Set-up: Continuous, surjective fn

$$f: P \longrightarrow Y$$

from convex polytope P (e.g. Δ_n) s.t. f maps $\mathrm{int}(P)$ homeomorphically to $\mathrm{int}(Y)$.

Step 1: Perform "collapses" on ∂P, each preserving regularity and homeomorphism type, via continuous, surjective collapsing functions $P \to P$ yielding $P/\sim$ with fewer cells s.t. $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$

(to accomplish IDs we know are needed by collapsing all non-reduced faces)

Step 2: Prove $\overline{f}: P/\sim \longrightarrow Y$ is homeomorphism by new regularity criterion

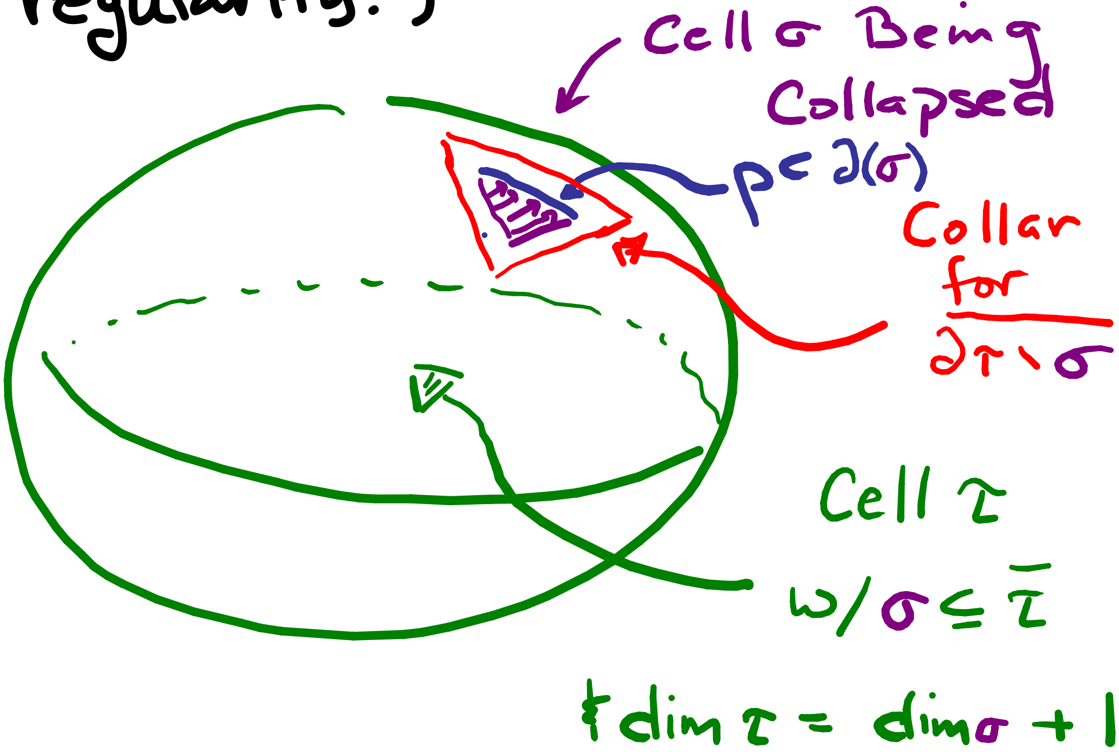
(to prove no further IDs are needed)

Step 1 Collapsing Cell σ onto Cell $\rho \subseteq \overline{\sigma}$ within $\partial\tau$

Thm (M. Brown; Connelly): Any topological manifold with boundary ∂M has a collar (i.e. a nbh'd homeomorphic to $\partial M \times [0,1]$).

e.g. $\overline{\partial\tau \setminus \sigma} =$ 

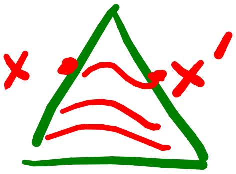
Plan: Collapse $\overline{\sigma}$ onto $\overline{\rho} \subseteq \partial\sigma$, stretching collar for $\overline{\partial\tau\setminus\sigma}$ to cover $\overline{\sigma}\setminus\overline{\rho}$, preserving top'l manifold structure (& homeom. type & regularity!)



(Mainly Combinatorial) Conditions Allowing Such Face Collapses Across Curves

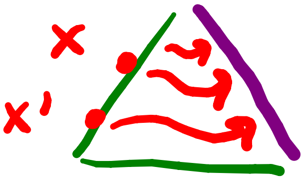
$$\begin{tikzcd}
 & & \Delta & & & \\
 & K_0 \arrow[r, "g_1"'] \arrow[d, phantom, sloped, "="] & K_1 \arrow[u, phantom, sloped, "="] \arrow[r, "g_2"'] \arrow[d, phantom, sloped, "\cong"] & K_2 \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \cdots \arrow[r] & K_i \\
 & \Delta & \Delta/\sim_1 & \Delta/\sim_2 & & \\
 \text{polytope!} \arrow[uur, bend left] & & & & &
\end{tikzcd}$$

- collapse face in K_i across images of parallel line segments in K_0 satisfying:
  - Distinct endpoints condition (DE):



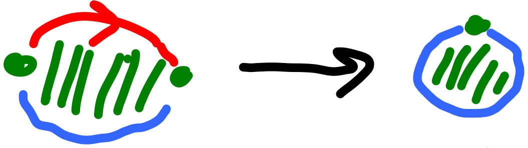
$x \neq x'$

  - Distinct initial points condition (DIP):



$x \neq x'$

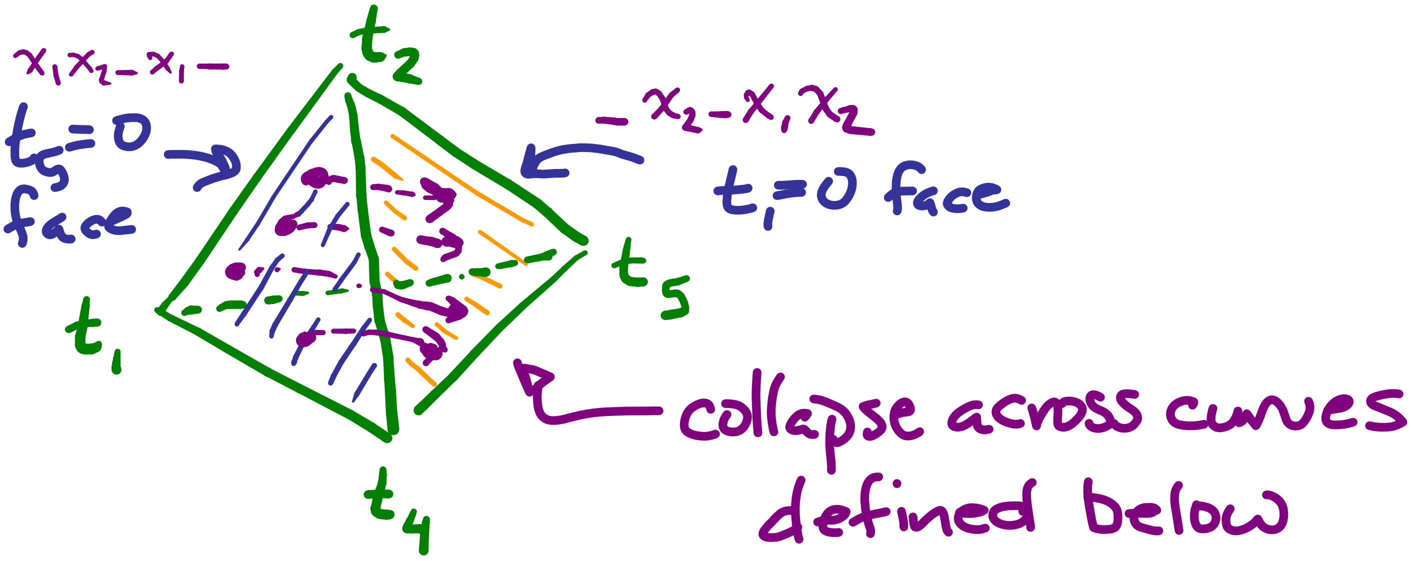
  - Least upper bound condition (LUB) to preclude



(conditions checkable via 0-Hecke algebra)

Example of a Face Collapse

e.g. $f_{(1,2,3,1,2)}$ face with $t_3=0$



$$(t_1,t_2,0,t_4,t_5) \mapsto x_1(t_1)x_2(t_2)x_1(t_4)x_2(t_5)$$

$$\boxed{\text{for } t_1+t_4>0} \quad = x_2(t_1')x_1(t_2')\underbrace{x_2(t_4')x_2(t_5)}$$

$$= x_2(t_1')x_1(t_2')x_2(t_4'+t_5)$$

for $t_1' = \frac{t_2t_4}{t_1+t_4}$ $\quad t_2'=t_1+t_4$ $\quad t_4' = \frac{t_1t_2}{t_1+t_4}$

Curves: $t_1'=k_1$ & $t_2'=k_2$ & $t_4'+t_5=k_3$

Step 2: Proving that induced

map $\bar{f}_{(i_1,\dots,i_d)} : \Delta_n/\sim \;\longrightarrow \left\{\begin{matrix}\text{space}\\ \text{of}\\ \text{matrices}\end{matrices}\right\}$

on quotient

space

is a homeomorphism.

identifications
from
collapsing
non-reduced
faces

New Regularity Criterion for Finite CW Complexes

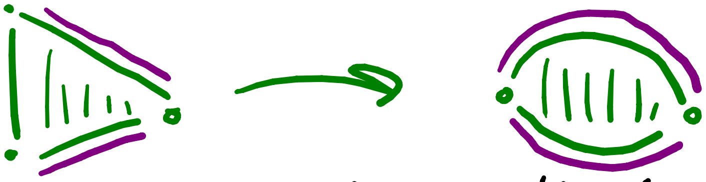
Preparatory Lemma (H.): Let K be a finite CW complex w/ characteristic maps $\{f_\alpha\}$. Suppose:

(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example:

(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example:



Then $F(K)$ is graded by cell dimension.

Insightful feedback (Quinn): Next theorem "spreads around" injectivity requirement.

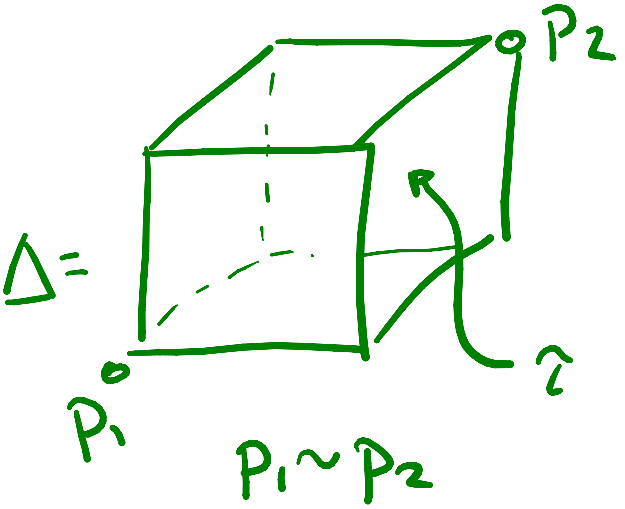
Thm (H.) Let K be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then K is regular w.r.t. $\{f_\alpha\}$ <=>

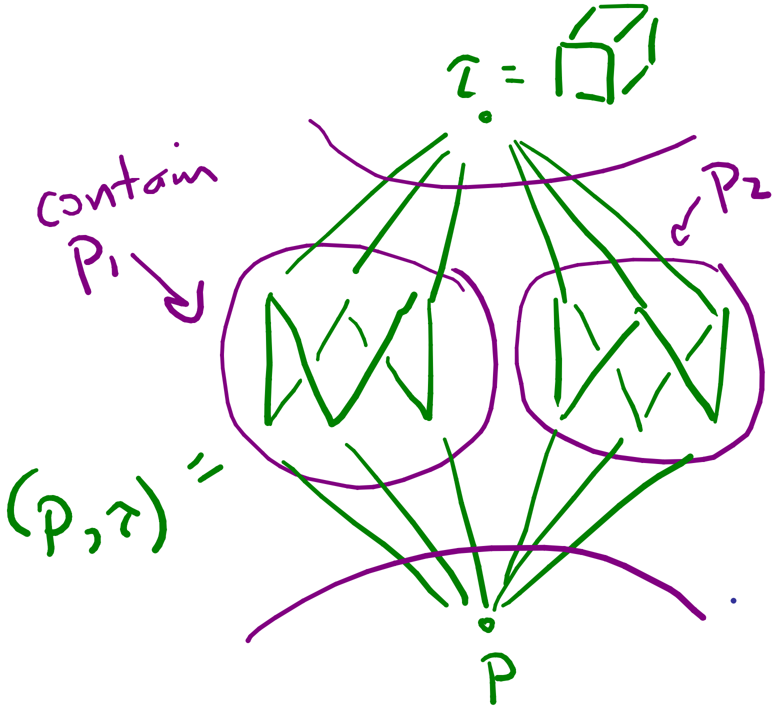
(1) K meets requirements of prop'n for $F(K)$ to be graded by cell dim.

(2) $F(K)$ is thin and each open interval (u,v) for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

Non-Example





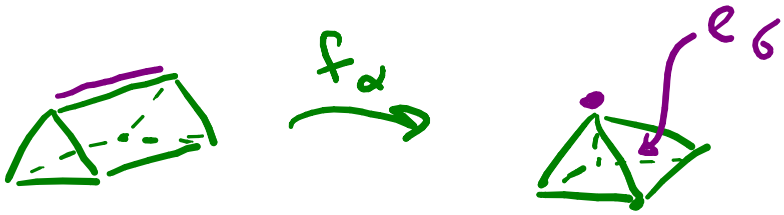
(3) For each α, the restriction of f_α to preimages of codim. one cells in $\overline{e_\alpha}$ is injective.

(topological condition)

Non-Example:

(4) $\forall\, e_\sigma \leq \overline{e_\alpha}$, f_σ factors as continuous inclusion $i: B^{\dim \sigma} \to B^{\dim \alpha}$ followed by f_α.

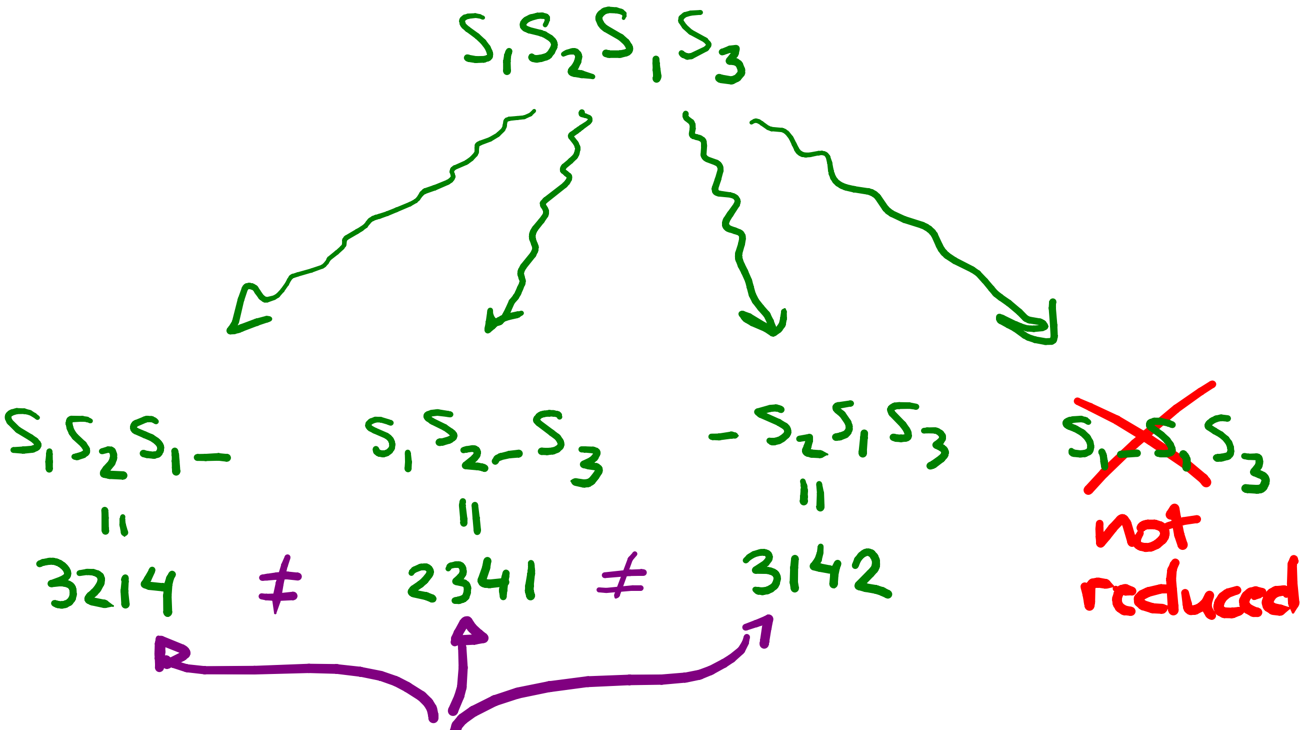
Non-Example:



Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one.

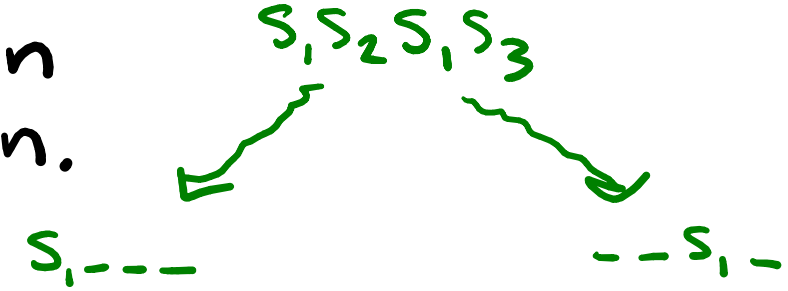
Proof: Induction on difference in dim.

Injectivity of Attaching Maps in Codimension One via Coxeter group strong exchange axiom



reduced subexpressions of reduced expression obtained by deleting one letter give **distinct** Coxeter group elements.

In contrast: fails in higher codimension.



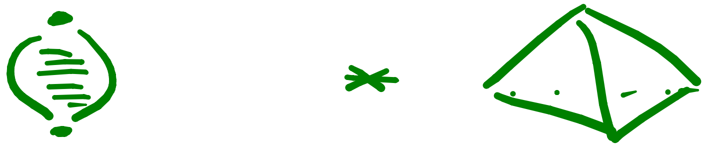
Additional Key Challenges

1. 0-Hecke algebra lacks inverses & lacks cancellation law.

   Key Idea: find ways to transfer properties from Coxeter gp

2. Need change-of-coord's for braid moves as homeom's on closed cells

   Key Idea: induction by embedding smaller instance



$$\underbrace{(i_1, i_2, i_3, \ldots\ldots i_{d'-1}}_{\text{all possible collapses done}}, \underbrace{i_{d'}, \ldots, i_d)}_{\text{not in any collapses yet}}$$

im$(f_{(i_1 \ldots i_{d'-1})})$

3. Need maps to extend to full complex

4. Tricky combinatorics to verify LUB at each collapsing step.

Other Stratified Spaces with Seemingly Similar Features

1. Totally nonnegative part of Grassmannian

Postnikov: polytope of plabic graphs w/ "measurement map" to $Gr_{\geq 0}$ + elaborate theory of plabic graphs

Postnikov-Speyer-Williams: $Gr_{\geq 0}$ is CW complex (via attaching maps that are not homeomorphisms)

2. Closed cells in totally nonnegative part of loop group

Lam-Pylyavskyy: developed theory of these spaces

3. Totally nonnegative part of flag variety

Rietsch: poset of closure reln's

Marsh-Rietsch: parametrization

Williams: poset is CW poset

Rietsch-Williams: CW complex w/ attaching maps via canonical bases.

Note: our spaces arise as links of cells

4. Stratified spaces of electrical networks

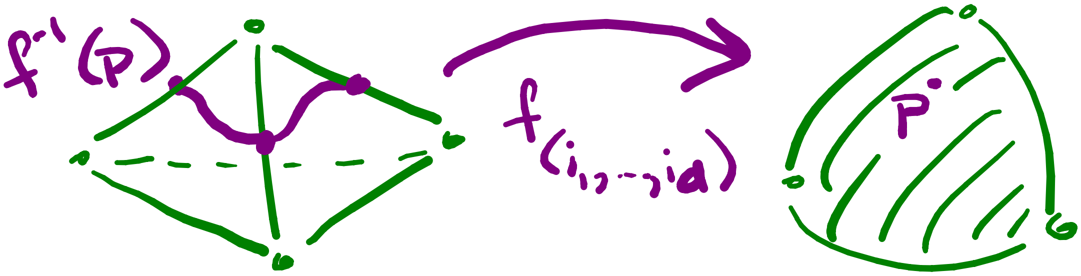
Kenyon-Propp-Wilson, Lam, Curtis-Ingerman-Morrow, Kenyon,...

Open Qn: homeomorphism type & other topol. structure for these spaces?

A Follow-up Project:

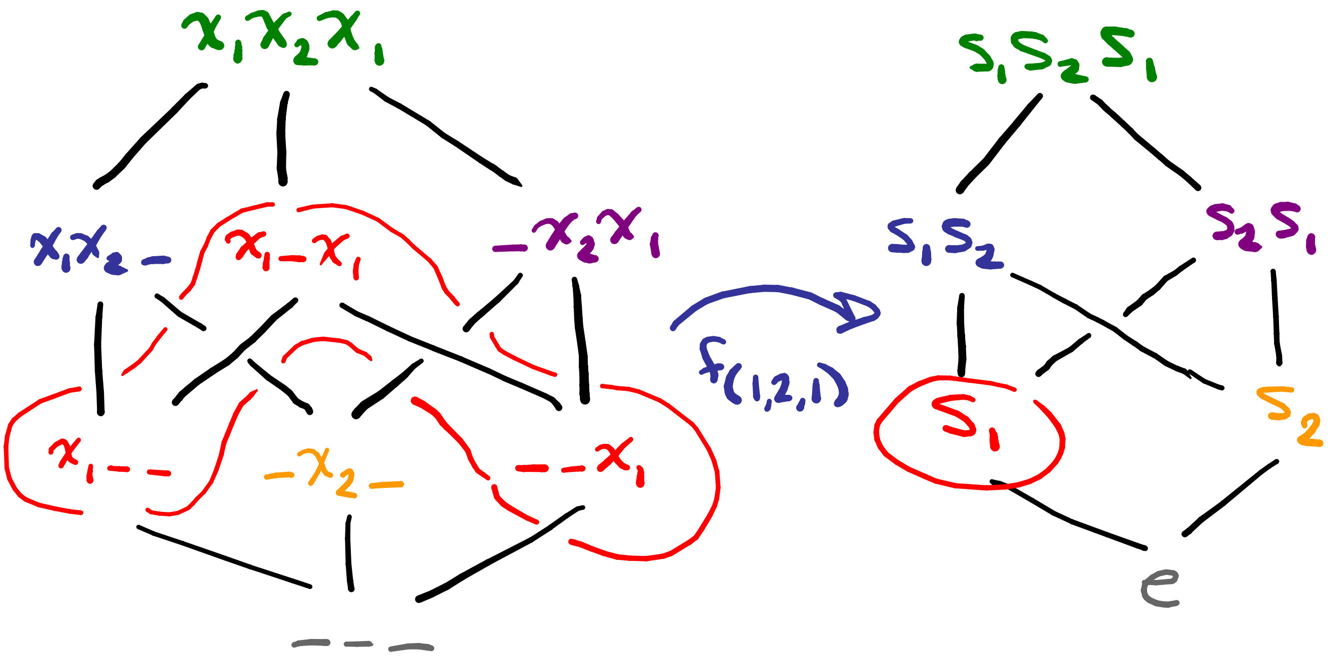
(with Jim Davis & Ezra Miller)

Conjecture (Davis-H.-Miller): $f^{-1}_{(i_1,\dots,i_d)}(p)$ for each $p \in Y_w^o$ is a regular CW complex homeomorphic to a ball with closure poset dual to face poset for interior of "subword complex" $\Delta((i_1,\dots,i_d),w)$.



Remark: Subword complexes introduced by Knutson & E. Miller as Stanley-Reisner complexes of initial ideals of coordinate rings assoc'd to matrix Schubert varieties.

A Poset Map (on Face Posets) induced by $f_{(i_1,\ldots,i_d)}$ (& implicit Def'n of Subword Complexes)



Boolean Algebra B_n Bruhat Order

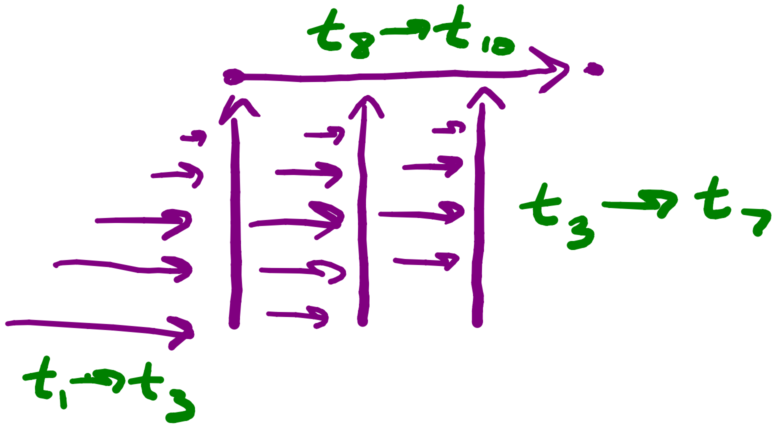
- Apply braid moves & $x_i^2 \to x_i$ to get reduced expression; replace x_i's by s_i's
- Fibers $f_{\geq}^{-1}(u) = \{x \in B_n \mid f(x) \geq u\}$ are dual to face posets of subword complexes (fibers as in Quillen's Lemma A)

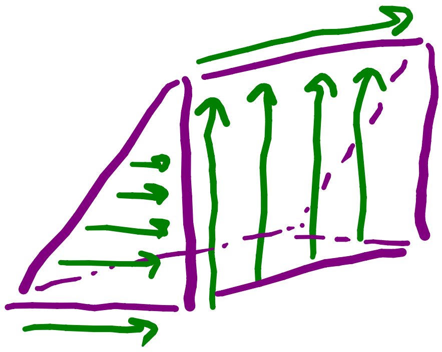
Description of fibers via Flow (Based on Collapses) to Base Point

e.g. $$\overset{1,\ 2,\ 3}{x_1 - x_1 x_2 x_1 - x_2 \; x_3 - x_3}$$

$$\underbrace{x_1(t_1)x_1(t_3)}_{x_1(t_1+t_3)}x_2(t_4)x_1(t_5)x_2(t_7)\underbrace{x_3(t_8)x_3(t_{10})}_{x_3(t_8+t_{10})}$$

$$\underbrace{x_1(t_1+t_3)x_2(t_4)x_1(t_5)x_2(t_7)}_{x_2(t_1')x_1(t_2')x_2(t_3'+t_7)}$$





Remarks:
- DHM Conjecture ⇒ FS Conjecture (via CE-Approx. Thm)
- Our proof factors $f_{(i_1,\dots,i_d)}$ as product of "nice" maps

Future (& Current) Further Questions:

1. Homeomorphism type for images & fibers of maps yielding:
    - totally nonnegative part of Grassmannian
    - totally nonnegative part of a loop group
    - totally nonnegative part of the flag variety
    - spaces of electrical networks

2. Combinatorics of generalized subword complexes as combinatorial model for fibers

   - Grassmannian?
   - loop group?

   Vertex decomposability?

   Gallery connectedness?

3. Combinatorics of generalized Bruhat order as combinatorial model for images

   - loop group?
   - electrical networks?

     Lam: proved thin

   In particular: are these CW posets?

4. Explicit polytope & map for flag variety (perhaps by developing combinatorics of suitable reduced & non-reduced objects for canonical bases explicitly)

5. Explanation for subword complexes arising in seemingly disparate settings?

6. More general setting explaining very strong analogies between reduced words, reduced plabic graphs, etc.

Thank you!