Posets Arising as 1-Skeleta of Simple Polytopes, Diameter Bounds on Polytopes & Poset Topology

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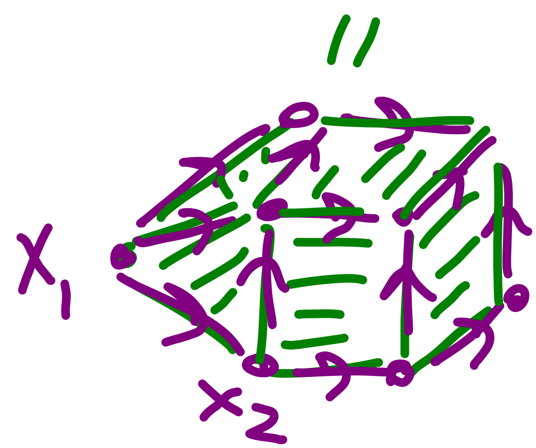
- with thanks to Karola Mészáros for fruitful discussions early in project

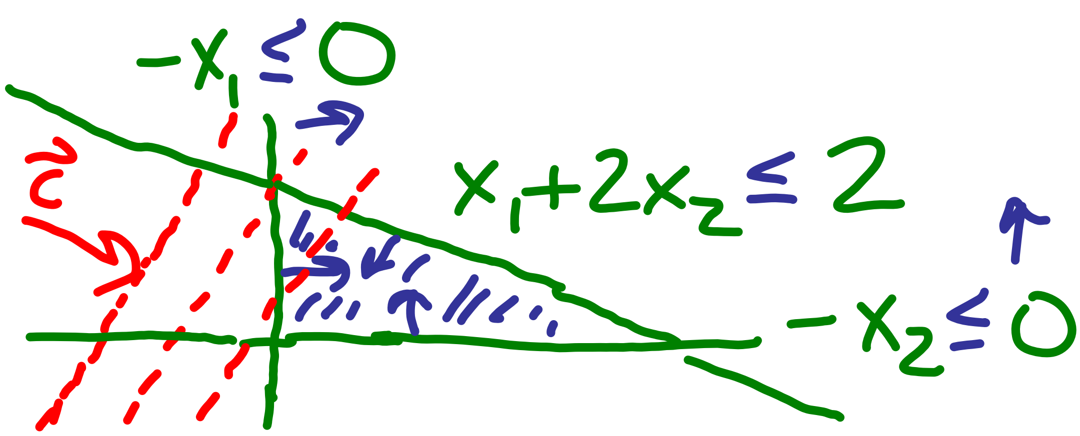
Linear Programming

- Given a matrix A & vectors $\vec{b}, \vec{c}$ seek $\max\{\vec{c}\cdot\vec{x} \mid A\vec{x} \le \vec{b}\}$
- $\{\vec{x} \mid A\vec{x} \le \vec{b}\}$ is polytope P if set is bounded



e.g.

$$\overbrace{\begin{pmatrix}-1 & 0\\ 0 & -1\\ 1 & 2\end{pmatrix}}^{A}\overbrace{\begin{pmatrix}x_1\\ x_2\end{pmatrix}}^{\vec{x}} \le \overbrace{\begin{pmatrix}0\\ 0\\ 2\end{pmatrix}}^{\vec{b}}$$



Solving Linear Programs via Simplex Method

- Define $G(P,\vec{c}) :=$ directed graph on 1-skeleton of P, i.e. on vertex-edge graph of P, with

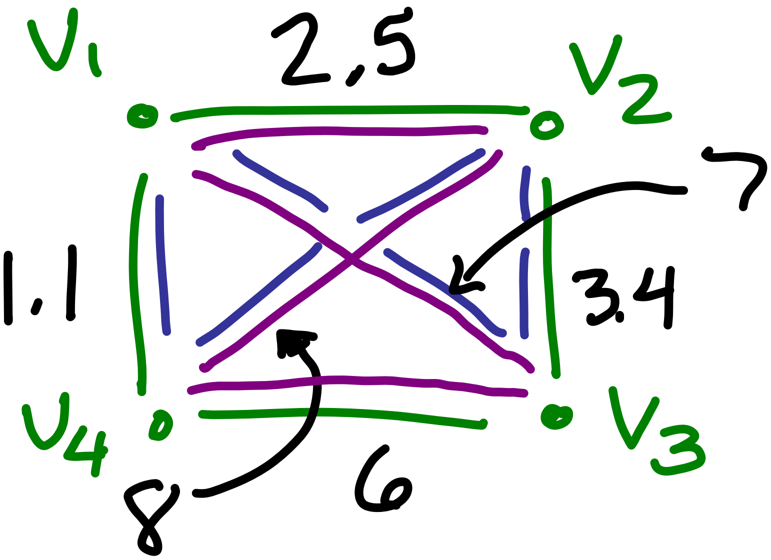
$$x_1 \to x_2 \iff \vec{c}\cdot\vec{x}_1 < \vec{c}\cdot\vec{x}_2$$

- $\max\{\vec{c}\cdot\vec{x} \mid A\vec{x} \le \vec{b}\} =$ sink of $G(P,\vec{c})$

Simplex Method: walk from some vertex $v \in G(P,\vec{c})$ following arrows $v \to v_1 \to v_2 \to \ldots \to s$ to sink s

- also may walk backwards to source of $G(P,\vec{c})$ to minimize $\vec{c}\cdot\vec{x}$

An Example: Traveling Salesman Problem

$G =$ 

Polytope Vertices:

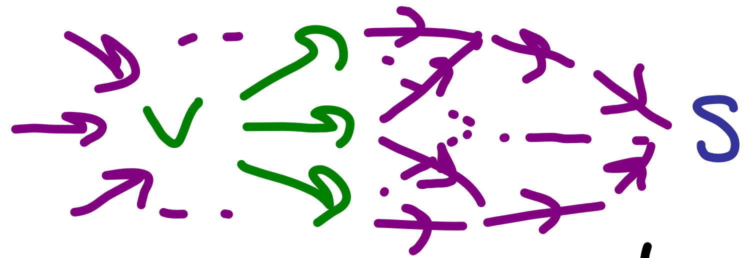
$(1,0,1,1,0,1)$, $(1,1,0,0,1,1)$,

e_{12} e_{14} e_{23} e_{34}

$(0,1,1,1,1,0)$

Cost Vector:

$$\vec{c} = (2.5, 7, 1.1, 3.4, 8, 6)$$

Pivot Rule: method to choose which out arrow 

to follow from v towards sink s.

Key Questions:

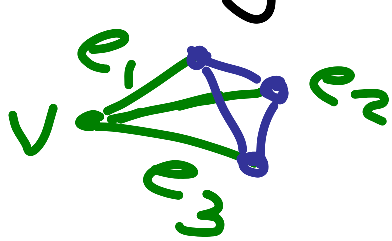
1. What is typical complexity of simplex method (path length)?

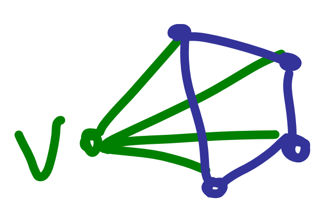
2. What is worst case? (i.e. diameter of $G(P, \vec{c})$)

Quick Background on Polytopes

- A polytope in $\mathbb{R}^d$ is convex hull of finite # vertices, or equiv'ly a bounded set that is an intersection of half spaces.

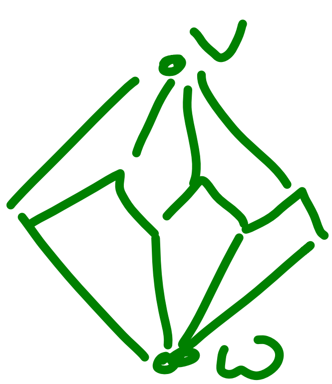
- A polytope is simple if for each vertex v and each collection $e_1, e_2, .., e_r$ of edges emanating out from v there is an r-dim'l face containing all these edges

e.g. 

but not 

Hirsch Conjecture: For d-dim'l polytopes with n facets (max'l faces), diameter of 1-skeleton graph, denoted $\Delta(d,n)$, satisfies $\Delta(d,n) \leq n-d$.

Francisco Santos: After many decades eluding many people, he constructed counterexamples ("spindles" := polytopes with vertices v, w s.t. each facet includes v or w.)



43-dim's, 86 facets, diam ≥ 44

Nonrevisiting path conjecture.

For each u, v in polytope P, there is path u to v not revisiting any facet it has left.

Non-Revis. Path Conj $\Rightarrow$ Hirsch Conj.

- nonrevisiting path leaves a facet at each step & still belongs to d facets at its conclusion

Strong Monotone Path Conjecture:

there exists directed path of length $\leq n-d$ from any vertex to vertex v maximizing $\vec{c} \cdot v$ with cost increasing each step.

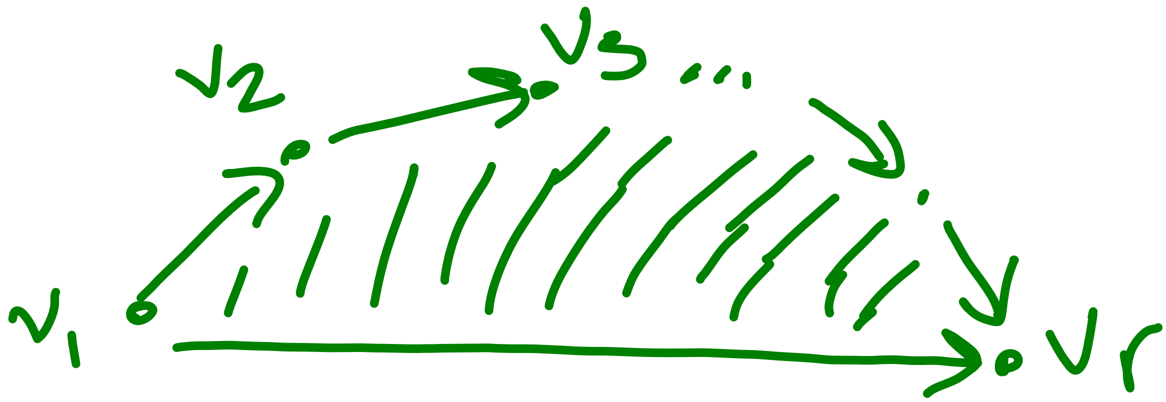
Our Plan

Impose further conditions on P and $\vec{c}$ that will imply a corollary of the following which we hope might also hold:

> For each $u, v \in P$, each directed path from u to v never revisits any facet it has left.

This property would make all pivot rules efficient for P and $\vec{c}$.

New Def'n: $G(P, \vec{c})$ has the **Hasse diagram property** if it is Hasse diagram of finite poset, i.e. $v_1 \to v_2 \to \cdots \to v_r$ for $r \geq 3$ directed path precludes $v_1 \to v_r$



Note: precludes d-simplices as faces for $d \geq 2$

Important Non-Examples:

"Klee-Minty Cubes"

e.g. $n=3$

$\vec{c}=(0,0,1)$

- path visits all vertices!
- first polytopes exhibiting inefficiency of simplex method

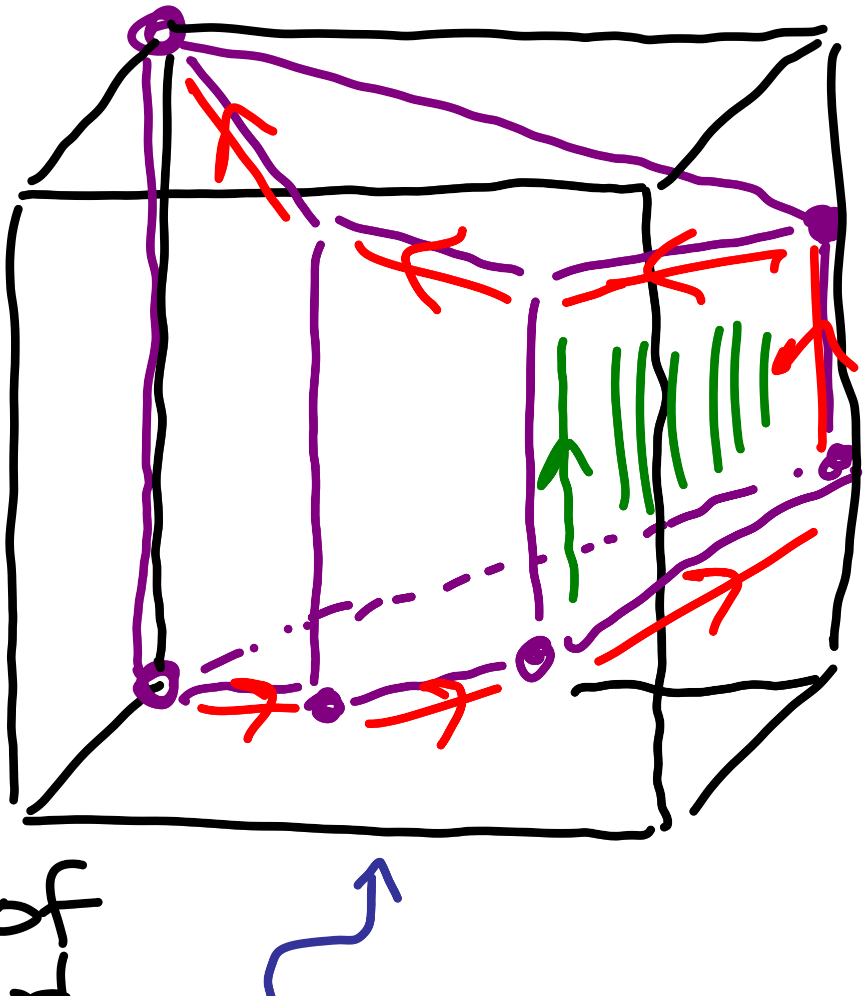


Figure modelled after one in Gärtner-Henk-Ziegler paper

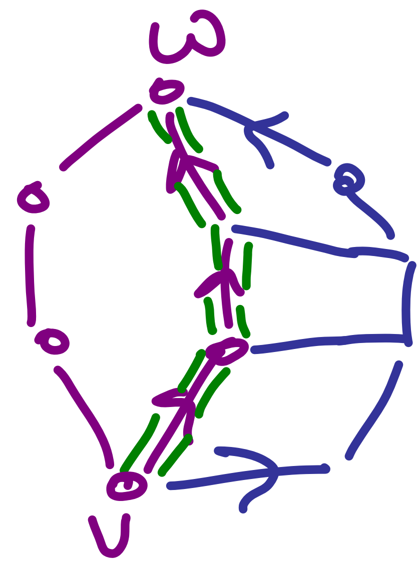
n-dimensional Klee-Minty cube

$$:= \left\{ (x_1,\dots,x_n) \,\middle|\, 0 \le x_1 \le 1 \,;\ \begin{array}{l} \varepsilon x_{i-1} < x_i < 1-\varepsilon x_{i-1} \\ \text{for } i>1 \,;\ 0<\varepsilon<\tfrac{1}{2} \end{array} \right\}$$

Note: Klee-Minty cubes violate Hasse diagram property in way that seems to be at the heart of what leads to existence of "long" directed path (visiting all 2^n vertices) in it

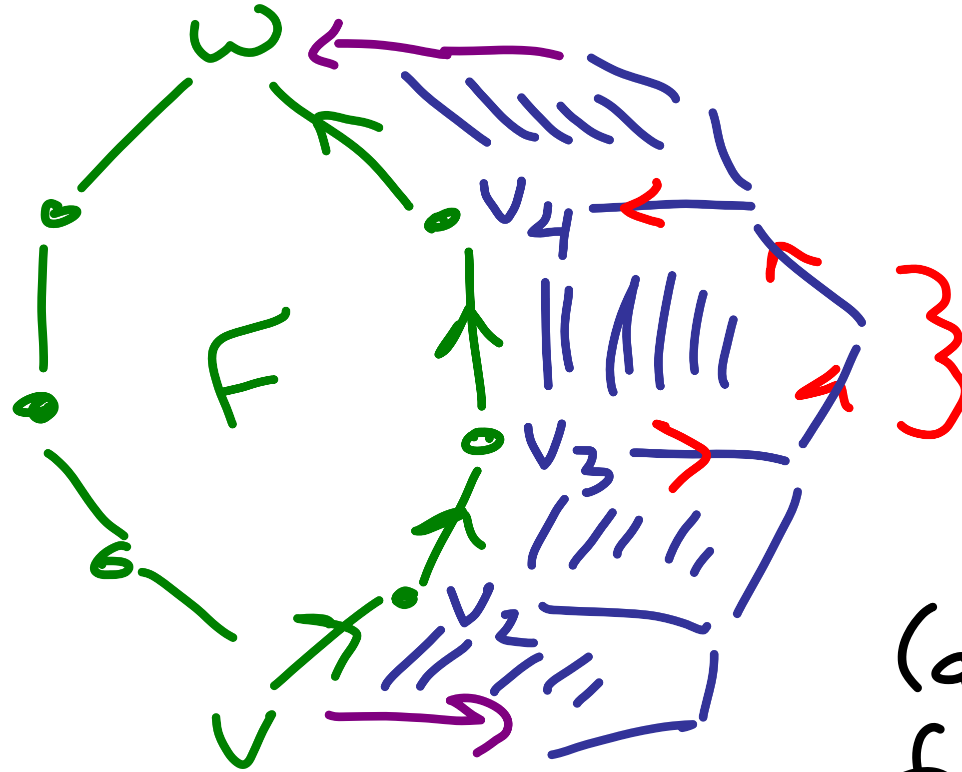
Our hope: Hasse diagram property precludes such issues.

Lemma: Given $F \subseteq G$ with $\dim(G) = \dim(F) + 1$ in simple polytope P w/ generic $\vec{c}$ s.t. $G(P, \vec{c})$ is a Hasse diagram, then there does not exist $v, w \in F$ with directed path P_F from v to w in F,



outward oriented edge from v to $G \setminus F$ and inward oriented edge $G \setminus F$ to w.

Corollary: Monotonicity of out-degrees & partic. outward directions.

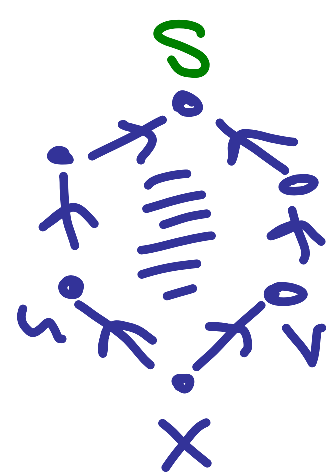


violate Hasse diagram (at switch from out to in)

Corollary: For each face $F \subseteq P$ with $\hat{0} \in F$ or $\hat{1} \in F$, directed paths cannot revisit F after departing from it.

Recall: A poset L is a **lattice** if for each $u, v \in L$ there exists unique least upper bound ("**join**") for u and v, denoted $u \vee v$, and unique greatest lower bound for u and v ("**meet**"), $u \wedge v$.

Note: for P simple & $G(P, \vec{c})$ Hasse diagram, an upper bound for u, v both covering x is sink s of unique 2-face containing x, u, v



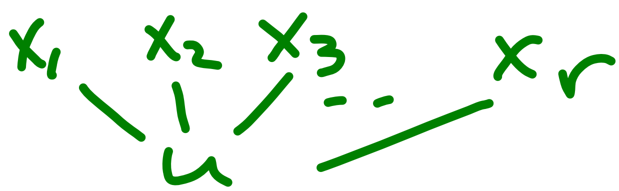
"Pseudo-joins" in a Polytope

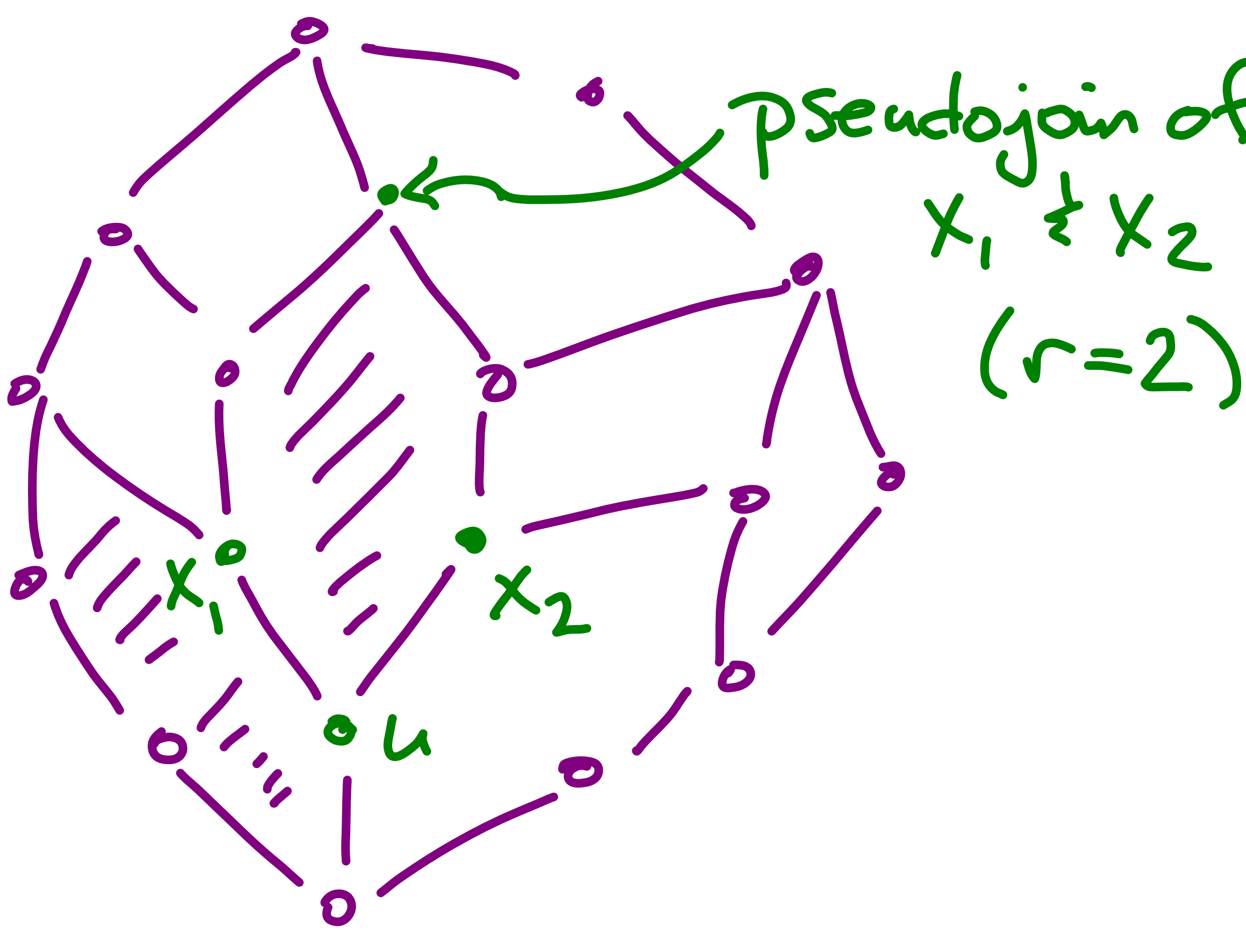
Let P be simple polytope w/ generic cost vector $\vec{c}$ such that $G(P,\vec{c})$ is Hasse diagram of poset L with $x_1, x_2, \ldots, x_r \in L$ s.t. there exists $u \in L$ with $u \lessdot x_i$ for $i = 1, 2, \ldots, r$.

Recall:

- $u \lessdot v$ means $u < v$ with no possible z s.t. $u < z < v$ "cover rel'n"
- a_i is atom of $[u,v]$ for $u \lessdot a_i$

Defn: The pseudo-join of $x_1, x_2, \ldots, x_r$ is the sink of the unique r-face of simple polytope P containing





Lemma: For P a simple polytope & $\vec{c}$ generic cost vector s.t. $G(P,\vec{c})$ is Hasse diagram of poset L, let $S,T \subseteq \{a_1,\dots,a_n\}$ be distinct sets of atoms. Then $\text{pseudojoin}(S) \neq \text{pseudojoin}(T)$.

For L a lattice, this also holds for atoms in each interval $[u,v] \subseteq L$.

Corollary: Subposet of pseudojoins is isomorphic to poset of subsets of $\{a_1,\dots,a_n\}$

Idea:

(1) Reduce to $S \subsetneq T$ with $|T| = |S| + 1$

- $S_1 \subsetneq S_2 \subsetneq S_3$ with $psj(S_1) = psj(S_3)$
  $\Rightarrow psj(S_2) = psj(S_3)$
- $S_1 \not\subseteq S_2$ & $S_2 \not\subseteq S_1$ then use
  $S_1 \cap S_2 \subsetneq S_1$ with $psj(S_1 \cap S_2) = psj(S_1)$

(2) Use codim one nonrevisiting lemma

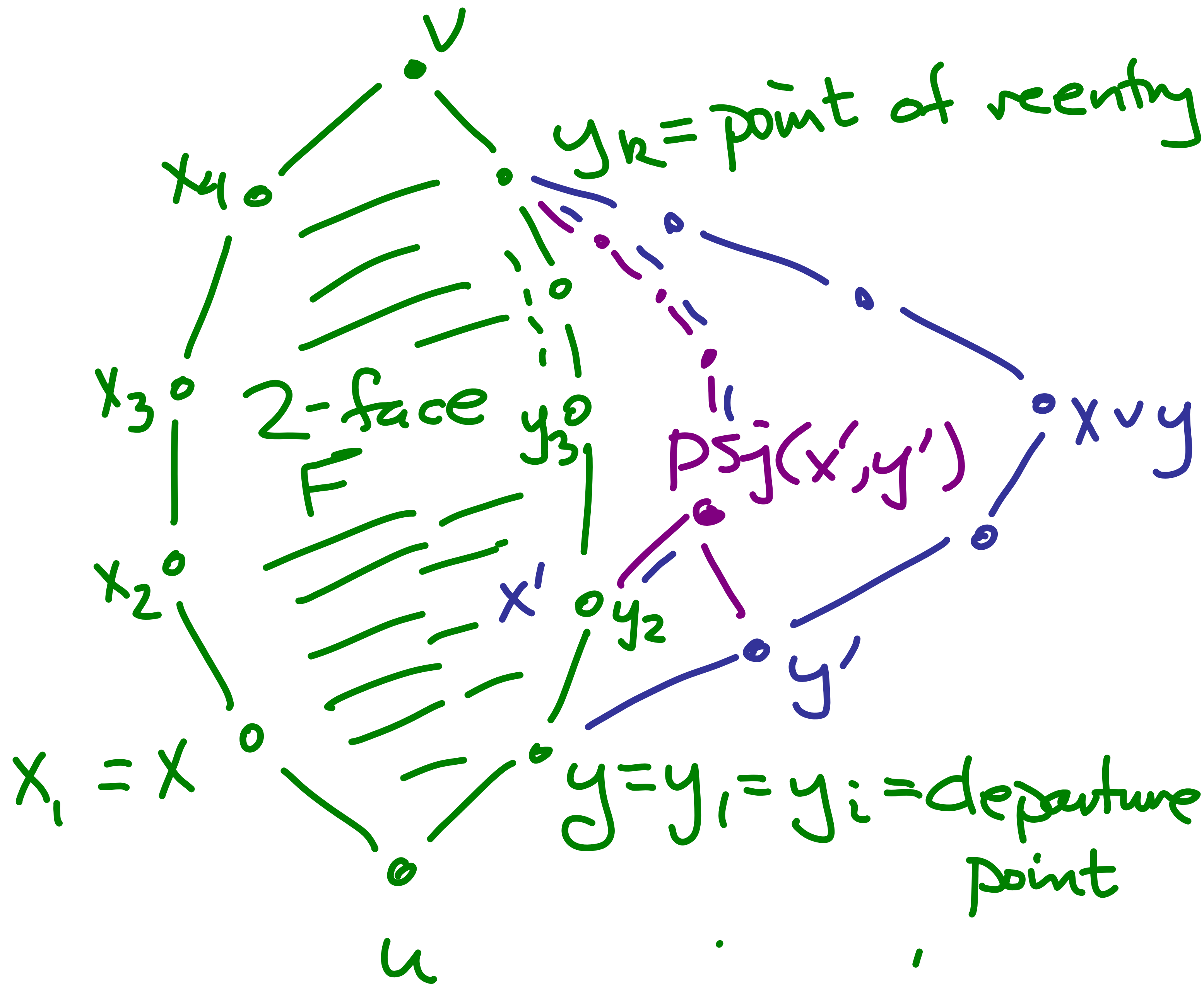
(3) For $[u,v]$, use that v is an upper bound for the atoms of $[u,v]$ w/ l.u.b. in $[u,v]$

Note: Since pseudo-join of $x_1, \dots, x_r$ is an upper bound, there exists directed path from $x_1 \vee \dots \vee x_r$ to pseudo-join$(x_1, \dots, x_r)$

Thm: Let P be a simple polytope and $\vec{c}$ be generic cost vector with $G(P, \vec{c})$ Hasse diagram of finite lattice. Then pseudo-join$(x_1, x_2, \dots, x_r) = x_1 \vee \dots \vee x_r$

Pf: induction on r with $r=2$ base case especially tricky part.

Idea for r=2 case:



• $y' \notin F \Rightarrow psj(x',y') \notin F$

$\quad\quad\quad\quad\quad\quad\quad\quad \parallel$ ← by induction on length longest path to $\hat{1}$

strict inequality ⇝ $\text{join}(x',y')$

$\quad\quad\quad\quad\quad\quad\quad\quad \wedge\!|$

$\Rightarrow \exists$ smaller $k-i \Rightarrow\Leftarrow$ $\quad y_k \in F$

Idea for Inductive Step:

Induct on $|S|$ with $|S| = 2$ base case as just discussed.

$T \subsetneq S$

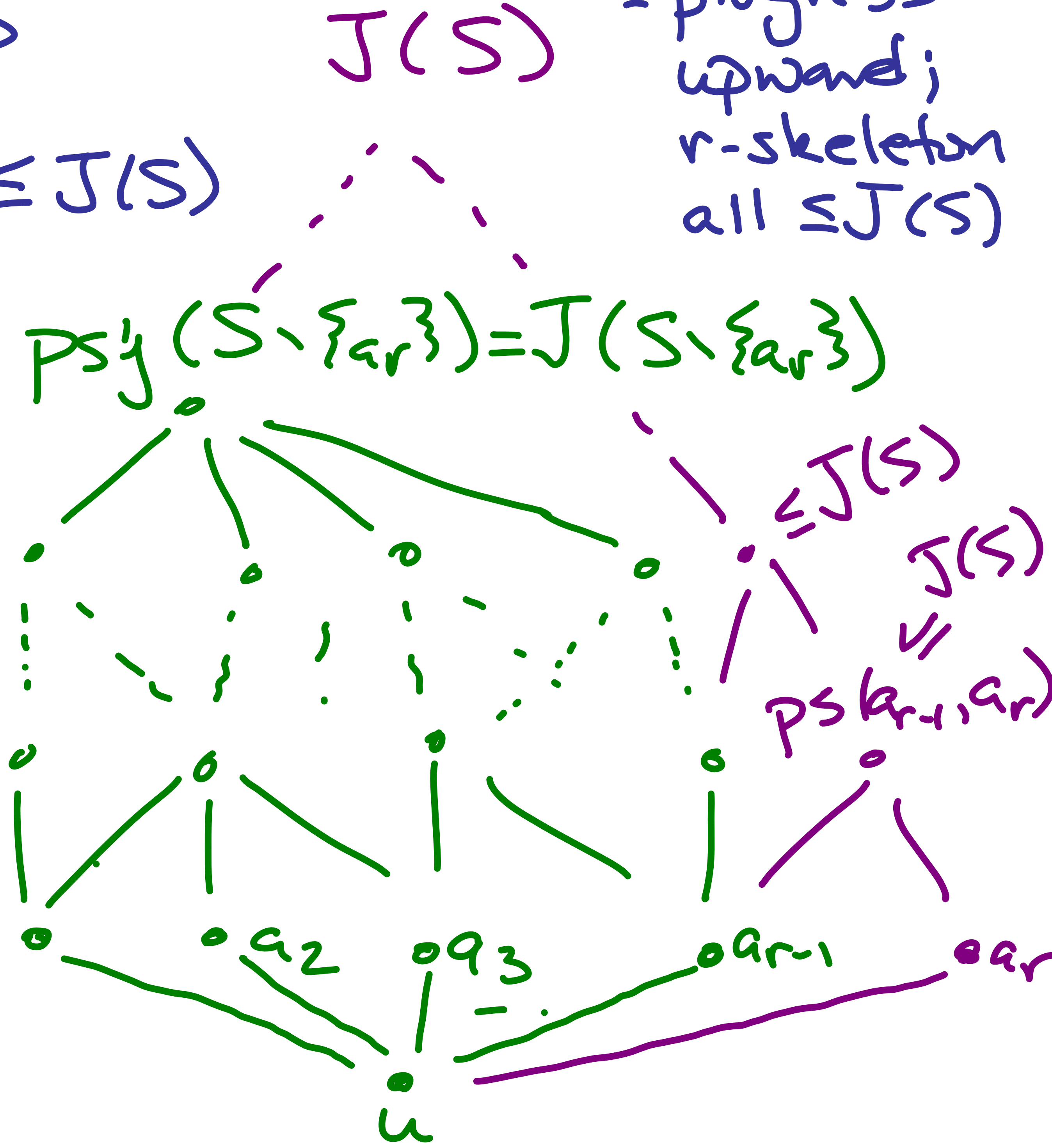
$\Downarrow$

$J(T) \le J(S)$

$\parallel$

$ps(T)$

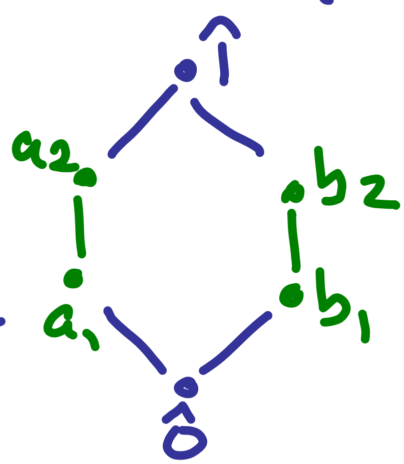
- progress upward; r-skeleton all $\le J(S)$



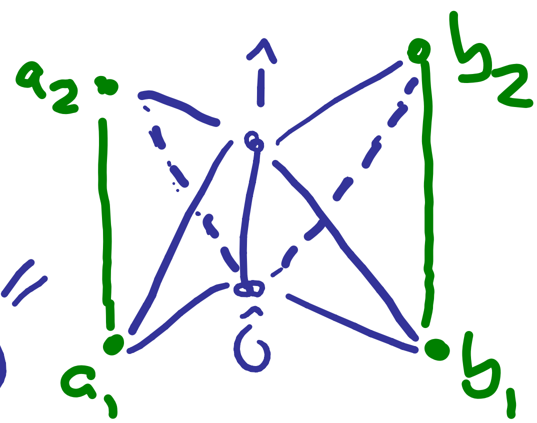
Defn: The order complex (or nerve) of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-dim'l faces are the $(i+1)$-chains $v_0 < v_1 < \dots < v_i$ in P.

e.g.

$P =$



$\Delta(P) =$



Thm (Hall; Popularized by Rota):

$$\mu_P(u,v) = \tilde{\chi}(\Delta(\underbrace{(u,v)}_{=\ \text{subposet } \{z \in P \mid u < z < v\}}))$$

A topological-combinatorial tool:

Quillen Fiber Lemma: Given a poset map $f: P \to Q$ s.t. $q \in Q \Rightarrow \Delta(\{p \in P \mid f(p) \leq q\})$ is contractible, then $\Delta(P) \simeq \Delta(Q)$.

Remark: Used extensively in finite group theory (to characterize groups via subgp lattice) & in topological combinatorics.

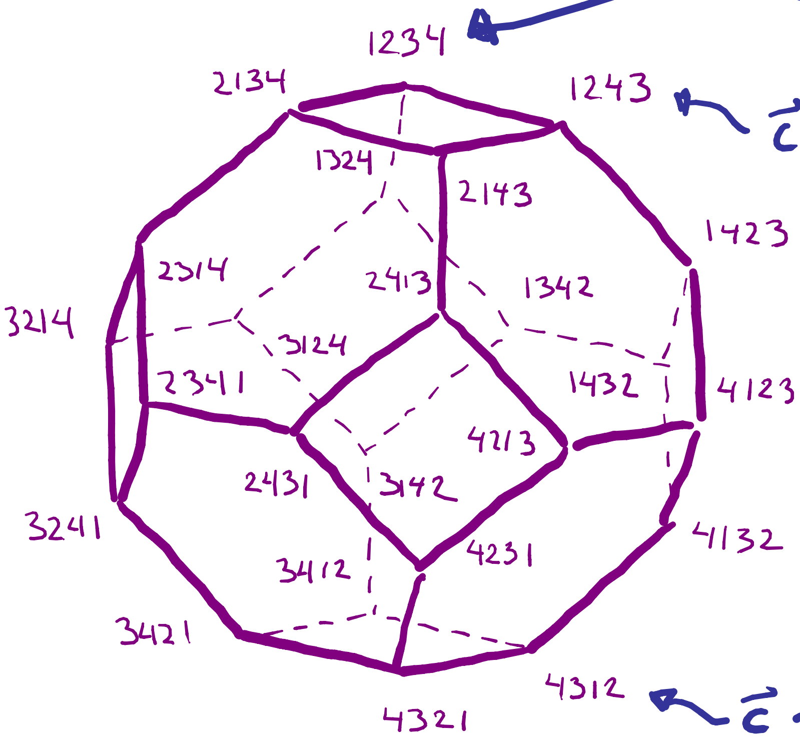
Thm: Let P be a simple polytope with generic cost vector $\vec{c}$ such that $G(P,\vec{c})$ is the Hasse diagram of a finite lattice L. Then each open interval $(u,v)=\{z\in L \mid u<z<v\}$ has order complex homotopy equivalent to a ball or a sphere

Applications:

- permutahedra $\rightsquigarrow$ weak order
- associahedra $\rightsquigarrow$ Tamari lattice
- generalized associahedra $\rightsquigarrow$ Cambrian lattices

Permutahedron as Weak Order

- cost vector $\vec{c}$ any strictly ascending vector such as $\vec{c} = (1,2,3,4)$.



$\vec{c} \cdot (1,2,3,4) = 30$

$\vec{c} \cdot (1,2,4,3) = 29$

$\vec{c} \cdot (1,4,2,3) = 27$

$\vec{c} \cdot (4,1,2,3) = 24$

$\vec{c} \cdot (4,1,3,2) = 23$

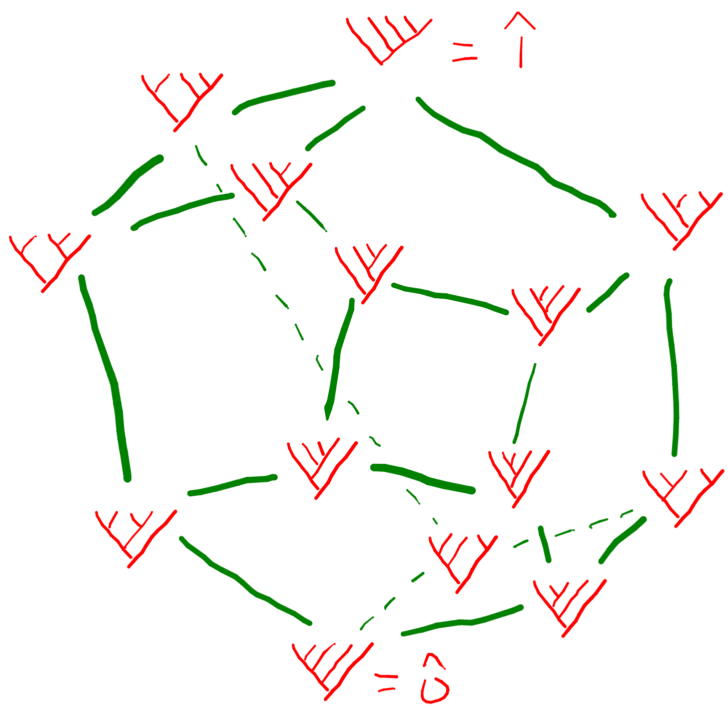
$\vec{c} \cdot (4,3,1,2) = 21$

$\vec{c} \cdot (4,3,2,1) = 20$

- Homotopy type 1st due to Edelman (type A) & Björner

Associahedron as Tamari Lattice

- Use Loday's realization
- Poset of binary trees with cover relations: $((a,b),c) \lessdot (a,(b,c))$



- Homotopy type 1st due to Björner & Wachs via nonpure lexicographic shellability

Checking the Hasse Diagram Property

(thanks to Lou Billera)

- let $A :=$ directed adjacency matrix for $G(P, \vec{c})$
- $(A^r)_{i,j} :=$ # directed paths of length r from v_i to v_j
- Calculate $A, A^2, \ldots, A^{|V(G)|-1}$
- Want $\mathrm{trace}(A^T \cdot A^r) = 0$ for $r = 2, 3, \ldots, |V(G)|-1$

Some Further Questions

Qn 1: Does P simple + $G(P, \vec{c})$ Hasse diagram of lattice $\Rightarrow$ no directed path can revisit face it has departed? (If not, variations?)

Qn 2: Variations on these hypotheses? Non-simple polytopes?

Qn 3: Structure of posets of joins/pseudo-joins for non-simple polytopes?

Qn 4: Other interesting examples?

Qn 5: Non-Hasse $\Rightarrow$ 2-face failure?