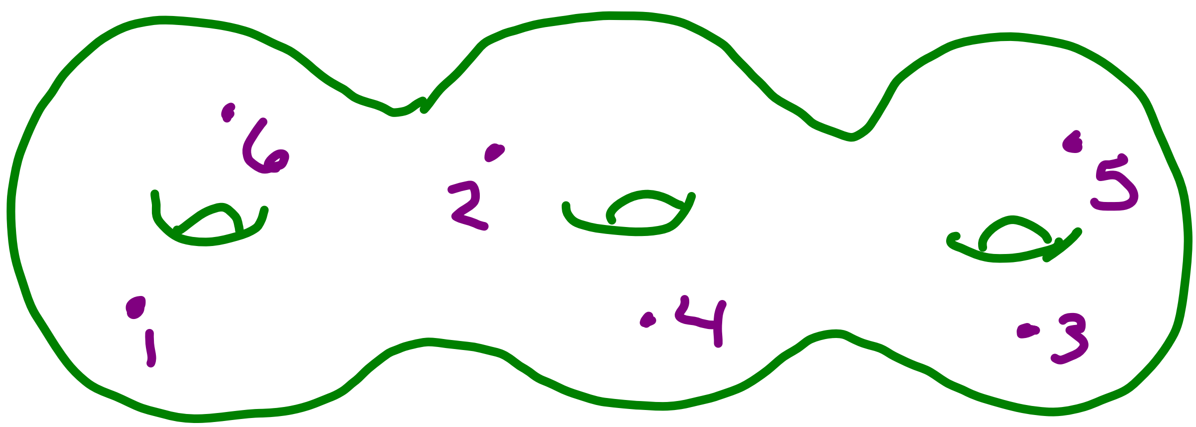
Representation Stability & the S_n-module Structure in the Homology of the Partition Lattice

Patricia Hersh
North Carolina State University

- joint work with Vic Reiner

(based on paper to appear in IMRN, "Rep'n stability for cohomology of configuration spaces in $\mathbb{R}^d$" & on work in progress)

A "Point" in a Configuration Space with S_n-rep'n's on Cohomology



- Manifold = 3-holed torus
- $n = 6 = $ # distinct labeled points
- S_n acts freely on configuration space by permuting pt. labels, inducing rep'n on each cohomology group

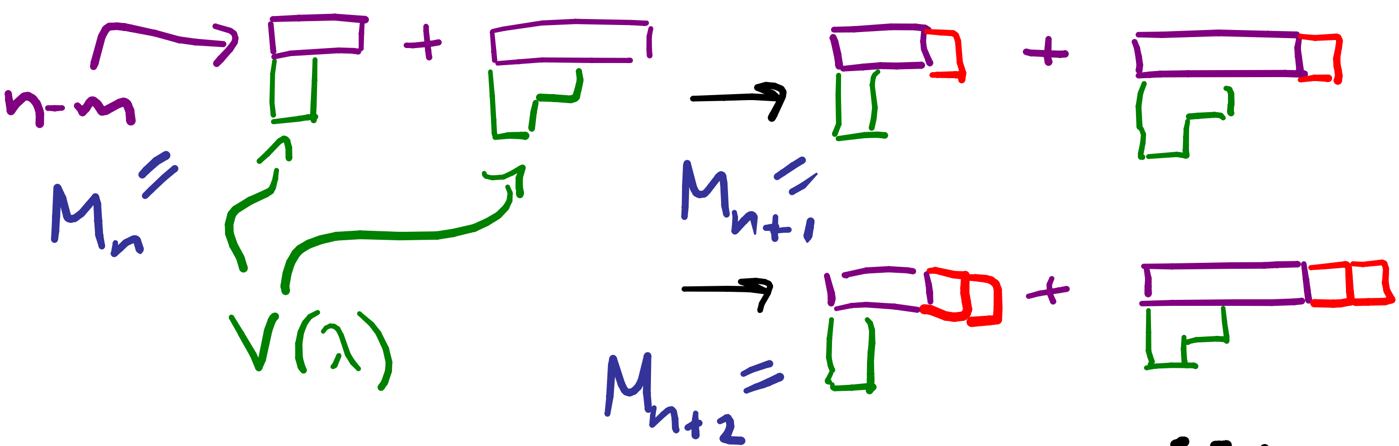
Representation Theoretic Stability

Defn (Church, Farb): A series of S_n-modules $M_1, M_2, \ldots$ for $n = 1, 2, \ldots$ **stabilizes at $B > 0$** if for each $n > B$, we have

$$M_n = \sum_{\lambda \vdash m \leq B} c_\lambda V(\lambda) \quad \text{where} \quad V(\lambda) \cong S^{(n-m,\lambda)}$$

and where c_λ does not depend on n

e.g.



Our Focus: S_n-rep'ns from partition lattice

Our Starting Point:

Thm (Church-Farb): $H^i(M_n, \mathbb{Q})$ stabllizes for $n \geq 4i$ where M_n is configuration space of n distinct points in plane & i is held fixed.

Thm (Church-Farb): More generally, letting M_n^d be the configuration space of n distinct labeled points on connected orientable d-manlfold, $H^i(M_n^d, \mathbb{Q})$ stabilizes for

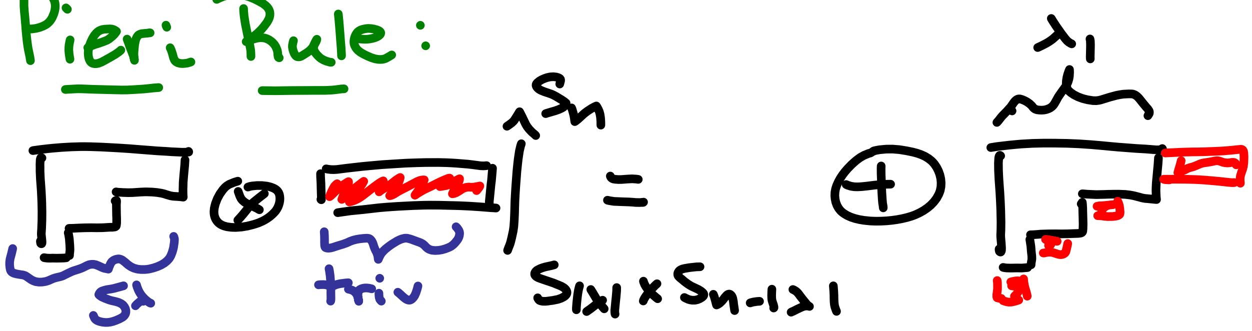
$$\begin{cases} n \geq 4i & \text{if } d = 2 \\ n \geq 2i & \text{if } d > 2 \end{cases}$$

Our First Objective: Sharpen these bounds for $M^d = \mathbb{R}^d$

How Representation Stability Typically Arises

- Finite number of irred. rep'ns S^λ; S^λ 1st appearing in $M_{|\lambda|}$
- Each M_n with $n \geq |\lambda|$ likewise includes $S^\lambda \otimes \mathrm{triv}_{n-|\lambda|} \big\uparrow_{S_{|\lambda|} \times S_{n-|\lambda|}}^{S_n}$
- Church-Ellenberg-Farb prove stability bounds of $n = 2\max|\lambda|$
- H-Reiner prove certain sharp stability bounds at $n = \max(|\lambda| + \lambda_1)$

Pieri Rule:



Motivations from Number Theory:

- Church-Ellenberg-Farb & Matchett/Wood-Vakil, & others:

$$\langle H^i(\mathrm{PConf}_n(\mathbb{C})), V\rangle_{S_n} = \dim_{\mathbb{Q}_\ell} H^i_{et}(\mathrm{Conf}_n; \mathcal{V})$$

coefs twisted by V

yielding various counting formulas over finite field via "Grothendieck-Lefschetz formula" & counting fixed pts of Frobenius map

e.g. $\lim_{n\to\infty}(\#\ \square\text{-free degree } n \text{ polys}) = q^n - q^{n-1}$

Remark: Applications to number theory focus on $M = \mathbb{R}^2$ case

Qn: Relationship to results of Björner-Ekedahl & Athanasiadis?

Church-Farb Method for Orientable Manifolds

- Use Totaro's E_2-page of Leray spectral sequence showing cohom. of manifold M + $H^i(M_n(\mathbb{R}^d))$ determines cohomology of config. space of n distinct pts on M as follows:

$$E_2^{p,(d-1)q} = \bigoplus_{\substack{S \text{ with} \\ |S| = n-q}} H^{q(d-1)}(\underbrace{C_S(\mathbb{R}^d)}_{\text{product of subspace arrangement complements}}) \otimes H^p(M^S)$$

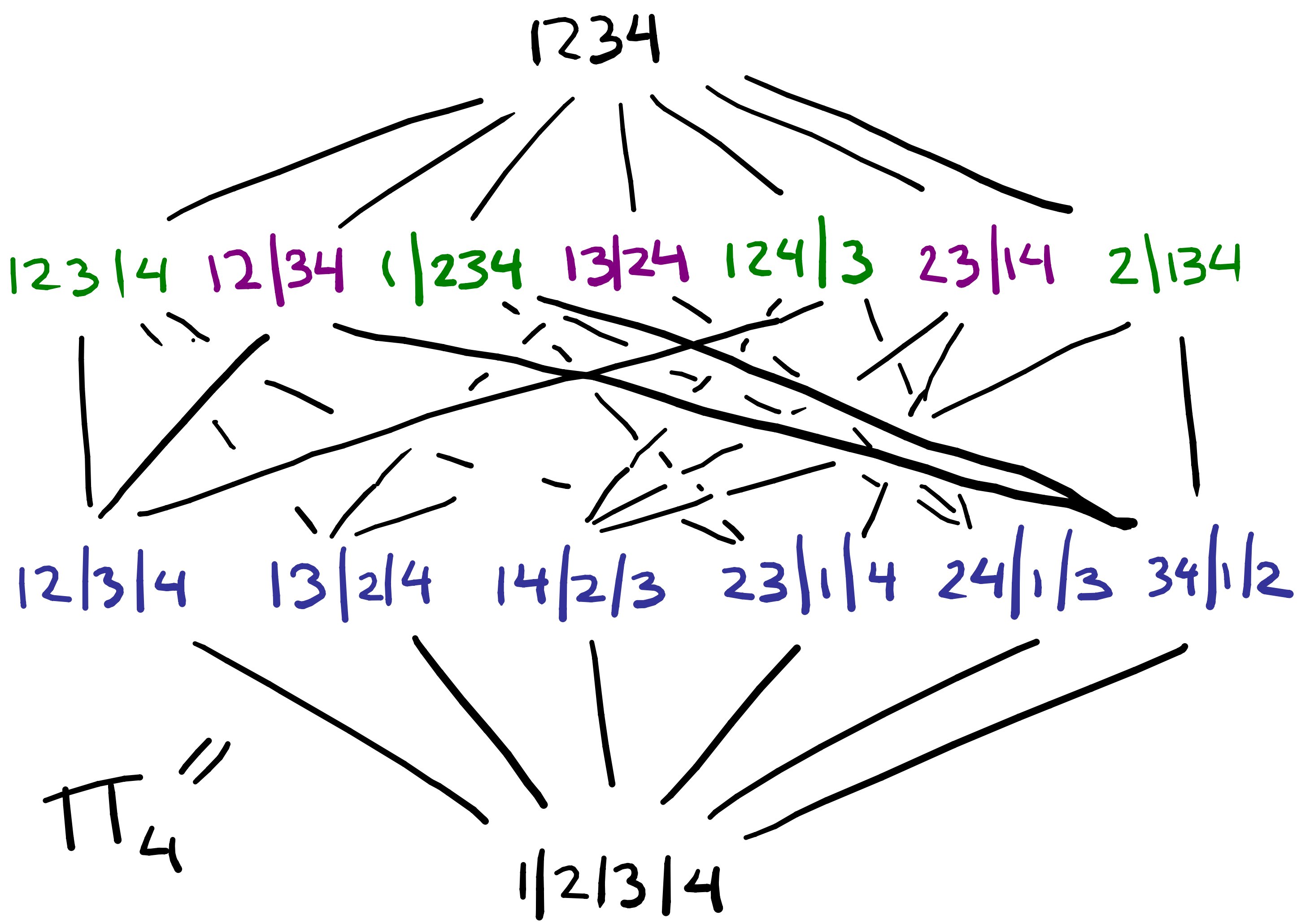
for set partition S with $|S|$ parts &

e.g. for $S = \{1,3\}\{2,4,5\}$

- $C_S(M) := \{\underline{x} \in M^5 \mid x_1 \neq x_3; x_2 \neq x_4 \neq x_5\}$
$$= C_{\{1,3\}}(M) \times C_{\{2,4,5\}}(M)$$
- $M^S := \{\underline{x} \in M^5 \mid x_1 = x_3; x_2 = x_4 = x_5\}$

& $E_2^{p,q} = 0$ for $d-1 \nmid q$

Partition Lattice Π_n & its S_n-representions



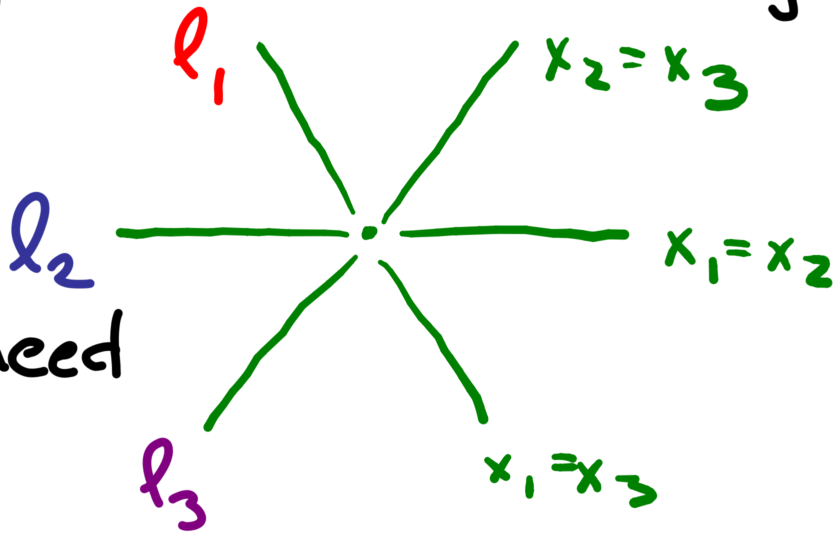
- S_n acts by permuting values

e.g. $(13)[12|3|45] = 32|1|45$

Reinterpreting via Subspace Arrangement Complements

- M_n = complement of type A (complex) braid arr't $\{x_i = x_j \mid 1 \le i < j \le n\}$

Warning: figure is $\mathbb{R}$-picture, need $\mathbb{C}$-picture



(Config space pt $p_i \longleftrightarrow x_i \in \mathbb{C}$)

- Π_n = intersection poset $\mathcal{L}(A_{n-1})$
  - S_n-module structure for $H^i(M_n)$ will translate to "Whitney homology" in Π_n.

Goresky-MacPherson formula

(for cohomology of subspace arr't)

$$\underbrace{\tilde{H}^i(M_{\mathcal{A}})}_{\text{subspace arr't complement}} \underset{\text{as groups}}{\cong} \bigoplus_{x \in \underbrace{L_{\mathcal{A}}^{>0}}_{\text{intersection lattice}}} \tilde{H}_{\mathrm{codim}(x)-2-i}(\hat{0}, x)$$

Plan: Apply to braid arrangement using upcoming S_n-equivariant versions due to Sundaram-Welker, yielding Whitney homology. (See also Blagojević, Lück, Ziegler for more general versions)

$$\mathcal{L}(A_2) = \begin{tikzcd}[column sep=small]
& {x_1=x_2=x_3 \atop (\ell_1 \cap \ell_2 \cap \ell_3)} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
{x_1=x_3 \atop (\ell_1)} \arrow[dr, dash] & {x_1=x_2 \atop (\ell_2)} \arrow[d, dash] & {x_2=x_3 \atop (\ell_3)} \arrow[dl, dash] \\
& \mathbb{R}^2 = \text{empty intersection} &
\end{tikzcd}$$

$\mathcal{L}(A_2)$ = poset of intersections of subspaces

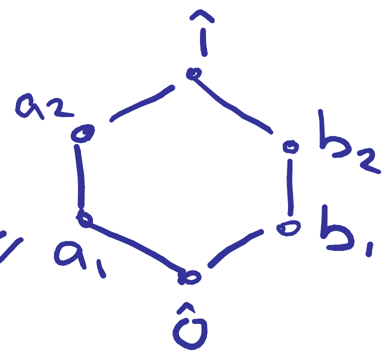
$$\cong$$

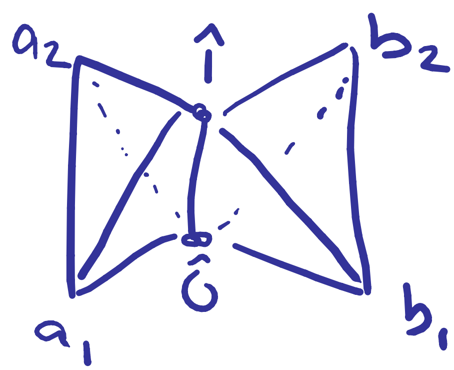
$$\Pi_3 = \begin{tikzcd}[column sep=small]
& 123 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
13|2 \arrow[dr, dash] & 12|3 \arrow[d, dash] & 23|1 \arrow[dl, dash] \\
& 1|2|3 &
\end{tikzcd}$$

Π_3 = lattice of set partitions

$x_i = x_j \Longleftrightarrow i,j$ in same block

Def'n: The order complex of a finite poset P is the simplicial complex $\Delta(P)$ whose i-dim'l faces are the $(i+1)$-chains in P.

e.g. $P =$ 

$\Delta(P) =$ 

- Let $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$ e.g. for Π_n

Convention: When we speak of topological properties (homology, etc.) of poset P, we mean $\Delta(P)$ or $\Delta(\bar{P})$.

Poset rank := # steps from bottom

S_n-Representations on Chains (i.e on Faces) & on Homology

- S_n action on set partitions is order-preserving & rank-preserving

- Hence, induces S_n-action α_S on

$\{$chains $u_1 < u_2 < \cdots < u_j \mid \text{rank}(u_r) = i_r$ for $1 \le r \le j$ and $S = \{i_1, \ldots, i_j\}\}$

$\Updownarrow$

$\{$faces of $\Delta(\overline{\Pi}_n)$ with vertices colored S, where vertices in $\Delta(\overline{\Pi}_n)$ colored by poset rank$\}$

- S_n-action on chains commutes with simplicial boundary map

$$d(u_0 < \dots < u_r) = \sum_{0 \le i \le r} (-1)^i (u_0 < \dots < \hat{u}_i < \dots < u_r)$$

- Thus, S_n-action on i-faces (i^{th} chain gp) induces rep'n on i^{th} homology

- But homology of Π_n concentrated in top degree due to "EL-shellability" of Π_n:

Thm (Stanley + Björner): Π_n is supersolvable, hence is EL-shellable.

- Likewise, homology of $\Pi_n^S = \{u \in \Pi_n \mid rk(u) \in S\}$ also concentrated in top degree by:

  Thm (Björner): P graded & EL-shellable $\Rightarrow P^S$ also EL-shellable. In particular, P^S is Cohen-Macaulay.

- The virtual rep'n $\beta_S := \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T$ is actual S_n-rep'n on top homology of $\Pi_n^S := \{u \in \Pi_n \mid rk(u) \in S\}$

Thm (H-Reiner): $\beta_S(\Pi_n)$ stabilizes at $n \geq 4\max(S)$ for any fixed S.

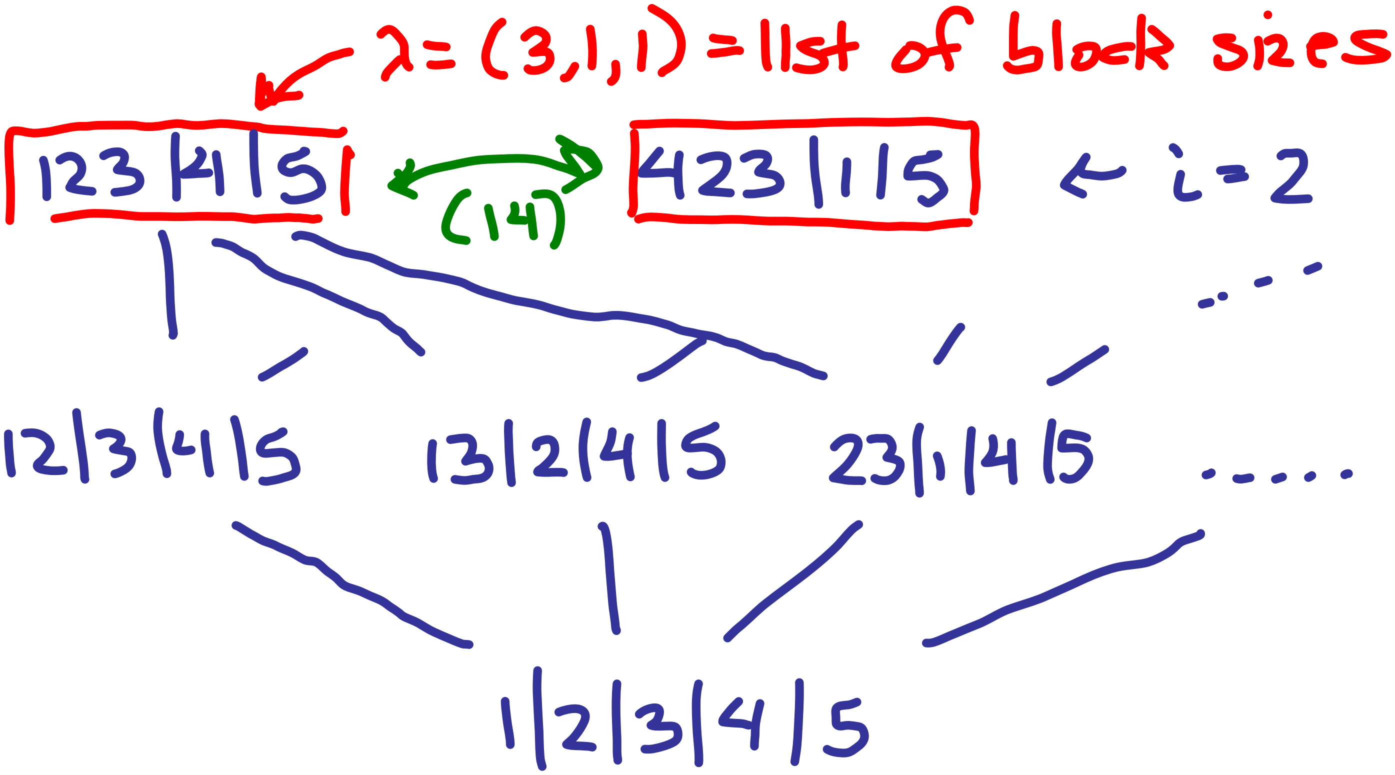
Upcoming Case for Config. Spaces on Manifolds: $S = \{1,2,..,i\}$, where we will do better...

Whitney Homology

$WH_i(P) :=$ "i-th Whitney homology of P"

$$= \bigoplus_{rk(u)=i} \tilde{H}_{i-2}(\hat{0}, u) = \bigoplus_{\lambda \text{ has } n-i \text{ blocks}} WH_\lambda(P)$$

$$WH_\lambda(P) := \bigoplus_{\substack{u \in P \\ type(u)=\lambda}} \tilde{H}_{top}(\hat{0}, u)$$



Fact (Sundaram): $WH_i(P) \cong \beta_{1..i}(P) + \beta_{1..i-1}(P)$

G-Equivariant Enrichment of Goresky-MacPherson Formula

Thm (Sundaram-Welker): Let A be a G-arrangement of $\mathbb{C}$-linear subspaces in $\mathbb{C}^n$ for G a finite subgroup of $GL_n(\mathbb{C})$. Then

$$\tilde{H}^i(M_A) \cong_G \bigoplus_{x \in (L_A^{>0})/G} \mathrm{Ind}_{\mathrm{Stab}(x)}^{G} \tilde{H}_{\mathrm{codim}(x) - i - 2}(\hat{0}, x)$$

(in our case) $= WH_i(L_{A_n}) = WH_i(\Pi_n)$

Note: there are numerous variations, e.g. allowing us also to handle config. spaces in $\mathbb{R}^{2d+1}$!

Upshot for Stability:

- $\beta_{1,j,i}(\Pi_n)$ stabilizes at $B>0$

$\Leftrightarrow$ $WH_i(\Pi_n)$ stabilizes at $B>0$

$\Leftrightarrow$ $H^i(M_n)$ stabilizes at $B>0$

Thm (H-Reiner): $H^i(M_n)$ stabilizes sharply at $3i+1$. More generally, $H^i(M_n^{2d})$ for M_n^{2d} = config. space of n distinct pts in $\mathbb{R}^{2d}$ stabilizes sharply for $n \geq 3\frac{i}{2d-1}+1$.

$H_i(M_n^{2d+1})$ = config. space of n distinct pts in $\mathbb{R}^{2d+1}$ stabilizes sharply for $n \geq 3\frac{i}{2d}$.

Thm (H-Reiner): $\langle H^i(M_n^{2d}), S^{(n-|\nu|,\nu)}\rangle$ vanishes for $|\nu| \le 2i$ and becomes constant for $n \ge n_0 := \begin{cases} |\nu|+i & \text{for } d \text{ odd} \\ |\nu|+i+1 & \text{for } d \text{ even} \end{cases}$

Thm (H-Reiner): $\langle \beta_S(\Pi_n), \text{triv}\rangle$ is constant for $n \ge 2\max(S) - \binom{|S|-1}{2}$

Note: This follows from partitioning of $\Delta(\Pi_n)/S_n$ giving combinatorial interpretation for $\langle \beta_S(\Pi_n), \text{triv}\rangle$ (i.e. from 2003 result of H.), our point of entry to this topic.

Proof Techniques & Results We'll Use

Thm (Hanlon-Stanley): $\Pi_n \cong \mathrm{sgn} \otimes \left(\zeta_n \uparrow_{C_n}^{S_n}\right)$

Method: Calculate $\mu_{\Pi_n^g}(\hat{0}, \hat{1}) = \chi_{\Pi_n}(g)$

Thm (Joyal): $\mathrm{lie}_n \cong \zeta_n \uparrow_{C_n}^{S_n}$

Thm (Barcelo): Explicit S_n-equiv't bijection yielding $\Pi_n \cong \mathrm{sgn} \otimes \mathrm{lie}_n$

Thm (Kraskiewicz & Weyman):

$$\mathrm{lie}_n \cong \bigoplus_{\substack{T \text{ SYT} \\ \text{w/ } \mathrm{maj}(T) \equiv 1 \ (\mathrm{mod}\ n)}} S^{\lambda(T)}$$

Key Properties of Symmetric Functions

- $S^\lambda \overset{ch}{\longleftrightarrow}$ schur fn $s_\lambda = \sum_{T \text{ SSYT shape } \lambda} x^T$

  "Frobenius charact." isom.

$$T = \begin{array}{|c|c|c|c|} \multicolumn{4}{c}{\overbrace{\hphantom{1\;1\;2\;2}}^{\lambda_1}} \\ \hline 1 & 1 & 2 & 2 \\ \hline 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \rightsquigarrow x_1^2 x_2^2 x_3 x_4 = x^T$$

$\Rightarrow$ s_λ includes monomial divisible by $x_1^{\lambda_1}$ but not $x_1^{\lambda_1+1}$.

- Wreath product of rep'ns $\longleftrightarrow$ plethysm of symmetric fns

$\Rightarrow$ f includes x_1^a & g includes x_1^b then $f \cdot g$ includes x_1^{a+b} while $f[g]$ cannot have $x_1^{(\deg f)b+1}$

Thm (Sundaram):

S_j-rep'n on top homology of Π_j

$$ch(WH_\lambda) = \prod_{j \text{ odd}} h_{m_j}[\Pi_j] \prod_{j \text{ even}} e_{m_j}[\Pi_j]$$

$$= \underbrace{\left(h_{m_1}\right)}_{\substack{\| \\ ch(triv_{m_1})}} \underbrace{\left(\prod_{\substack{j \text{ odd} \\ j>1}} h_{m_j}[\Pi_j]\right)\left(\prod_{j \text{ even}} e_{m_j}[\Pi_j]\right)}_{\text{“}\widehat{Wh_\lambda}\text{” has degree } \leq 2i \text{ by } *}$$

where ch = "Frobenius characteristic" isom.

$ch(f) = \sum_\mu f(\mu) \frac{p_\mu}{z_\mu}$ from S_n class functions to ring of symmetric fn's

$$h_n := \sum_{1 \leq i_1 \leq i_2 \leq \cdots} x_{i_1} x_{i_2} \cdots x_{i_n} = ch(\text{trivial rep'n})$$

$$e_n := \sum_{1 \leq i_1 < i_2 < \cdots} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_n} = ch(\text{sgn rep'n})$$

$$M\left(\sum c_\mu x^\mu\right) := \sum c_\mu \left(x^\mu \otimes \mathbb{1}_{n-|\mu|} \uparrow_{S_{|\mu|} \times S_{n-|\mu|}}^{S_n} \right)$$

* **Key Fact for Stability:** $u \in \Pi_n$ of rank i has at most $2i$ letters in nontrivial blocks

Significance: Gives upper bound of $2i$ on $|\lambda|$, where sharp stability bound is $\max\{|\lambda| + \lambda_1\}$

12|34|56|78 ← max # letters in nontriv. blocks = 2-rank = $2i$

$\lambda = (2,2,2,2)$

12|34|56|7|8

$\lambda = (2,2,2,1,1)$

12|34|5|6|7|8

123|4|5|6|7|8

$\lambda = (3,1,1,1,1,1)$

12|3|4|5|6|7|8

1|2|3|4|5|6|7|8

Wiltshire-Gordon Conjectures
& Related Results

Defn (Wiltshire-Gordon):

$$V_n^k = \bigoplus_{\substack{|\lambda|=n \\ \ell(\lambda)=n-k \\ \lambda \text{ has no parts of size } 1}} WH_\lambda(\Pi_n) = \text{"essential part" of } WH_k(\Pi_n)$$

(i.e. $S^\lambda \otimes \mathrm{triv}_0$ part of $H^k(PConf(\mathbb{R}^3)))$

Thm (H-Reiner):

$$\mathrm{Ind}(\mathrm{Res}(V_n^k) \oplus V_{n-1}^k) \cong \mathrm{Res}(V_{n+1}^{k+1})$$

(conjectured by Wiltshire-Gordon)

Key Question: Decompose V_n^k into irreducible rep'ns, since this would exactly give the S^λ irrep's yielding $S^\lambda \otimes \mathrm{triv}_{n-|\lambda|}$ rep'ns comprising k-th cohomology for config. space of n distinct, labeled pts in $\mathbb{R}^2$.

Progress (Upcoming Thm): Answer instead for $\bigoplus_k V_n^k$.

Open Qn: Analogous results for $\mathbb{R}^d$ for $d > 2$?

Thm (H-Reiner):

$$V_n = \mathrm{ch}\Big(\bigoplus_k V_n^k\Big) \cong \bigoplus S^{\lambda(T)}$$

T is "Whitney generating" SYT

where T is **Whitney generating** if either

(1) $T = \emptyset$ or $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$ or $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$

or

(2) $T|_{\{1,2,3,4\}}$ is one of the four shapes:

$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$ $\qquad T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$

$$T_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad T_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

with the following further restrictions:

(a) If T_3, then the first ascent k with $k \geq 4$ is odd

(b) If T_4, then the first ascent k with $k \geq 4$ is even

ascent := i such that $i+1$ in weakly higher row

Idea: Both sides satisfy same recurrence: categorified $d_n = n d_{n-1} + (-1)^n$

Qn: Refined recurrence for each k where k = #cycles in a derangement?

Sharp Stability of Rank-Selected Homology in Π_n?

Conjecture (H-Reiner): for fixed $S \subseteq \{1,2,\ldots,n-2\}$ with $i = \max(S)$, the rank-selected homology $\beta_S(\Pi_n)$ stabilizes sharply at $n = 4i - |S| + 1$.

Rk: For $S = \{1,2,\ldots,i\}$, this yields $3i+1$ & for $S = \{i\}$, this yields $4i$.

Evidence: "$WH_{S,i}(\Pi_n)$" $:= \beta_S + \beta_{S \setminus \{i\}}$ for $i = \max S$ has component that does not stabilize earlier.

Thm (H.-Reiner): $\langle 1, \beta_S(\Pi_n)\rangle$ stabilizes for $n \geq 2\max S - \left(\frac{|S|-1}{2}\right)$

Idea: Partitioning for $\Delta(\Pi_n)/S_n$ in (H., 2003) $\Rightarrow$ combinatorial interpretation for

$$\langle 1, \beta_S(\Pi_n)\rangle = \sum_{T \subseteq S} (-1)^{|S-T|} \langle 1, \alpha_T(\Pi_n)\rangle$$

$$= \sum_{T \subseteq S} (-1)^{|S-T|} f_T(\Delta(\Pi_n)/S_n)$$

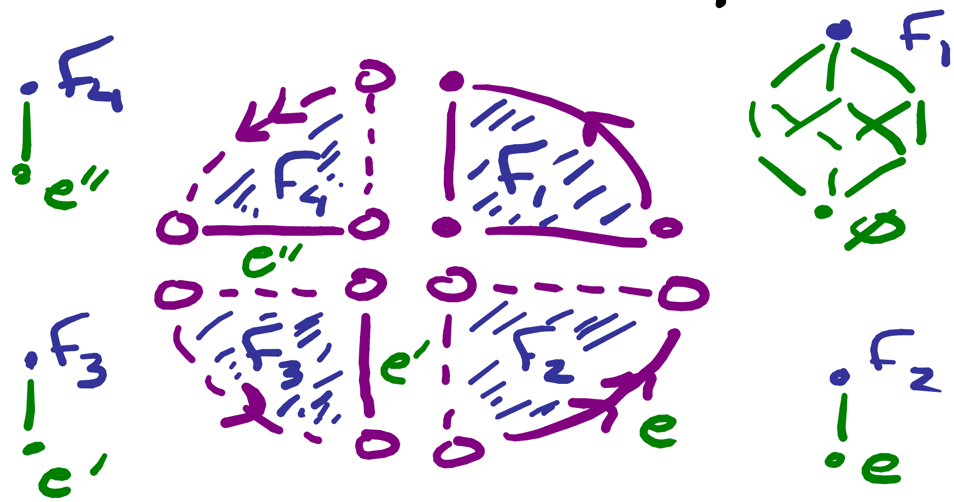
$$= h_S(\Delta(\Pi_n)/S_n)$$

- Injection $\varphi_n : \left\{\begin{array}{c}\text{facets} \\ \text{contrib.} \\ \text{to } b_S(n)\end{array}\right\} \to \left\{\begin{array}{c}\text{facets} \\ \text{contrib} \\ \text{to} \\ b_S(n+1)\end{array}\right\}$ eventually also a surjection.

Rk: This is sharp for $S = \{i\}$

Partitioning: Δ is partitionable if face poset $F(\Delta)$ decomposes into disjoint union of boolean algebras w/ facets as top elements

e.g.

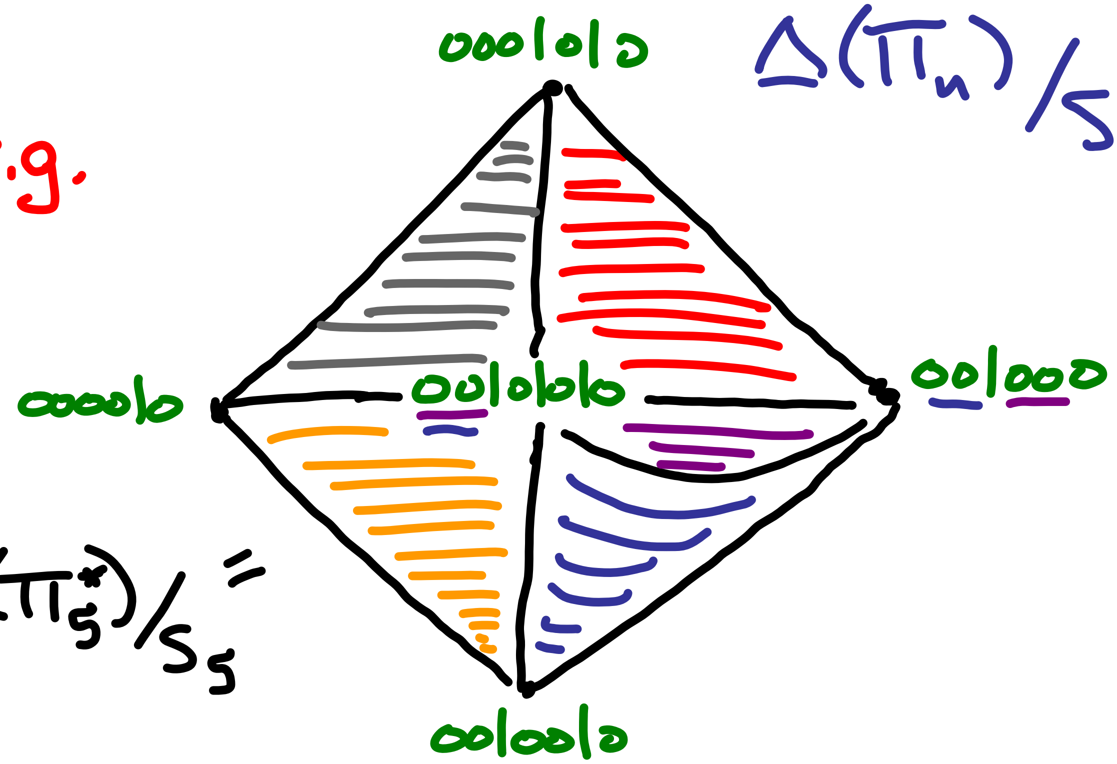


- $h_S(\Delta(\Pi_n)/S_n)$ as # saturated chain orbits in balanced complex "chain labeling" with "descent set" S.
- $\Delta(\Pi_n)/S_n$ "almost" lex. shellable, but $lk_\Delta(F) \cong \mathbb{RP}^n$ for some F

"Chain Labeling" on Balanced Complex

$\Delta(\Pi_n)/S_n$

e.g.



$\Delta(\Pi_5^*)/S_5 =$

- put labels on edges with endpts having consec. colors
- idea subsequently generalized to non-graded case by Hultman

e.g. $\lambda(0|00|00|00) = 351\ldots$

3 1 2

insert bars as far left as possible

= sequence of bar positions

Translating "Polynomial Characters" into Symmetric fns (to get Improved "Power Saving Bounds")

- Any polynomial $P(x_1, x_2, x_3, \ldots)$ gives a class fn for S_n by letting $x_i = \#$ i-cycles in conjugacy class
- The elements $\binom{X}{\lambda} = \binom{x_1}{m_1}\binom{x_2}{m_2}\cdots$ where λ has m_i parts of size i form a basis for $\mathbb{Q}[x_1, x_2, x_3, \ldots]$

Prop'n (H-Reiner): $ch(\chi_P) = \begin{cases} \frac{p_\lambda}{z_\lambda} h_{n-|\lambda|} & \text{for } n \geq |\lambda| \\ 0 & \text{otherwise} \end{cases}$

for $P = \binom{X}{\lambda} = \binom{x_1}{m_1}\binom{x_2}{m_2}\cdots$

Combining with Earlier Results...

Prop'n (H-Reiner): $ch(x_P) = \begin{cases} \frac{p_\lambda}{z_\lambda} h_{n-|\lambda|} & \text{for } n \geq |\lambda| \\ 0 & \text{otherwise} \end{cases}$

for $P = \binom{X}{\lambda} = \binom{x_1}{m_1}\binom{x_2}{m_2}\cdots$

- guarantees for all $P \in \mathbb{Q}[x_1, x_2, \ldots]$, $x_P = M\left(\sum_\mu c_\mu x^\mu\right)$ s.t. $|M| \leq \deg(P)\ \forall \mu$.
- analyze $\langle x_P, H^i(M_n^{2d})\rangle$ via:

Thm (H-Reiner): $\langle H^i(M_n^{2d}), S^{(n-|\nu|, \nu)}\rangle$ vanishes for $|\nu| \leq 2i$ and becomes constant for $n \geq n_0 := \begin{cases} |\nu| + i & \text{for } d \text{ odd} \\ |\nu| + i + 1 & \text{for } d \text{ even} \end{cases}$

Upshot: $\langle x_P, H^i(PConf(\mathbb{C}))\rangle_{S_n}$ is constant for $n \geq \max\{2\deg(P), \deg(P) + i + 1\}$.