Regular Cell Complexes in Total Positivity

Patricia Hersh

North Carolina State University

(Paper with same title, in:
Inventiones Math., 197 (2014), 57-114.)

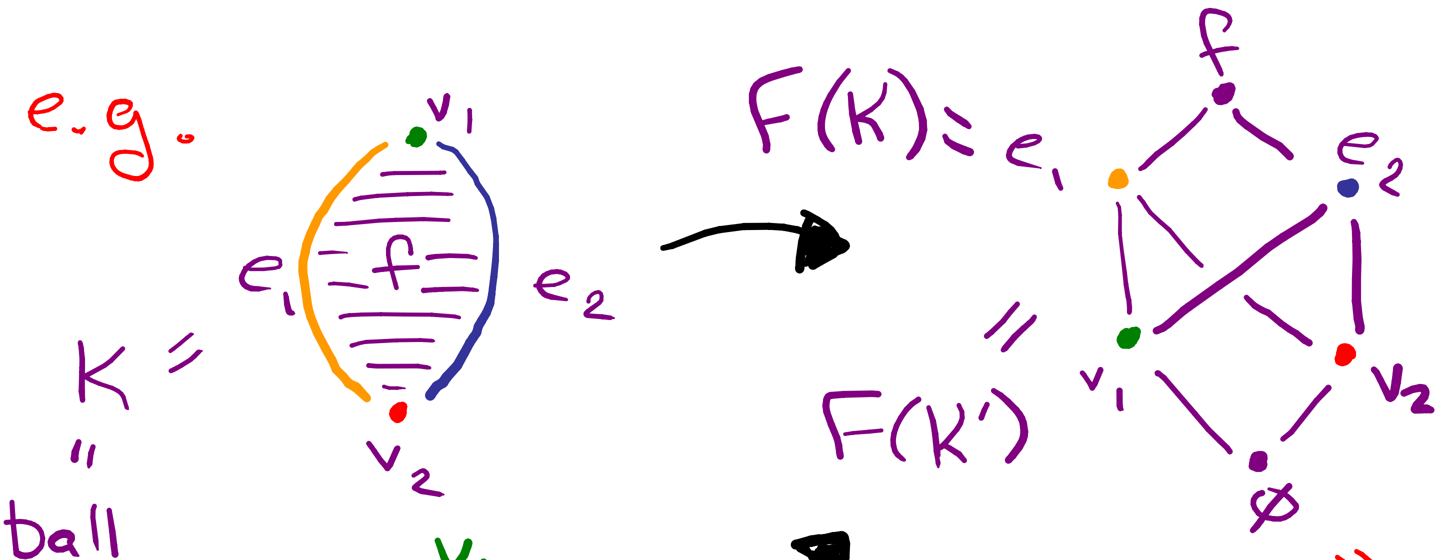
(See http://www4.ncsu.edu/~plhersh
for slides, including appendix with
more details)

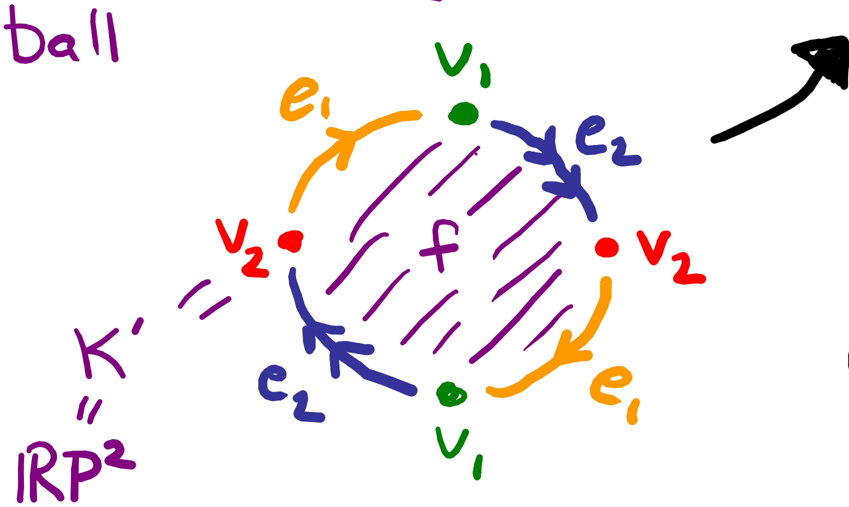
Topological Aspects of Total Positivity

- Lusztig initiated study of Totally nonnegative, real part of (matrix) Schubert varieties, flag varieties, ...
  (i.e. part with minors all nonnegative in spaces of matrices or of flags)
- Conjecturally/provably homeomorphic to closed balls (after deconing)
- Proving this:
  - puts restrictions on relations among (exponentiated) Chevalley generators.
  - reveals structure in canonical bases; a motivation for cluster algebras.
- Main Result of Talk: Proof of Fomin-Shapiro Conjecture via new tools exploiting interplay of combinatorial data & topological data.

Deducing Topological Structure from Combinatorics + Codim One Topology?

e.g.



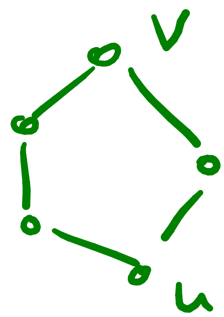


"closure poset" or "face poset"

$(u \le v \iff u \le \bar{v})$

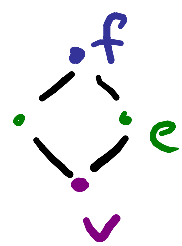
Notations: A **CW complex**: cells e_α, characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \to \bigcup_{\beta \in \bar{e}_\alpha} e_\beta$ & attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

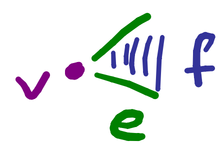
Recall: a poset is graded if $u \le v$ in P implies minimal paths u to v all same length.

e.g. 

is not graded

- A graded poset is thin if each rank 2 interval has exactly 4 elements.





Recall: A CW complex is regular if the attaching map for each cell is a homeomorphism, i.e. cell closures are closed balls.

- K regular $\Rightarrow K \cong \Delta(F(K) \setminus \{\hat{0}\}) = \mathrm{sd}K$

(setting where combin. determines top.)

Defn (Björner): A finite, graded poset P is CW poset if

- P has unique min'l elt. $\hat{0}$
- P has additional element(s)
- $x \neq \hat{0}$ $\Rightarrow$ $\underbrace{\Delta(\hat{0}, x)}$ $\cong S^{\text{rank}(x)-2}$
  $\hat{P}$

nerve, or "order complex" of P
i.e. simplicial complex whose faces are chains $u_1 < \dots < u_i$.

Thm (Björner): P is CW poset $\Leftrightarrow$ there exists regular CW complex K with $P = F(K)$.

A Goal of Mine: Use combinatorics of $F(K)$ + manageable topological info (codim. one cell incidences) to understand K.

Some Examples of CW Posets

- Shellable & thin (Danaraj-Klee)

- Bruhat order (Björner & Wachs)

- Closure poset for double Bruhat decomp. of totally nonneg. part of flag variety (Williams)

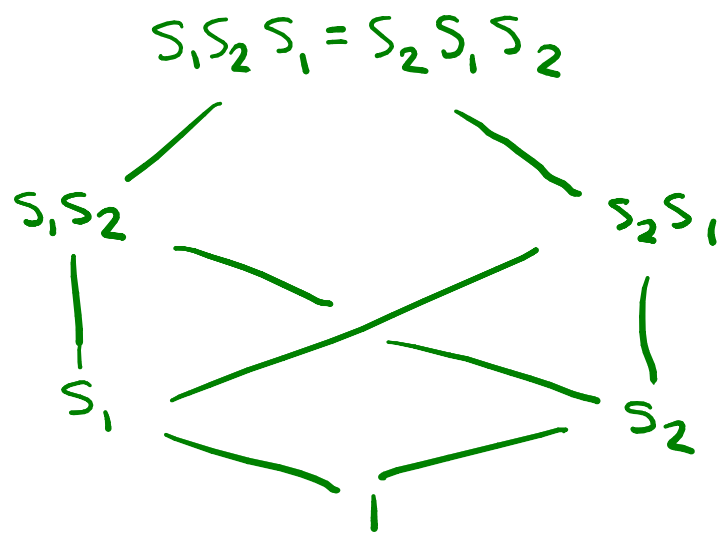
- Closure poset of triangulation of double suspension of homology sphere with "big cell" glued in (due to work of J. Cannon & R. Edwards)

  (hence the focus of CW posets on intervals $(\hat{0}, u)$)

The Bruhat order is partial order on Coxeter group W with $u \le v \iff$ there exists reduced expressions (i.e. products of minimal # adjacent transpositions) $r(u)$ and $r(v)$ with $r(u)$ subexpression of $r(v)$.

e.g. $W = S_3$ with generators

$s_1 = (1,2)$
$s_2 = (2,3)$



- Closure poset for Schubert cell decompositions of flag varieties G/B
- reduced word $(i_1, \ldots, i_d)$ for $s_{i_1} s_{i_2} \cdots s_{i_d}$

Question (Bernstein): Find regular CW complexes naturally arising from rep'n theory which are homeomorphic to closed balls and have the (lower) Bruhat intervals as closure posets.

Conjectural Solution (Fomin & Shapiro)

The Bruhat stratification of $lk(id)$ in totally nonnegative, real part of unipotent radical in semisimple, simply connected algebraic group defined and split over $\mathbb{R}$.

Thm (Fomin-Shapiro): This has Bruhat order as closure poset. Has desired homological properties.

Theorem (H.): Fomin-Shapiro Conjecture indeed holds.

Special Case (Running Example for Talk): Space of totally nonnegative upper triangular matrices with 1's on diagonal & entries just above diagonal summing to fixed, positive constant, stratified by which minors are positive and which are 0.

Concrete Realization: products of certain elementary matrices, by results of Whitney & Lusztig.

The Totally Nonnegative Part of a Space of Matrices

- $$x_i(t) = \underbrace{I_n + tE_{i,i+1}}_{\text{(type A)}} = \begin{array}{cc} \begin{array}{c} \text{column } i+1 \end{array} & \\ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & t & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} & \begin{array}{c} \text{row } i \end{array} \end{array}$$

  $$x_i(t) = \underbrace{\exp(te_i)}_{\text{(general finite type)}}$$

- $$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$$

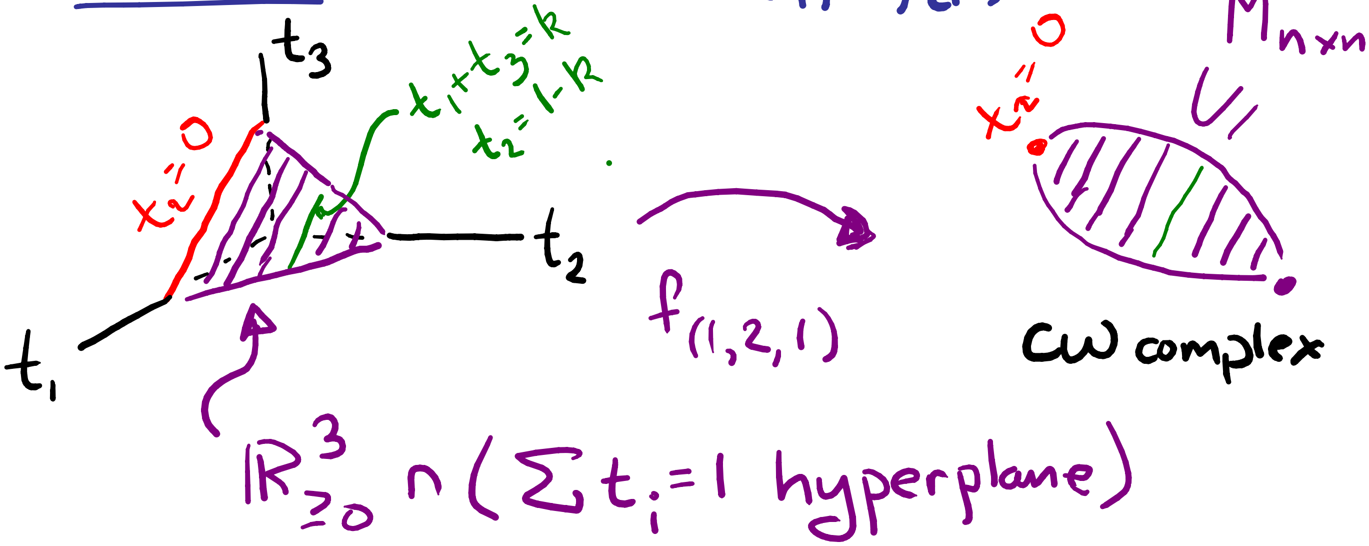
  $$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

e.g. $$f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1,\dots,i_d)}$



$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow{\scriptstyle t_2=0}$$

$$x_1(t_1)\cdot x_1(t_3)$$

$$f_{(1,2,1)}(t_1,0,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

Non-injectivity: results from "modified nil-moves" $x_i(u)x_i(v) \to x_i(u+v)$ directly & after "long braid moves" in 0-Hecke algebra.

1st Ingredient to Fomin-Shapiro Conj:

0-Hecke Algebra Captures which Simplex Faces have Same Image under $f_{(i_1,\ldots,i_d)}$

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$ "nil-move"

$\Big\downarrow$ suppress parameters

$x_i^2 = x_i$ (0-Hecke alg. reln, up to sign)

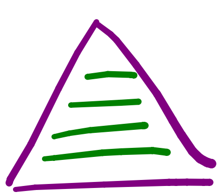
(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1t_2}{t_1+t_3}\right)$

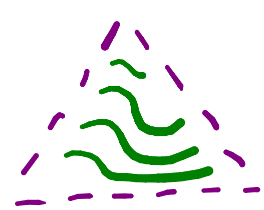
$\Big\downarrow$ (type A) assuming $t_1+t_3>0$

$x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1}$ "long braid move" with enrichment from parameters

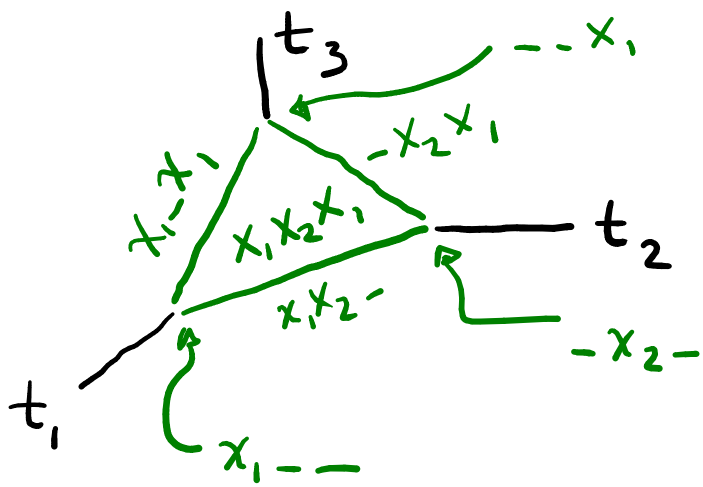
(similar relation holds outside type A)

Fibers as Curves:

(1): 

(2) then (1): 

Indexing Faces of Preimage by Words in 0-Hecke Algebra



Key Observation About $f_{(i_1,\dots,i_d)}$:

$$\mathrm{im}(f_1)=\mathrm{im}(f_2) \iff \underbrace{x(f_1)=x(f_2)}_{\text{equal as 0-Hecke algebra elements}}$$

Thm (Lusztig): If $(i_1,\dots,i_d)$ is reduced, then $f_{(i_1,\dots,i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

Upshot: $f_{(i_1,\dots,i_d)}$ restricts to homeomorphism on each face given by reduced subword. Faces indexed by non-reduced subwords (& some reduced ones) are redundant.

Properties of Change-of-Coordinates Map Given by Braid Moves

e.g. $(t_1, t_2, t_3) \mapsto \left(\frac{t_2 t_3}{t_1+t_3}, t_1+t_3, \frac{t_1 t_2}{t_1+t_3}\right)$

in type A

- Tropicalizes to change-of-basis map for Lusztig's canonical bases:

$$(a,b,c) \mapsto (b+c-\min(a,c), \min(a,c), a+b-\min(a,c))$$

- A motivation for development of cluster algebras (& mutation)

Exercise: check this is an involution.

Proof Strategy (for FS-Conjecture & for images of "nice" maps from polytopes)

Set-up: Continuous, surjective fn

$$f : P \to Y$$

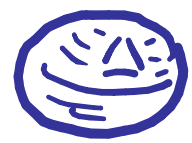
from convex polytope P (e.g. Δ_n) s.t. f maps $\mathrm{int}(P)$ homeomorphically to $\mathrm{int}(Y)$.

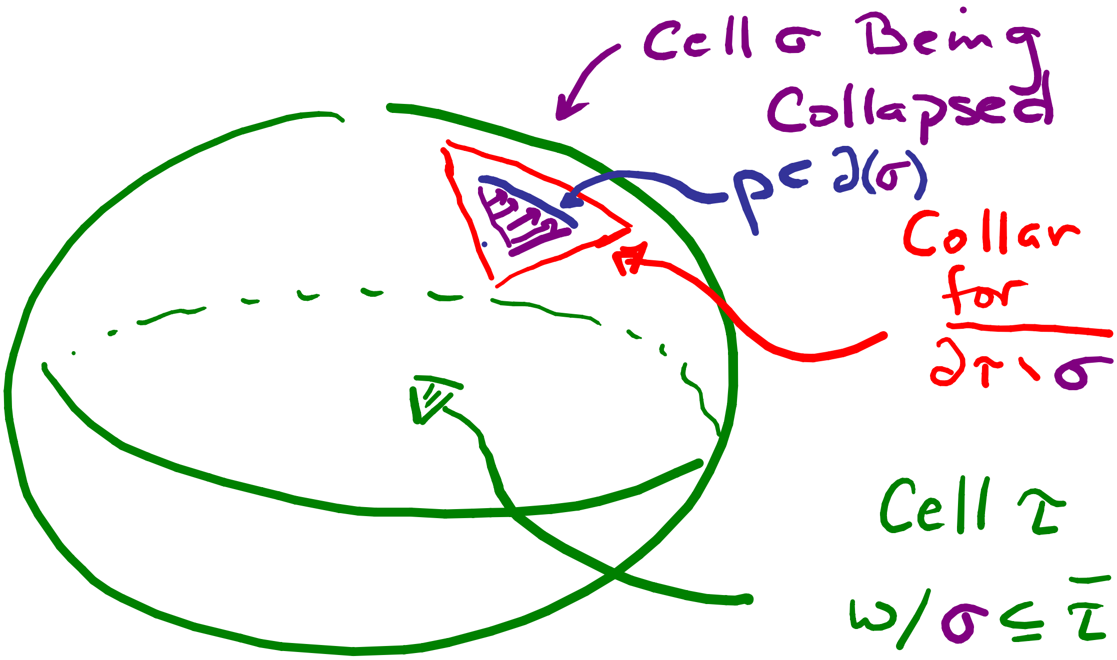
Step 1: Perform "collapses" on ∂P, each preserving regularity and homeomorphism type - via continuous, surjective collapsing functions $P \to P$ yielding $P/\sim$ with fewer cells s.t. $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$

Step 2: Prove $\bar{f} : P/\sim \to Y$ is homeomorphism by new regularity criterion

Collapsing Cell σ onto Cell $\rho \leq \bar{\sigma}$ within $\partial\tau$

Thm (M. Brown; Connelly): Any topological manifold with boundary ∂M has a collar (i.e. a nbh'd homeomorphic to $\partial M \times [0,1]$).

Fact: Our collapses will preserve this (hence existence of collar) for: $\overline{\partial\tau \setminus \sigma} =$ 



& $\dim \tau = \dim \sigma + 1$

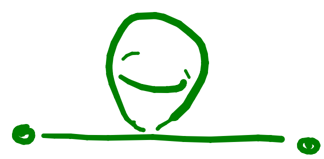
Plan: Collapse $\bar{\sigma}$ onto $\bar{\rho} \leq \partial\sigma$, stretching collar for $\overline{\partial\tau \setminus \sigma}$ to cover $\bar{\sigma} \setminus \bar{\rho}$.

2nd Ingredient: New Regularity Criterion

Preparatory Lemma (H.): Let K be a finite CW complex w/ characteristic maps $\{f_\alpha\}$. Suppose:

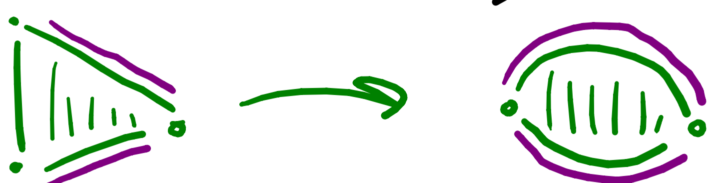
(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example:



(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example:



Then $F(K)$ is graded by cell dimension.

Insightful feedback: Next theorem "spreads around" injectivity requirement.

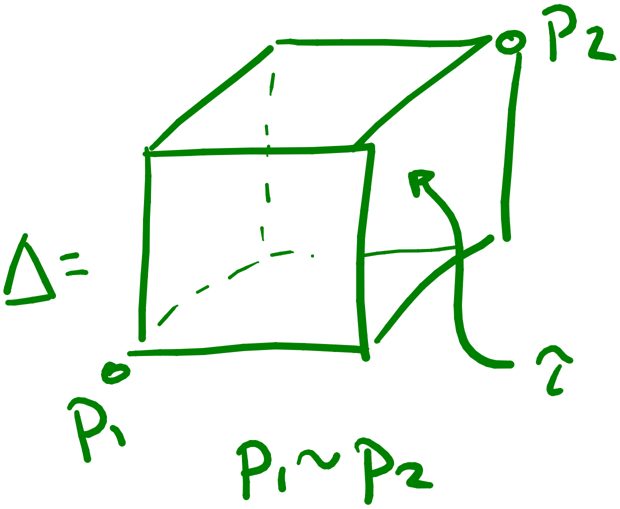
Thm (H.) Let K be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then K is regular w.r.t. $\{f_\alpha\} \iff$

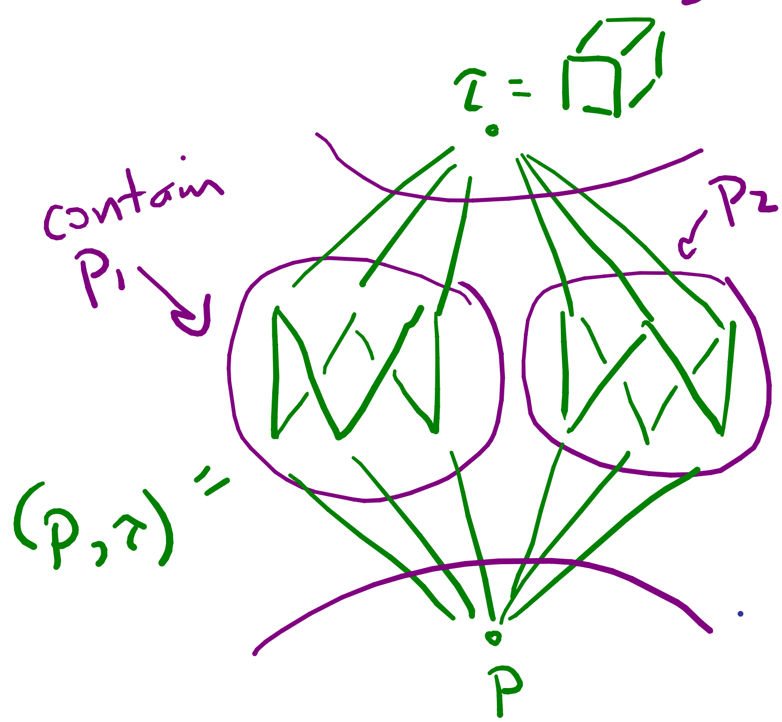
(1) K meets requirements of prop'n for $F(K)$ to be graded by cell dim.

(2) $F(K)$ is thin and each open interval (u,v) for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

Non-Example





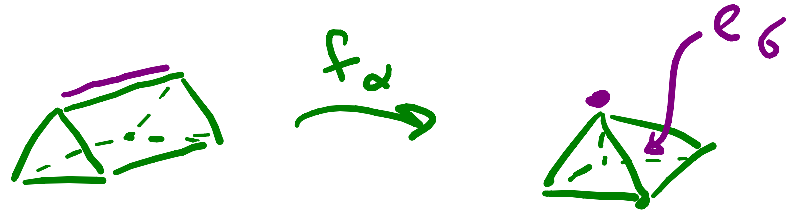
(3) For each α, the restriction of f_α to preimages of codim. one cells in $\overline{e}_\alpha$ is injective.

(topological condition)

Non-Example:

(4) $\forall\, e_\sigma \subseteq \overline{e}_\alpha$, f_σ factors as continuous inclusion $i: B^{\dim \sigma} \to B^{\dim \alpha}$ followed by f_α.

Non-Example:



Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one.

Proof: Induction on difference in dim.

3rd Ingredient: Injectivity of Attaching Maps in Codimension One via Coxeter group exchange axiom

$$s_1 s_2 s_1 s_3$$

$$\begin{array}{cccc}
s_1 s_2 s_1 - & s_1 s_2 - s_3 & - s_2 s_1 s_3 & \cancel{s_1 - s_1 s_3} \\
\shortparallel & \shortparallel & \shortparallel & \text{not reduced} \\
3214 \;\neq & 2341 \;\neq & 3142 &
\end{array}$$

reduced subexpressions of reduced expression obtained by deleting one letter give distinct Coxeter group elements.

In contrast: fails in higher codimension.

$$s_1 s_2 s_1 s_3$$

$$s_1 - - - \qquad\qquad - - s_1 -$$

4th Ingredient: (Mainly Combinatorial)

Requirements Enabling Collapses Across Curves

There is a series of earlier free collapses

"$= \Delta$" (as g_1 is surjection onto Δ)

$$K_0 \xrightarrow{g_1} K_1 \xrightarrow{g_2} K_2 \rightarrow \cdots \rightarrow K_i$$

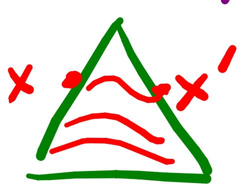
$K_0 = \Delta$ — polytope!

$K_1 \cong \Delta/\sim_1$ (new cell structure)

$K_2 \cong \Delta/\sim_2$

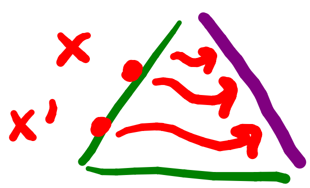
with closed cell of K_i covered by images of parallel line segments in K_0 with family $\mathcal{C}_i$ of "parallel-like" curves satisfying:

- Distinct endpoints condition (DE):



$x \neq x'$

- Distinct initial points condition (DIP):



$x \neq x'$

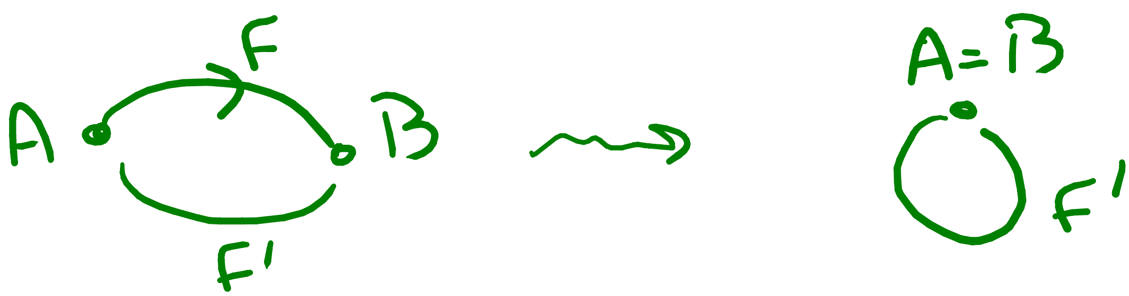
- "Least upper bound condition" (LUB)...

LUB: Condition to ensure Regularity is Preserved

(suggested by David Speyer)

If A & B are IDed via face collapse of F, then all least upper bounds for A & B just prior to collapse of F must also be collapsed in this step.

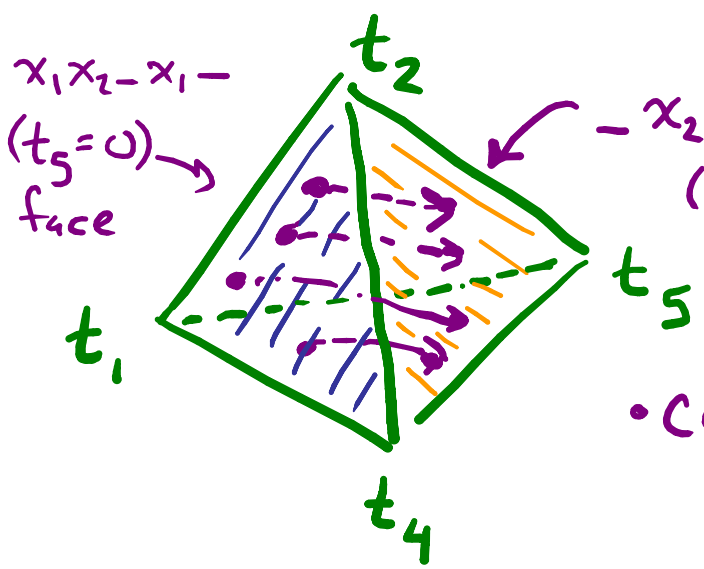
e.g. Want to prevent:



Note: conditions on which cells IDed yet; checkable with combinatorics of reduced/nonreduced words in 0-Hecke algebra.

Collapsing "non-reduced" Face Across Curves

e.g. $f_{(1,2,3,1,2)}$ face with $t_3=0$



- collapse across curves identifying $t_1t_2t_4$ & $t_2t_4t_5$ faces

$$(t_1,t_2,0,t_4,t_5) \mapsto x_1(t_1)x_2(t_2)x_1(t_4)x_2(t_5)$$

$$= x_2(t_1')x_1(t_2')\underbrace{x_2(t_4')x_2(t_5)}$$

only for $t_1+t_4>0$

$$= x_2(t_1')x_1(t_2')x_2(t_4'+t_5)$$

for $t_1'=\dfrac{t_2t_4}{t_1+t_4}$ $\quad t_2'=t_1+t_4 \quad t_4'=\dfrac{t_1t_2}{t_1+t_4}$

$\boxed{\text{curves within fibers of } f_{(1,\ldots,1)}}$: $t_1'=k_1$ & $t_2'=k_2$ & $t_4'+t_5=k_3$

5th Ingredient: Deletion Pairs: How to Transfer Coxeter Group Properties to 0-Hecke Algebra

In a non-reduced expression $s_{i_1} \cdots s_{i_d}$, let $\{s_{i_r}, s_{i_t}\}$ be a **deletion pair** if $s_{i_r} \cdots s_{i_{t-1}}$ and $s_{i_{r+1}} \cdots s_{i_t}$ are reduced expressions while $s_{i_r} \cdots s_{i_t}$ is nonreduced.

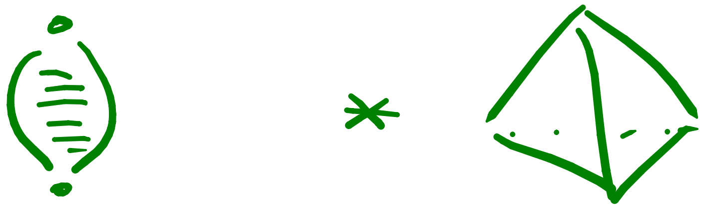
Key Coxeter Group Property: Any two reduced expressions for same $w \in W$ connected by series of braid moves - ensures nonreduced expressions admit modified nil-moves.

e.g. $\underbrace{x_1 x_2 x_1} \mid x_2 x_3 x_2$

$x_2 x_1 \underbrace{x_2 \; x_2} x_3 x_2$

$x_2 x_1 \quad x_2 \quad x_3 x_2$

6th Ingredient: Cell Collapsing Order (& Embedding) Enabling Proof by Induction on Word Length



$$(\underbrace{i_1, i_2, i_3, \dots\dots i_{d'-1}}_{\text{all possible collapses done}}, \underbrace{i_{d'}, \dots, i_d}_{\text{not in any collapses yet}})$$

$\operatorname{im}(f_{(i_1 \dots i_{d'-1})})$

"The Fine Print":

Collapsing Order: greedily choose:

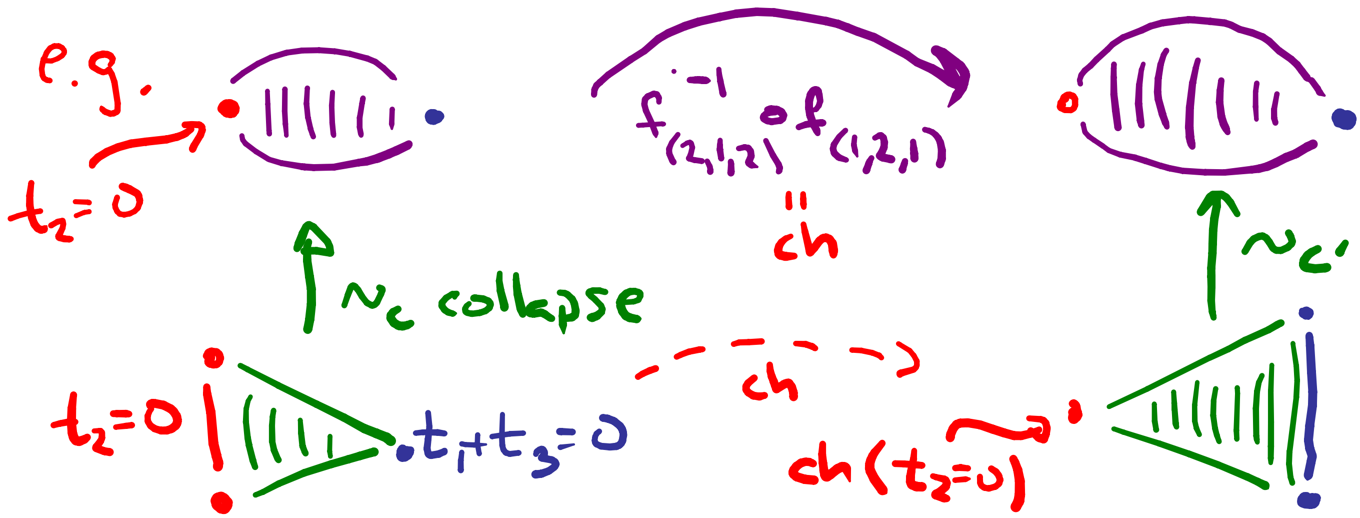
(1) leftmost deletion pair, then

(2) minimize $t-r$, then

(3) maximize cell dimension.

Repeat until all "non-reduced" cells collapsed.

Long Braid Move as Change of Coord's Homeomorphism on Closed Cell



Idea: Subwords of $(i,j,\ldots)$ and $(j,i,\ldots)$ do not admit any long braid moves. Thus:

$$
\begin{tikzcd}
\Delta^{m(i,j-1)}_{s_i s_j \ldots} / \sim_c \arrow[drr, "\bar{f}_{(i,j,\ldots)} = \text{homeom.}"] \arrow[drr, bend right] \arrow[dd, dashed, "f^{-1}_{(j,i,\ldots)} \circ f_{(i,j,\ldots)}"'] & & \\
& & Y_{s_0 s_1 \ldots} \arrow[dll, bend right, "\bar{f}^{-1}"'] \\
\Delta^{m(i,j)-1}_{s_j s_i \ldots} / \sim_{c'} \arrow[urr, "\bar{f}_{(j,i,\ldots)} = \text{homeom.}"'] & &
\end{tikzcd}
$$

Summary: 1. Eliminate all cells indexed by non-reduced subwords of $(i_1, \ldots, i_d)$ via explicit collapses.

2. Thereby also identify pairs of cells indexed by subwords that are reduced words for same $u \in W$.

3. Justify collapses preserve homeomorphism type & regularity via combinatorics of reduced words.

4. Deduce from regularity criterion that resulting quotient space is homeomorphic to $\mathrm{im}(f_{(i_1, \ldots, i_d)})$ once all non-reduced cells eliminated.

Thus, Fomin-Shapiro Conjecture holds.

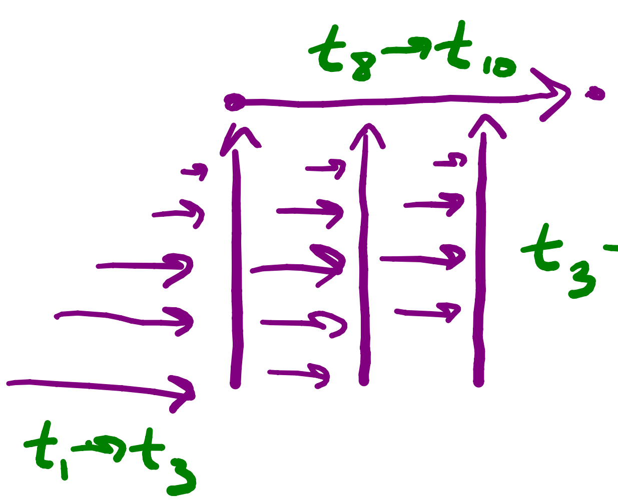
"Flow" on a Fiber From Collapsing Process to Base Point of Fiber

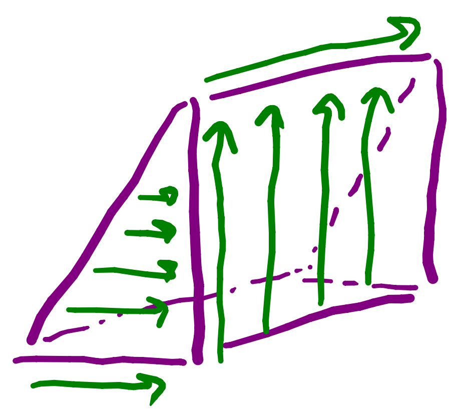
e.g. $x_1 - x_1 x_2 x_1 - x_2 x_3 - x_3$

$$x_1(t_1)\,x_1(t_3)\,x_2(t_4)\,x_1(t_5)\,x_2(t_7)\,x_3(t_8)\,x_3(t_{10})$$

$$x_1(t_1+t_3) \qquad\qquad x_3(t_8+t_{10})$$

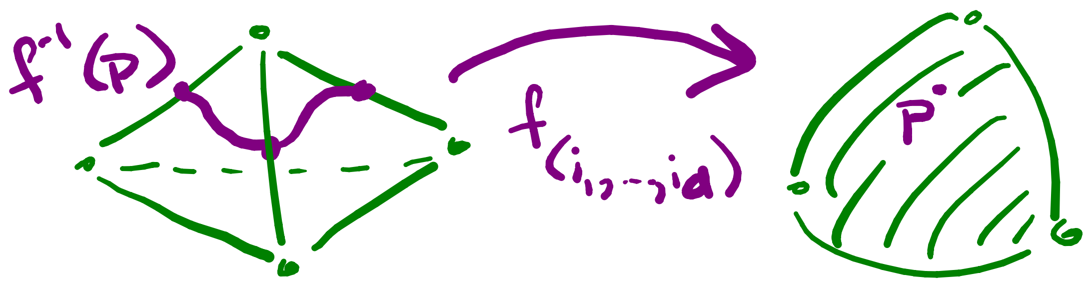
$$x_2(t_1')\,x_1(t_2')\,x_2(t_3'+t_7)$$





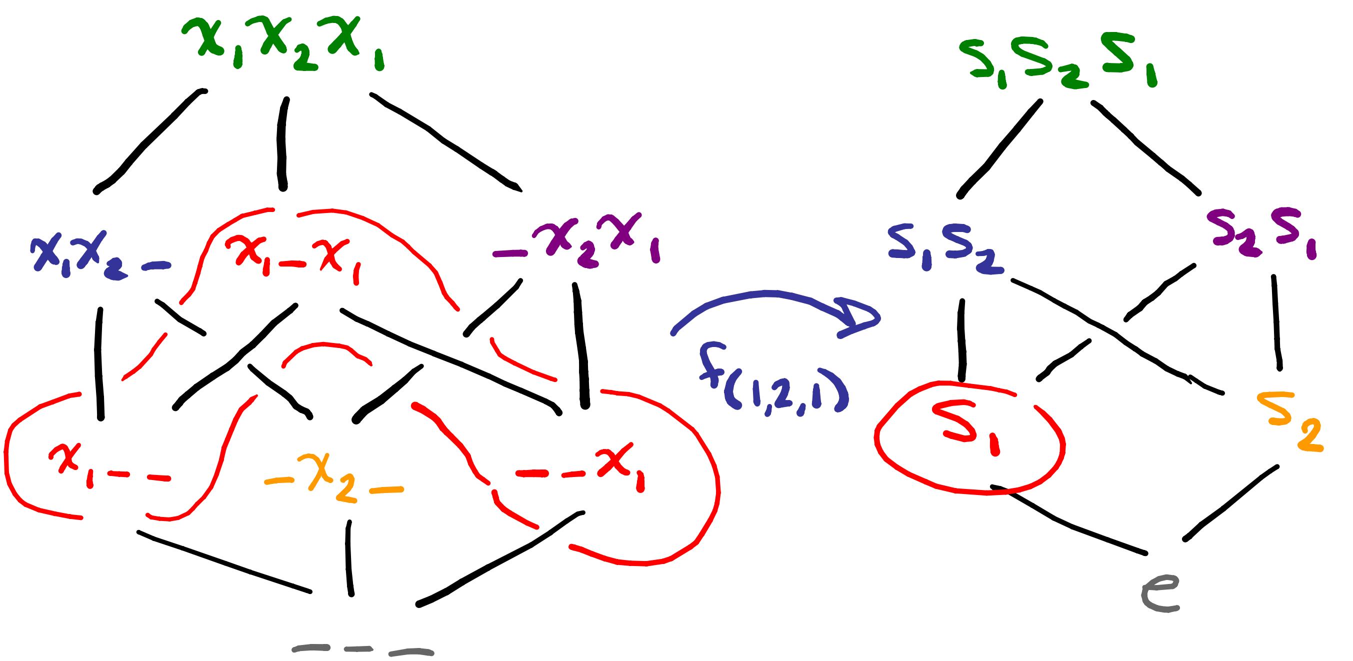
A Follow-up Project:
(with Jim Davis & Ezra Miller)

Conjecture (Davis-H.-Miller): $f^{-1}_{(i_1,\dots,i_d)}(p)$ for each $p \in Y^o_\omega$ is a regular CW complex homeomorphic to a ball with closure poset dual to face poset for interior of "subword complex" $\Delta((i_1,\dots,i_d),\omega)$.



Remark: Subword complexes previously arose as Stanley-Reisner complexes of initial ideals of coordinate rings assoc'd to matrix Schubert varieties.

A Poset Map (on Face Posets) induced by $f_{(i_1,\dots,i_d)}$ (& implicit Def'n of Subword Complexes)



Boolean Algebra B_n Bruhat Order

- Apply braid moves & $x_i^2 \to x_i$ to get reduced expression; replace x_i's by s_i's
- Fibers $f^{-1}_{\geq}(u) = \{x \in B_n \mid f(x) \geq u\}$ are dual to face posets of subword complexes (fibers as in Quillen's Lemma A)

Davis-H-Miller Conjecture implies Fomin-Shapiro Conjecture

(with Jim Davis & Ezra Miller)

Combining Top'l Results: Let $g: B \to Z$ be continuous surjection from ball B to Hausdorff space Z whose restriction to $\mathrm{int}(B)$ is an embedding. Suppose also:

(1) $g(\partial B) \cong \partial B = S^n$;

(2) $g(\partial B) \cap g(\mathrm{int}(B)) = \emptyset$;

(3) $g^{-1}(p)$ is contractible $\forall p \in g(\partial B)$.

Then $Z \cong B$.

(Based on Kirby-Siebenmann + local contractibility of $\mathrm{Homeo}(S^n, S^n)$.)

Existing Proof of Fomin-Shapiro viewed from this Perspective

- factors $f_{(i_1,\dots,i_d)}$ as composition of simple collapsing maps g_i

  (where requisite paths of homeomorphisms are easy to construct explicitly).

- regularity criterion shows

  $$f_{(i_1,\dots,i_d)} : g_n \circ \dots \circ g_1(\Delta_n) \to f_{(i_1,\dots,i_d)}(\Delta_n)$$

  is indeed a homeomorphism.

Further Questions:

1. Analogous story for totally nonnegative part of: Grassmannian? loop group? flag variety?

   (partial results of Postnikov, Rietsch, Williams, Speyer, Marsh, ...)

2. Explanation why subword complexes arising in distinct, but related settings? More general notion of subword complexes, perhaps as combinatorial models for fibers of "nice" maps.

Thank you!

Connection to Schubert Varieties & Bruhat Decompositions

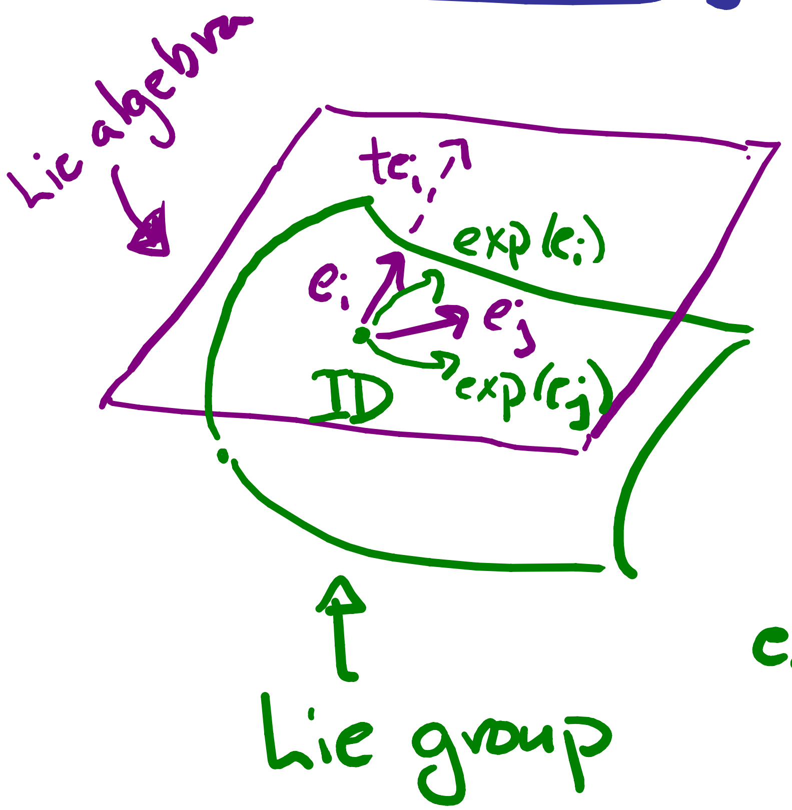
- Y_w° = image of $f_{(i_1,\dots,i_d)}: \mathbb{R}^d_{>0} \longrightarrow M_{n\times n}$

- $Y_w = \overline{Y_w^\circ}$ = image from $\mathbb{R}^d_{\geq 0}$
  = totally nonnegative part of $\overline{B^- w B^-} \cap \begin{pmatrix} \text{unipotent} \\ \text{radical of } B \end{pmatrix}$

- Y_{w_0} = totally nonnegative part of space of upper triangular matrices w/ 1's on diagonal

(old result of Whitney-type A)

A Motivation: Understanding Relations Among (Exponentiated) Chevalley Generators



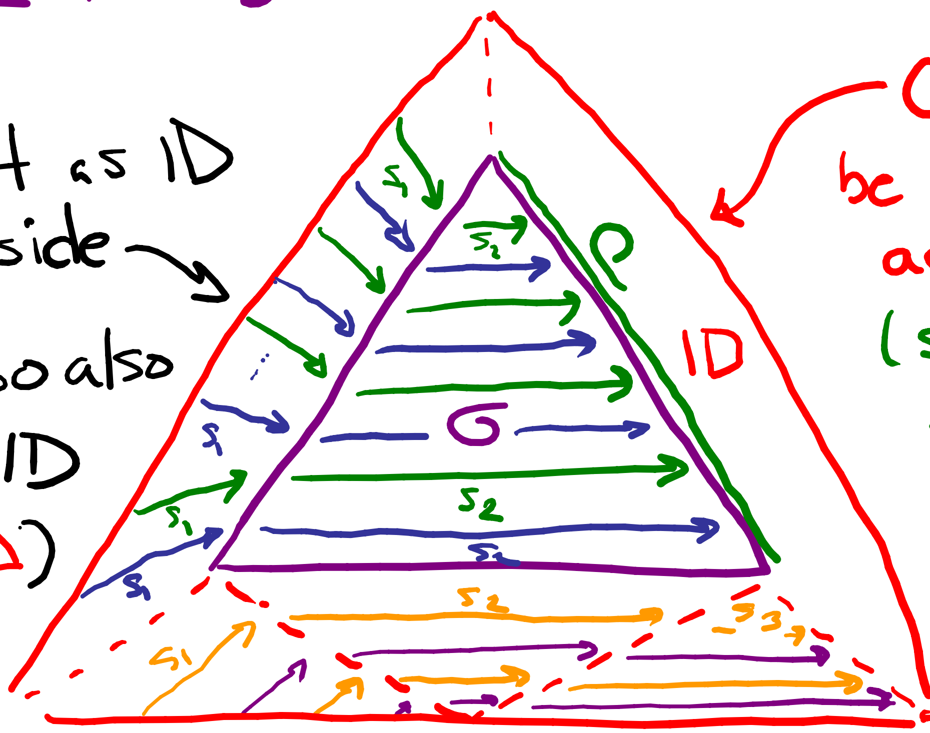
$$f_{(i,j)}(t_1, t_2) = \exp(t_1 e_i)\exp(t_2 e_j) = x_i(t_1)\, x_j(t_2)$$

$$\exp(te_i) = \boxed{ID + te_i} + \underbrace{\frac{t^2 e_i^2}{2}}_{0=} + \underbrace{\frac{t^3 e_i^3}{6}}_{0=} + \ldots$$

We Prove: Only the "obvious" relations occur

Collapsing a Cell $\bar{\sigma}$ onto a Cell $\bar{\rho}$

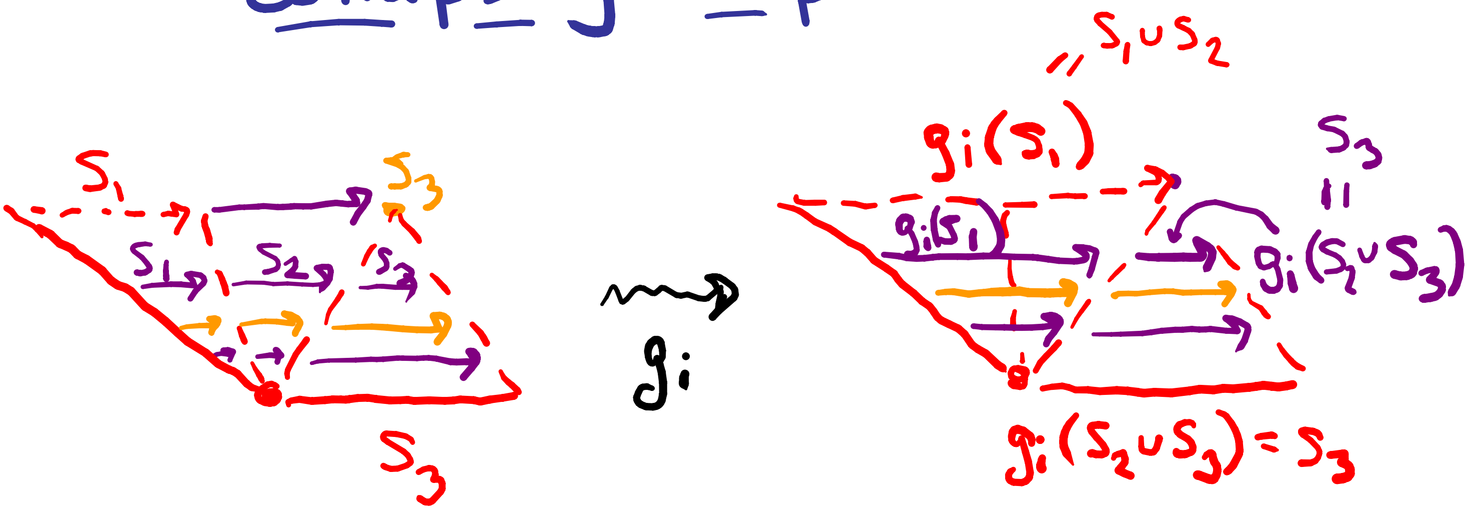
Act as ID outside $\triangle$, so also need ID at $\partial(\triangle)$



Collar to be stretched across σ (s_1 stretched to cover $s_1 \cup s_2$)

- Map segments s_2 in $\bar{\sigma}$ onto endpoint in $\bar{\rho}$, stretch extension $s_1 \subseteq$ collar to cover $s_1 \cup s_2$, act as ID on $\bar{\rho} \times [0,1] \subseteq$ collar.
- For $c \subseteq \partial\sigma$, collapsing map on $c \times [0,1]$ will stretch s_1 to cover $s_1 \cup s_2$ & shorten $s_2 \cup s_3$ to cover s_3, as depicted next.

"Close-up" of bottom part of collapsing map



Key Observations:

(1) This type of collapse makes sense more generally, relying on existence of continuous fn $\ln : \bar{\sigma} \to \mathbb{R}$ sending point to "length" of segment in $\bar{\sigma}$ containing it.

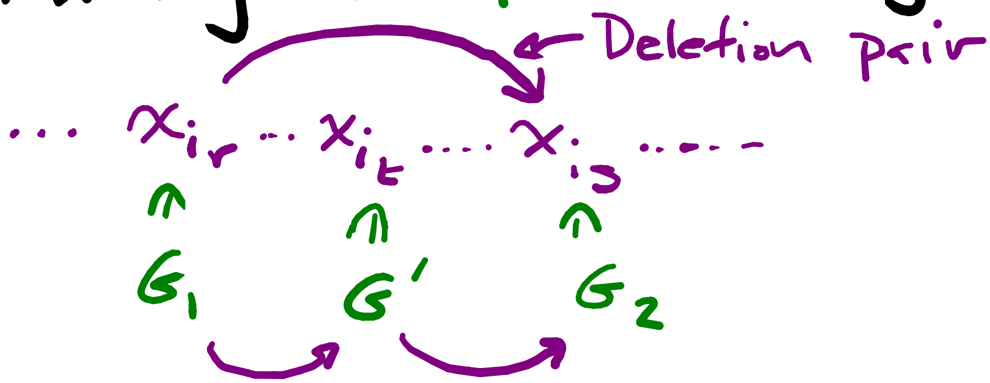
(2) These collapses are explicitly approximable by homeomorphisms:

Approximate

by

Verifying DE (Distinct Endpoints Condition) with Combinatorics (Gives Flavor of Many Lemmas)

Suppose collapse of F uses curves starting in G_1 and ending in G_2



If G_1 were already identified earlier with G_2 then there exists G' with earlier steps identifying G_1 with G' and G' with G_2. But the former would have also identified $G_1 \cup \{x_{i_s}\} = F$ with the cell $G' \cup \{x_{i_s}\} = F'$ which was already collapsed in step identifying G' with G_2 $\Rightarrow\Leftarrow$

Ingredients in Relationship Between Fibers & Image of "Nice" Map:

- CE-approx. theorem: $g : \partial B \to \partial B$ as above may be approximated by homeomorphisms
  - Armentrout: dim 3
  - Quinn: dim 4
  - Kirby-Siebenmann: dim ≥ 5 (& more generally)
- Local Contractibility of $\mathrm{Homeo}(S^n, S^n)$: two homeomorphisms "close enough" to each other may be connected by path of homeomorphisms

Idea: $B \cong$ metric ball $= \{\vec{0}\} \cup (0,1] \times \partial B$.

Use path of homeomorphisms converging to $g|_{\partial B}$ to construct $f : B \to B$ with $f^{-1}(p) = g^{-1}(p)\ \forall p \in B$ and $f|_{\partial B} = g|_{\partial B}$,

so $g(B) = B/\sim = f(B) \cong B$.

Checking Sphericity for $f_{(i_1,\ldots,i_d)}(\partial\Delta^{d-1})$

1. Stratification has Bruhat intervals as closure posets, thus CW posets.
2. Induction on dimension $\Rightarrow$ cell closures in $f_{(i_1,\ldots,i_d)}(\partial B)$ are balls, so $f_{(i_1,\ldots,i_d)}(\partial B)$ is regular CW complex "K".
3. Hence, $K \cong \Delta(F(K) \setminus \hat{0}) \cong$ sphere.

Subword Complexes (introduced by Knutson & Miller)

$Q :=$ (not necessarily reduced) expression

$w :=$ Coxeter group element

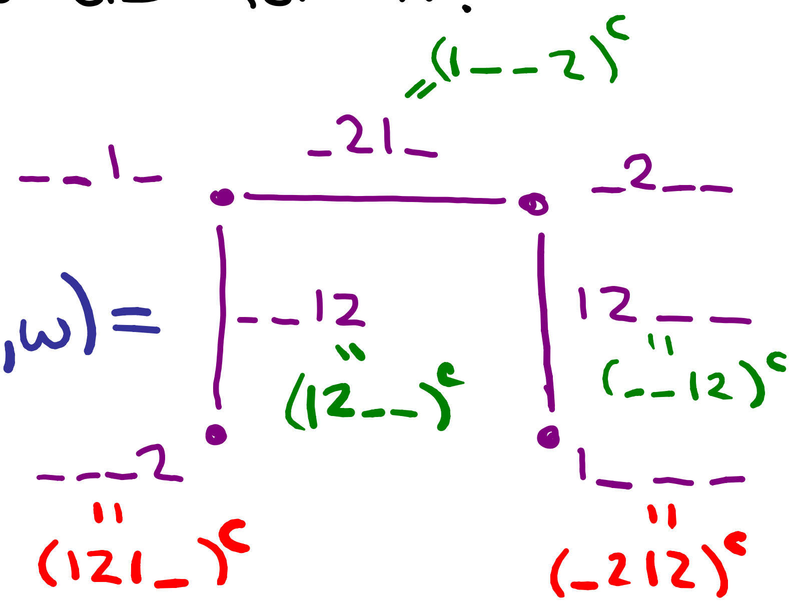
Facets of $\Delta(Q,w)$ are the subwords of Q whose complements are reduced words for w.

e.g.

$Q = (1,2,1,2)$

$w = s_1 s_2$

$\Delta(Q,w) =$



Thm (Knutson-Miller): $\Delta(Q,w)$ is vertex decomposable (hence shellable) ball or sphere.

More Generally? "Fibers" of Parametrization Maps for Nonneg flag variety, loop groups, etc.?

Homotopy Type of Bruhat Intervals:
New Proof by Quillen Fibre Lemma

Thm (Armstrong-H.): The poset map $f_{(i_1 \ldots i_d)}$ yields short proof of:

$$\Delta_{Bruhat}(u,v) \simeq S^{rk\,v - rk\,u - 2} \quad \text{for all } u \leq v$$

Idea: • fibers $f^{-1}_{\geq}(u) = \{x \in B_n \mid f(x) \geq u\}$ are dual to face posets of subword complexes – proven to be balls by Allen Knutson & Ezra Miller.

Subword complexes previously arose as:

Stanley-Reisner complex for Gröbner degeneration of matrix Schubert variety ideal (Knutson and Miller)