Counting by Inclusion-Exclusion Topologically

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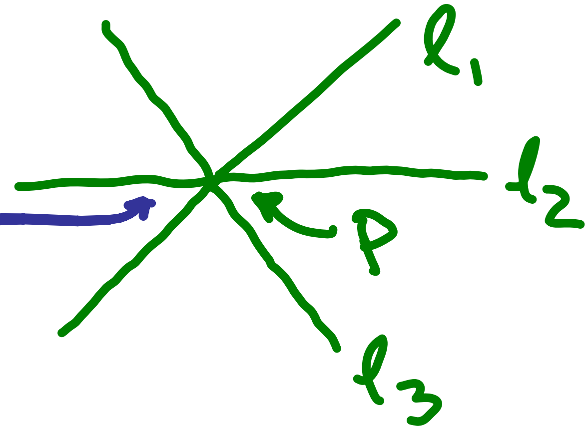
Counting by Inclusion-Exclusion

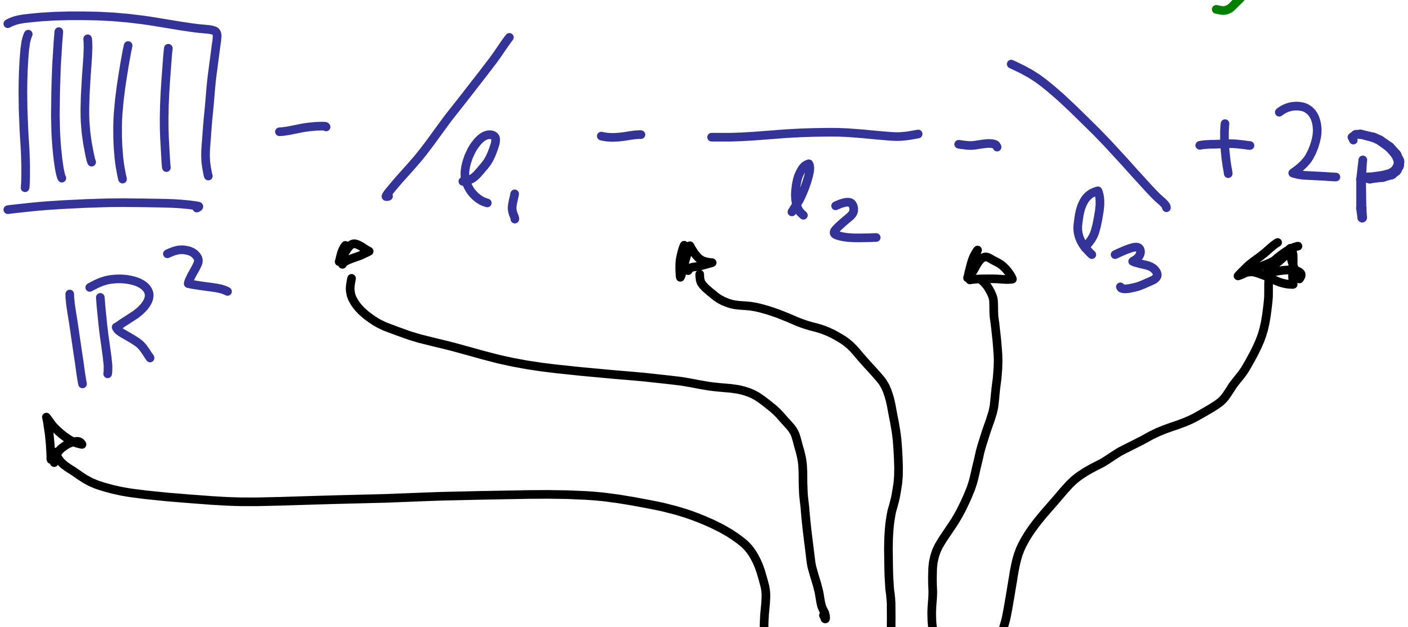
e.g. "counting" points in the $\mathbb{R}^2$

complement of $\rightsquigarrow$

yields:

counted
$1-1-1-1+2=0$
times





- Coefficients $1, -1, -1, -1, 2$ in such inclusion-exclusion counting formula given by "Möbius function" μ (upcoming)

Now let's define a function "M"
to calculate these coefficients
(& generalize this counting technique)

We need:

- $M(\mathbb{R}^2, \mathbb{R}^2) = 1 = $ coef. of $\mathbb{R}^2$
- $M(\mathbb{R}^2, \ell_1) = -1 = $ coef of ℓ_1

  (so $M(\mathbb{R}^2, \mathbb{R}^2) + M(\mathbb{R}^2, \ell_1) = 0$)

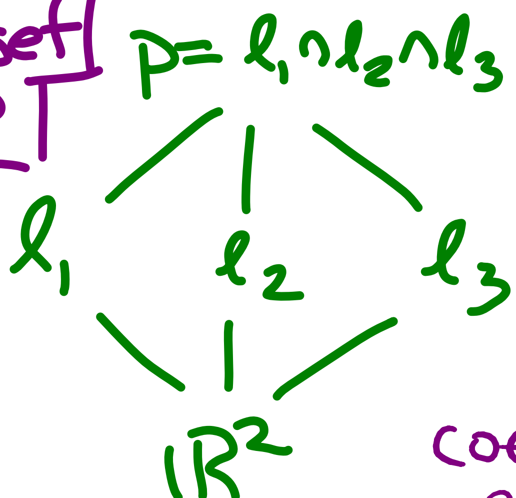
- $M(\mathbb{R}^2, \ell_2) = -1$
- $M(\mathbb{R}^2, \ell_3) = -1$
- $M(\mathbb{R}^2, p) = 2$

$$\left(\text{so } M(\mathbb{R}^2, \mathbb{R}^2) + \sum_{i=1}^{3} M(\mathbb{R}^2, \ell_i) + M(\mathbb{R}^2, p) = 0 \right)$$

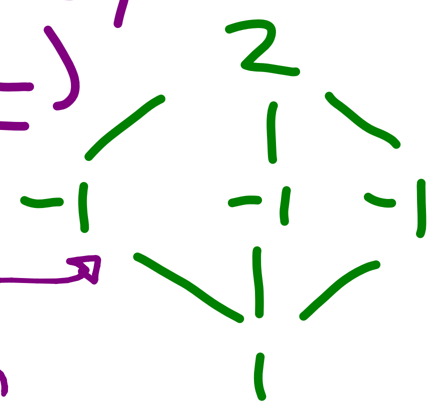
Def'n: Möbius function $\mu_P(x,y)$ of a "partially ordered set" P is defined recursively: $\mu_P(x,x)=1$

and $$\mu_P(x,y) = -\sum_{x \supseteq z \supsetneq y} \mu_P(x,z)$$

$\boxed{\text{Poset } P}$



$\boxed{\text{Its Mobius fn } \mu_P(\mathbb{R}^2, -)}$



coefficients in $\mathbb{R}^2 - \ell_1 - \ell_2 - \ell_3 + 2p$

(we say $u \leq v$ & draw upward path u to v for v subset of u)

Partially Ordered Sets (Posets) More Generally

Unlike the integers where any u, v satisfy $u \le v$ or $v \le u$ (or both if $u = v$), some sets only allow comparison of some of the pairs of elements

e.g. 1. subsets of $\{1, 2, \ldots, n\}$ with $S \le T \iff S \subseteq T$

if and only if

2. positive integers w/ $d \le n \iff d$ is a divisor of n

(e.g. $2 \le 6$ but $2 \not\le 5$)

Poset of Subsets of $\{1,2,\ldots,n\}$
its Möbius Function

$$(1-1)^n = 0^n = 0 \text{ for } n \geq 1$$

$$\| $$

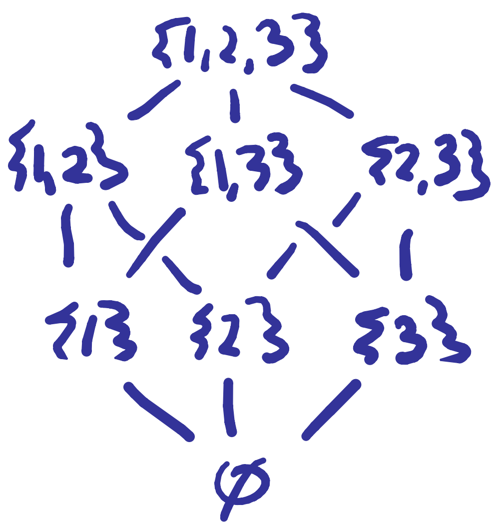
$$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}} \text{ for } x=1 \text{ \& } y=-1$$

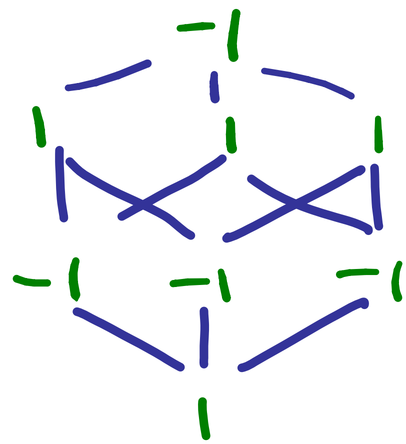
$$\| $$

$$\sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k} \quad \text{for } x=1 \text{ \& } y=-1$$

$$\| $$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}$$





e.g. $\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 0$

$$\| $$

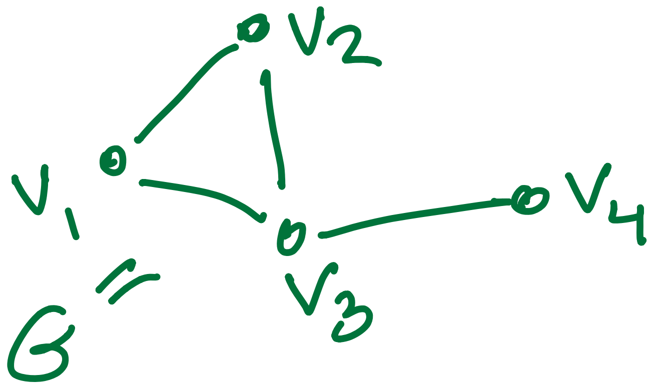
$$1 - 4 + 6 - 4 + 1$$

Proof by Induction yields: $\mu(\hat{0}, u) = (-1)^{rk(u)}$

Counting "Proper d-colorings" of a Graph

proper := no two adjacent nodes same color

e.g.



$d = 7$

$\#$ proper 7-colorings $\quad K : V(G) \to \{1, \ldots, 7\}$

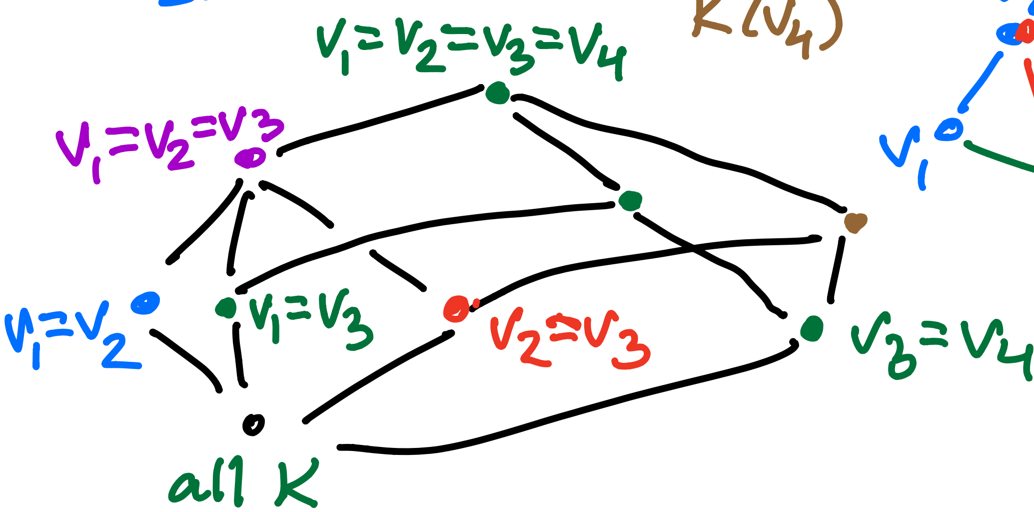
$\quad\ =$

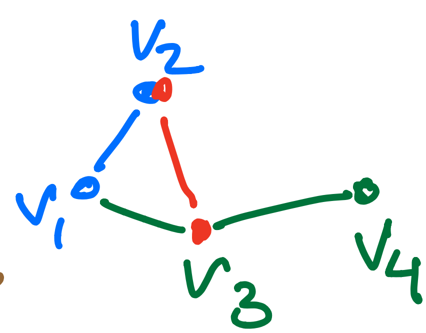
total $\#$ 7-colorings $-$ those with $K(v_1) = K(v_2)$

$-$ those with $K(v_2) = K(v_3)$

$- \ldots$

$$\#\text{proper 7-colorings} = \underbrace{7^4}_{\text{all colorings}} - \underbrace{7^3}_{\substack{\text{those with}\\ K(v_1)=K(v_2)}} - \underbrace{7^3}_{K(v_2)=K(v_3)} - \underbrace{7^3}_{K(v_1)=K(v_3)} - \underbrace{7^3}_{K(v_3)=K(v_4)} + \underbrace{7^2}_{K(v_2)=K(v_3)=K(v_4)} + \underbrace{7^2}_{K(v_1)=K(v_3)=K(v_4)} + \underbrace{2\cdot 7^2}_{K(v_1)=K(v_2)=K(v_3)} - \underbrace{7^1}_{K(v_1)=K(v_2)=K(v_3)=K(v_4)}$$

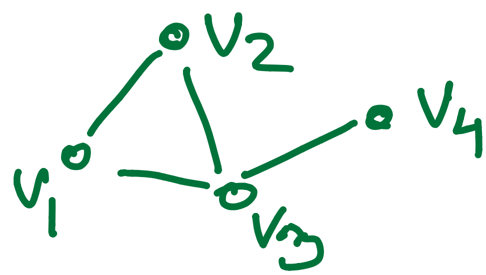




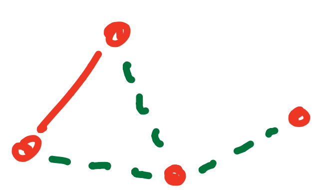
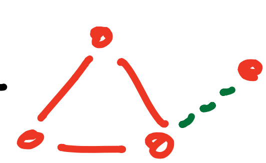
$$\chi_G(d) = \#\text{proper } d\text{-colorings of } G$$

$$= d^4 - 4d^3 + 4d^2 - d$$

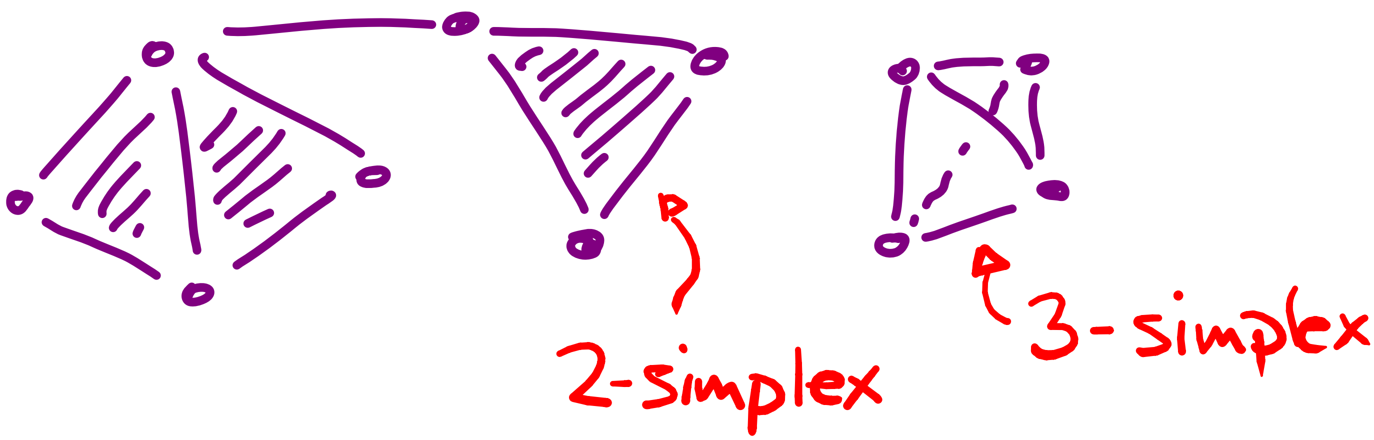
$$= \sum_{u \in \text{poset of "flats" of } G} \mu(\hat{0}, u)\, d^{\,rk(G) - rk(u)}$$

for $G =$ 

- a flat of G is a subset of edges s.t. adding any further edge decreasing # connected components

e.g.  or 

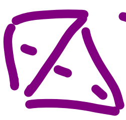
Simplicial Complexes



A simpllcial complex (e.g a triangulation) is made of vertices (called "0-simplices"), edges ("1-simplices"), solid triangles ("2-simplices"), solid tetrahedra ("3-simplices"), etc.

(Reduced) Euler Characteristic

- The reduced Euler characteristic of K, denoted $\tilde{\chi}(K) = -1 + \#\text{vertices} - \#\text{edges} + \#\text{triangles} - \dots$

e.g. $\tilde{\chi}($  $) = -1+4-6+4 = 1$

$\tilde{\chi}($  $) = -1+5-9+6 = 1$

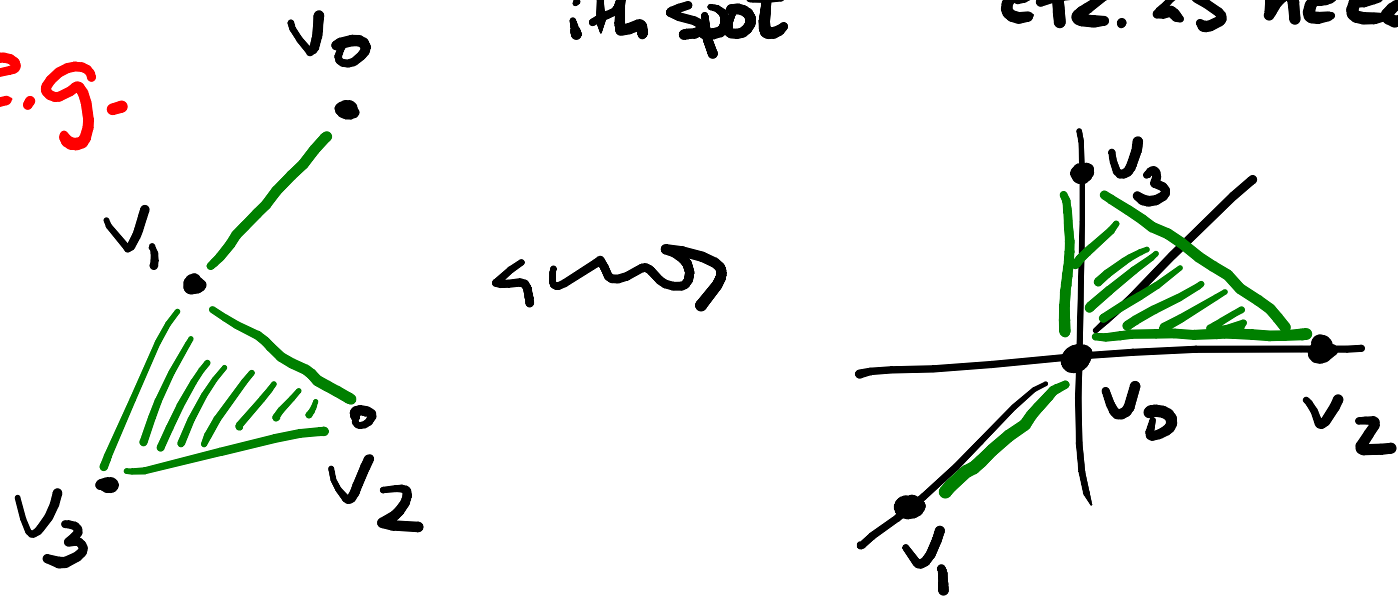
$\Big\}\ \tilde{\chi}(\text{2-sphere})$

Adding faces without changing "Topology" won't change $\tilde{\chi}$!

$$\tilde{\chi} = \underset{\overset{\shortparallel}{0}}{-1+3-3+1} = \underset{\overset{\shortparallel}{0}}{-1+4-5+2} = \underset{\overset{\shortparallel}{0}}{-1+5-8+4}$$

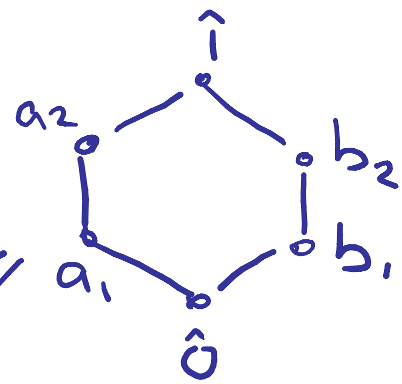
Realization: any simplicial complex with vertices $v_0, v_1, \ldots, v_n$ can be drawn in $\mathbb{R}^n$ by letting $v_0 = (0,0,\ldots,0)$ and for $1 \le i \le n$ letting $v_i = (0,0,\ldots,0,\underset{\substack{\uparrow \\ i\text{th spot}}}{1},0,\ldots,0)$ & adding edges, etc. as needed

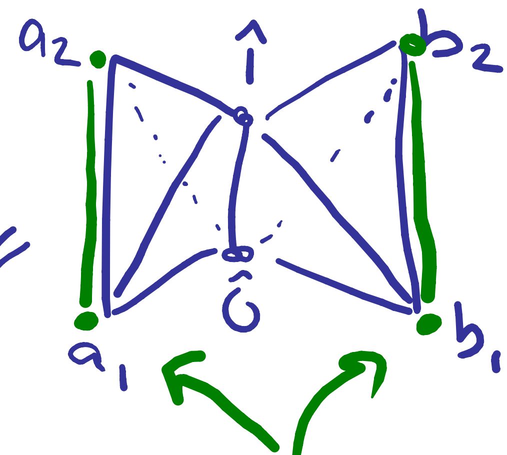
e.g.



- We can also think of simplicial complexes abstractly, letting a face be a collection of the vertices in it, requiring for S a face & $T \subseteq S$ then T must also be a face.

Def'n: The order complex of a poset P is the abstract simplicial complex, **denoted** $\Delta(P)$, whose i-dimensional faces are the $(i+1)$-"chains" $v_0 < \dots < v_i$ in P

e.g. $P =$ 

$\Delta(P) =$ 

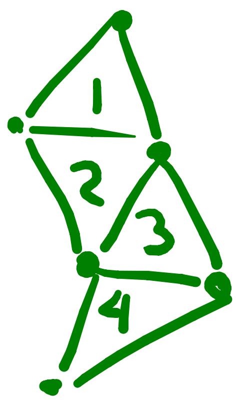
Key Property (due to Hall; popularized by Rota):

$$\mu_P(x,y) = \tilde{\chi}(\Delta_P(x,y)) = -1 + \#\text{vertices} - \#\text{edges} + \#\text{2-faces} \dots$$

$$= -1 + \beta_0 - \beta_1 + \beta_2 - \dots$$

- $\beta_i = \#$ i-dim'l hole boundaries
- $(u,v) = \{z \in P \mid u < z < v\}$
- $\Delta_P(u,v) = \Delta(\{z \in P \mid u < z < v\})$

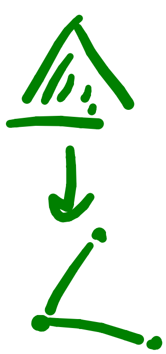
Techniques Yielding Möbius Functions (& Poset Topology)



- (Lexicographic) shellability
- EL-labelings (Anders Björner)
- CL-labelings (Anders Björner & Michelle Wachs)

$\Rightarrow \Delta(P) \simeq$ 

(telling us $\tilde{\chi}$ hence μ)



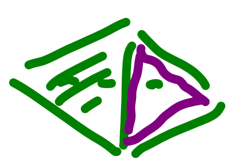
- Lexicographic discrete Morse functions (Babson-H.)

(for other topol. types)

Technique: Shellability

- simplicial complex is **pure** of dim. d if all maximal faces ("facets") are d-dimensional
- simplicial complex is **shellable** if there is total order $F_1, F_2, \ldots, F_k$, a **shelling**, on facets satisfying conditions guaranteeing we can build up the complex by attaching facets in this order so each step either leaves topology (homology) unchanged or closes off a sphere S^i, increasing β_i by 1.

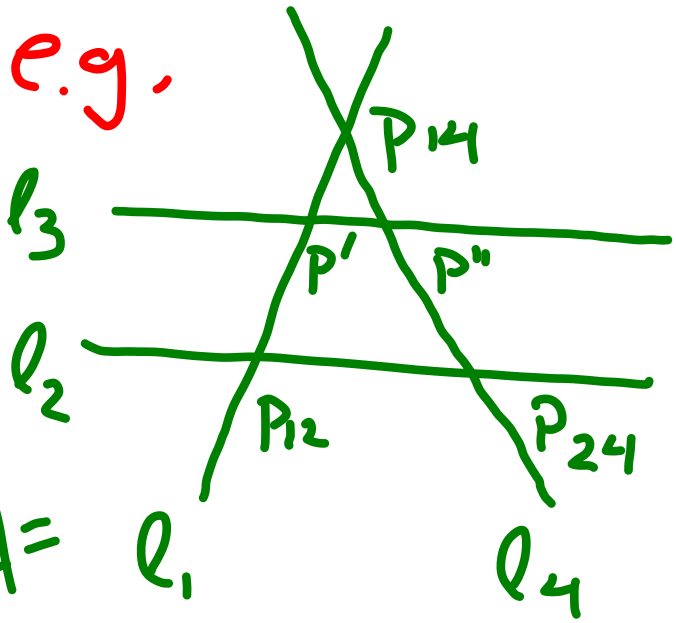
OR



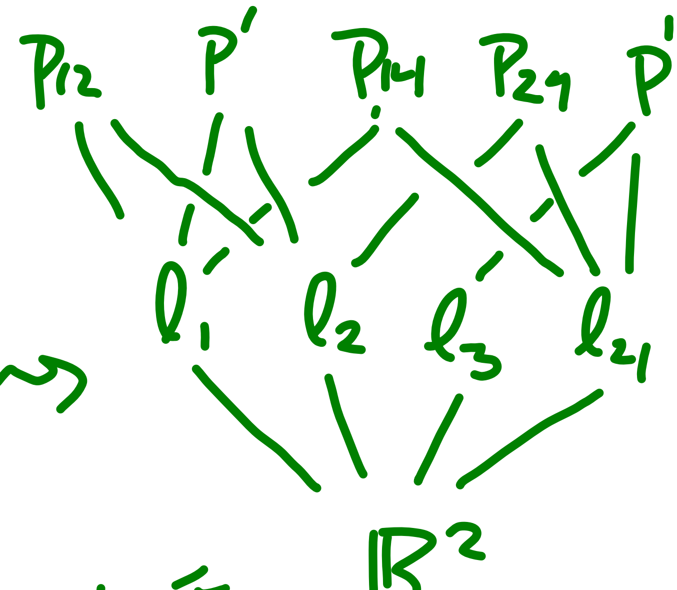
Intersection Posets

e.g.

$A =$

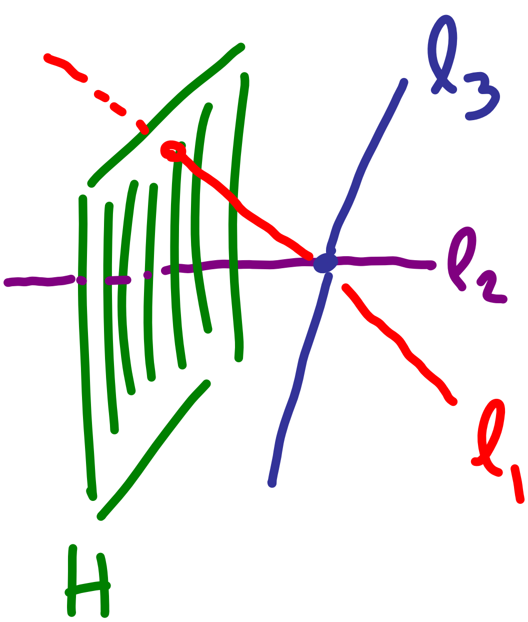


$\rightsquigarrow$

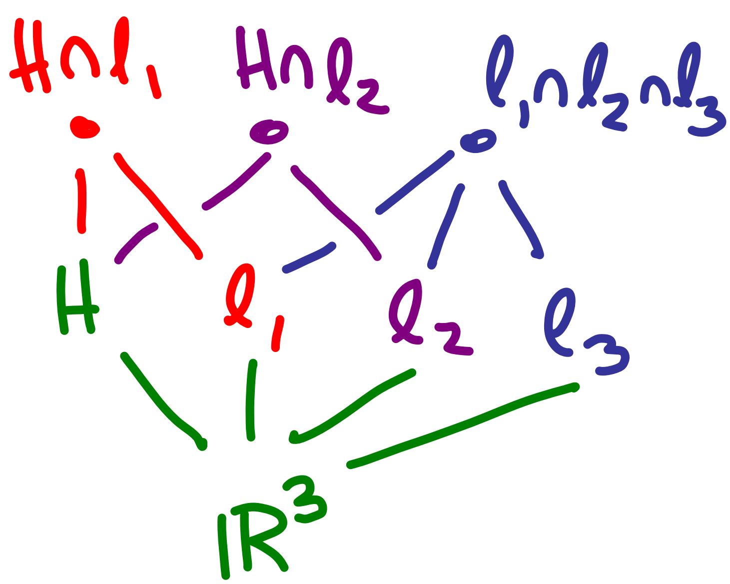


$L_A =$

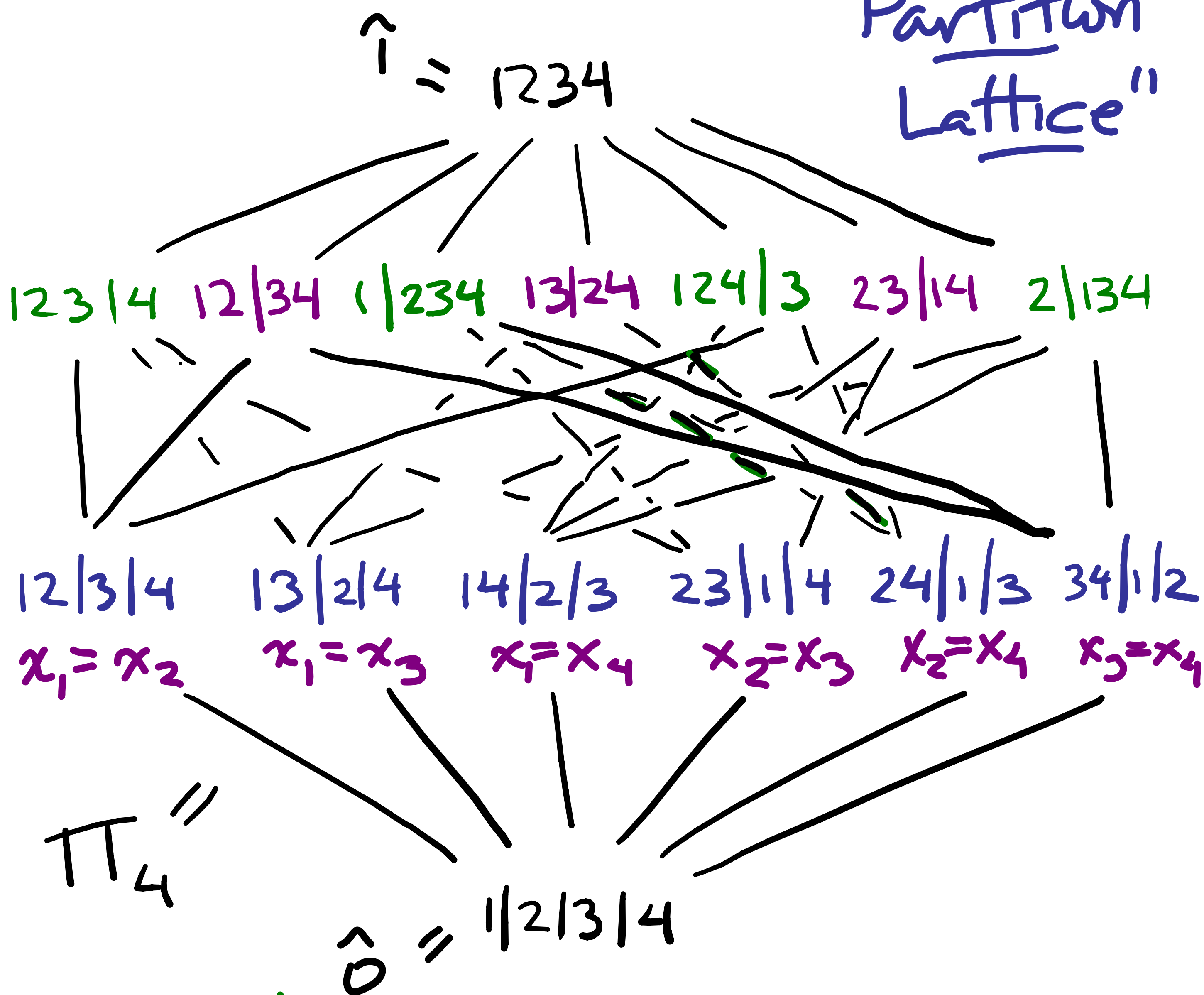
(intersection poset)



$\rightsquigarrow$



Intersection Poset L_A for $A = \{x_i = x_j \mid 1 \le i < j \le n\}$ the "Partition Lattice"



$\mu_{\Pi_4}(\hat{0}, \hat{1}) = -6$

Some Applications of Möbius Functions & "Shellability"

1. Shellability of intersection posets of hyperplane arrangements due to shellability of "geometric lattices"

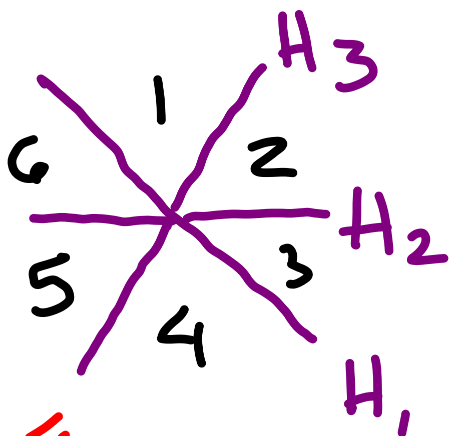
(Anders Björner) & "geometric semilattices"

(Michelle Wachs _ James Walker), yielding Möbius fns of "intersection posets" of hyperplane arrangements

{ useful e.g. for ...

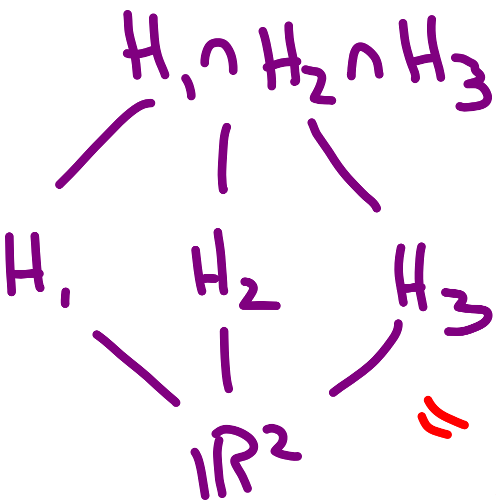
2. Tom Zaslavsky: region counting formulas for the complement of $\mathbb{R}$-hyperplane arr't $\mathcal{A}$

$\mathcal{A} =$



$$\#\text{regions} = \sum_{u \in L_{\mathcal{A}}} |\mu(\vec{0}, u)|$$

$$\#\text{bdd regions} = \left| \sum_{u \in L_{\mathcal{A}}} \mu(\vec{0}, u) \right|$$



$= L_{\mathcal{A}} =$ "intersection poset"

e.g. $\#\text{regions} = 1 + 3 + 2$

$\#\text{bdd regions} = 1 - 3 + 2$

$\mu(\mathbb{R}^2, \mathbb{R}^2) = 1$

$\mu(\mathbb{R}^2, H_i) = -1$ for $i = 1, 2, 3$

$\mu(\mathbb{R}^2, H_1 \cap H_2 \cap H_3) = 2$