Combinatorics Meets Topology: Möbius Functions, Euler Characteristic & Beyond

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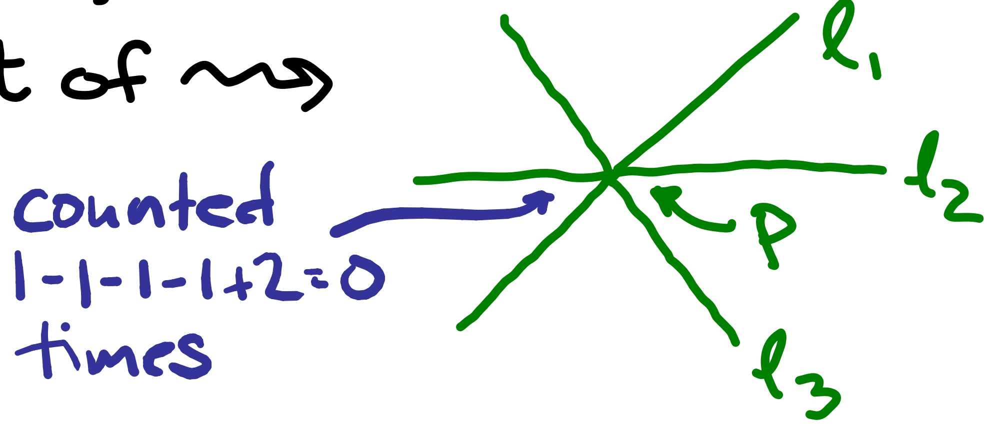
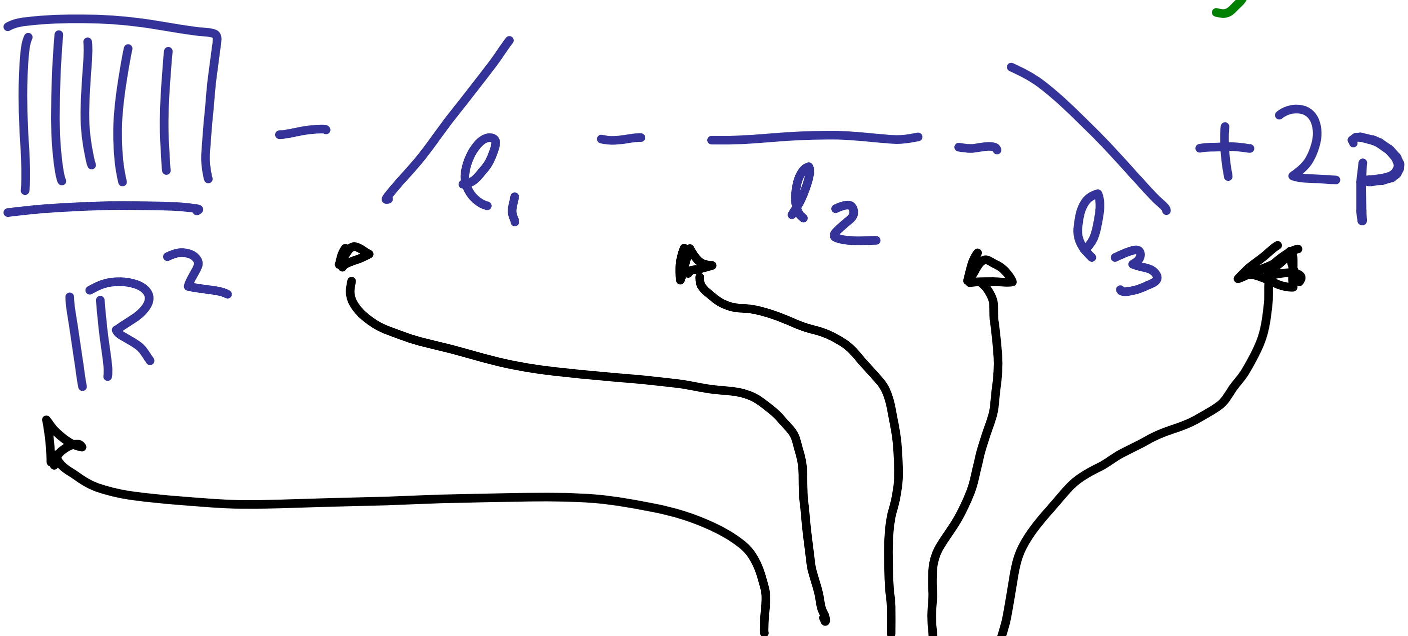
Plan: discuss how counting problems can be solved using topology.

Counting by Inclusion-Exclusion

e.g. "counting" points in the $\mathbb{R}^2$ complement of ⇝

yields:





- Coefficients 1,-1,-1,-1,2 in such inclusion-exclusion counting formula given by "Möbius function" μ (upcoming)

Now let's define a function "$\mathcal{M}$" to calculate these coefficients (& generalize this counting technique)

We need:

- $\mathcal{M}(\mathbb{R}^2, \mathbb{R}^2) = 1 =$ coef. of $\mathbb{R}^2$
- $\mathcal{M}(\mathbb{R}^2, \ell_1) = -1 =$ coef of ℓ_1

  (so $\mathcal{M}(\mathbb{R}^2, \mathbb{R}^2) + \mathcal{M}(\mathbb{R}^2, \ell_1) = 0$)

- $\mathcal{M}(\mathbb{R}^2, \ell_2) = -1$
- $\mathcal{M}(\mathbb{R}^2, \ell_3) = -1$
- $\mathcal{M}(\mathbb{R}^2, p) = 2$

  (so $\mathcal{M}(\mathbb{R}^2, \mathbb{R}^2) + \sum_{i=1}^{3} \mathcal{M}(\mathbb{R}^2, \ell_i) + \mathcal{M}(\mathbb{R}^2, p) = 0$)

Second Example:

Qn: How many students at Duke haven't studied any of the languages French, German or Spanish?

Ans: $|D| - |F| - |G| - |Sp|$

- $|D|$: size of Duke
- $|F|$: # French students
- $|G|$: # German students
- $|Sp|$: # Spanish students

$+ |F \cap G| + |F \cap Sp| + |G \cap Sp|$

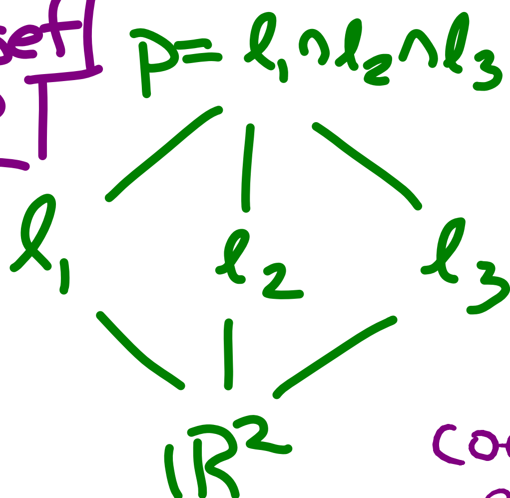
$- |F \cap G \cap Sp|$

Coefficients again calculated recursively.

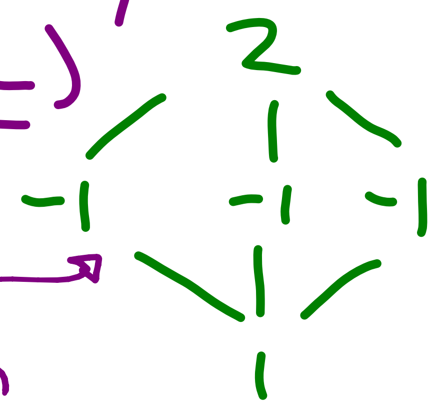
Def'n: Möbius function $\mu_P(x,y)$ of a "partially ordered set" P is defined recursively: $\mu_P(x,x) = 1$

and $$\mu_P(x,y) = -\sum_{x \le z < y} \mu_P(x,z)$$

Poset P



Its Möbius fn $\mu_P(\mathbb{R}^2, -)$



coefficients in $\mathbb{R}^2 - \ell_1 - \ell_2 - \ell_3 + 2p$

(we say $u \le v$ & draw upward path u to v for v subset of u)

Partially Ordered Sets (Posets) More Generally

Unlike the integers where any u, v satisfy $u \leq v$ or $v \leq u$ (or both if $u = v$), some sets only allow comparison of some of the pairs of elements

e.g. 1. subsets of $\{1, 2, \ldots, n\}$ with $S \leq T \iff S \subseteq T$

($\iff$ = if and only if)

2. positive integers w/ $d \leq n$ $\iff$ d is a divisor of n

(e.g. $2 \leq 6$ but $2 \not\leq 5$)

Poset of Subsets of $\{1,2,\ldots,n\}$

its Möbius Function

$$(1-1)^n = 0^n = 0 \text{ for } n \geq 1$$

$$\Vert$$

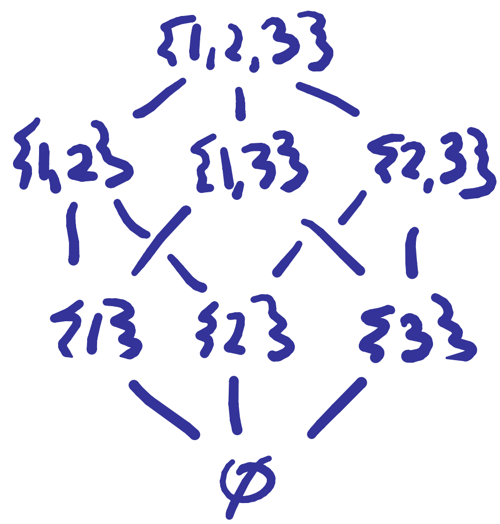
$$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}} \text{ for } x=1 \text{ \& } y=-1$$

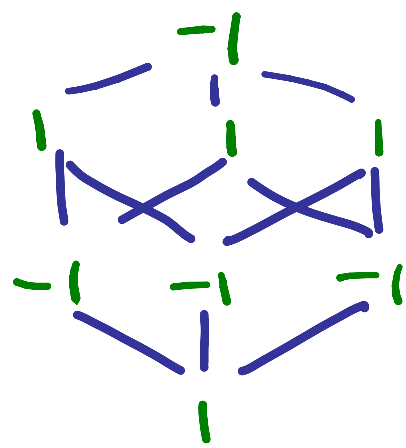
$$\Vert$$

$$\sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k} \quad \text{for } x=1 \text{ \& } y=-1$$

$$\Vert$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}$$





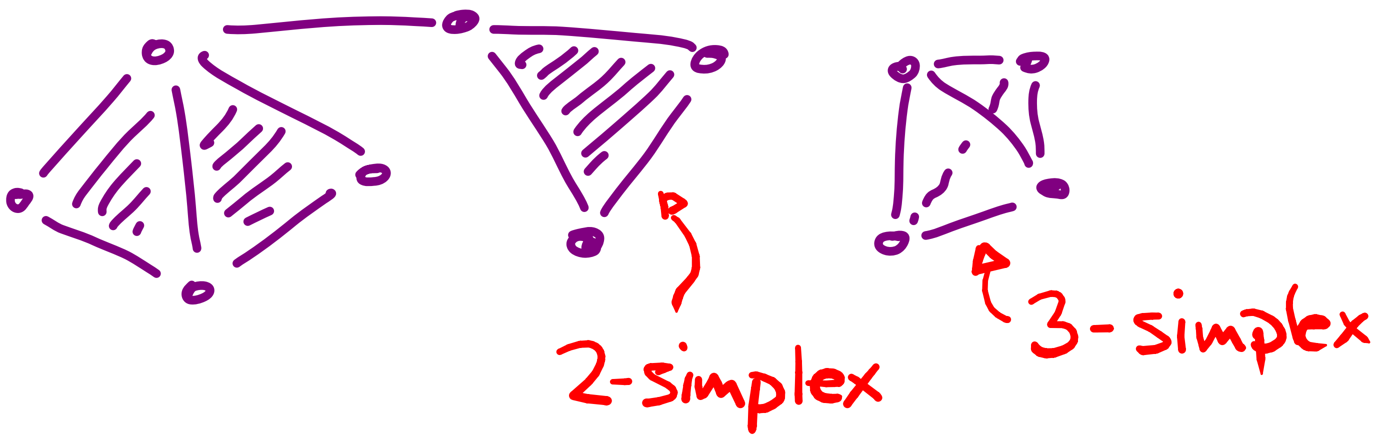
e.g. $\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 0$

$$\Vert$$

$$1 - 4 + 6 - 4 + 1$$

Proof by Induction yields: $\mu(\hat{0},u) = (-1)^{rk(u)}$

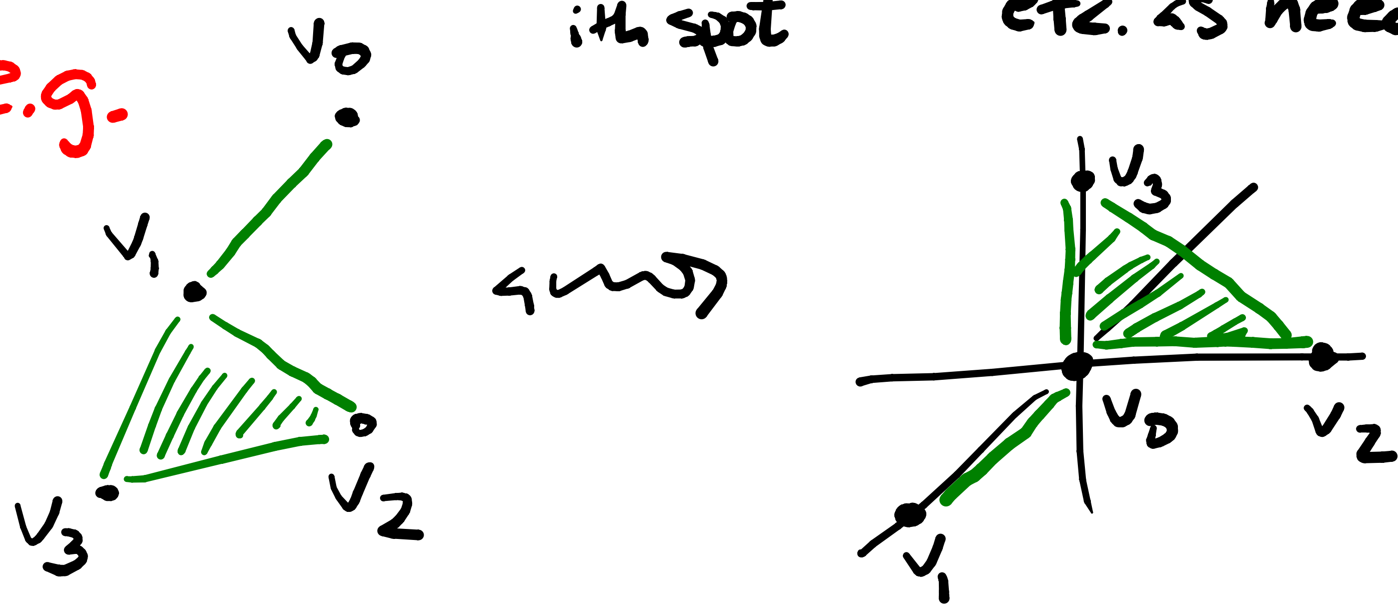
Simplicial Complexes



A simplicial complex (e.g a triangulation) is made of vertices (called "0-simplices"), edges ("1-simplices"), solid triangles ("2-simplices"), solid tetrahedra ("3-simplices"), etc.

Realization: any simplicial complex with vertices $v_0, v_1, \ldots, v_n$ can be drawn in $\mathbb{R}^n$ by letting $v_0 = (0, 0, \ldots, 0)$ and for $1 \le i \le n$ letting $v_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ & adding edges, etc. as needed
(1 in the ith spot)

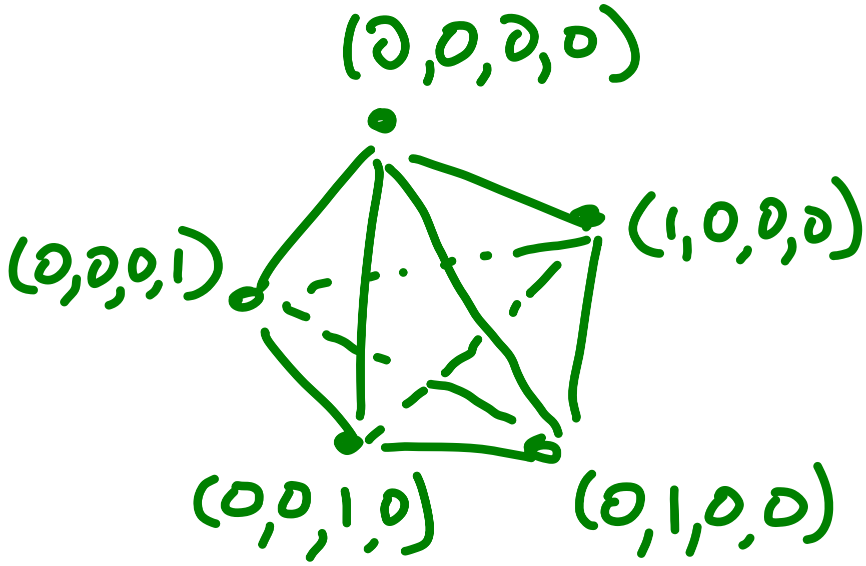
e.g.



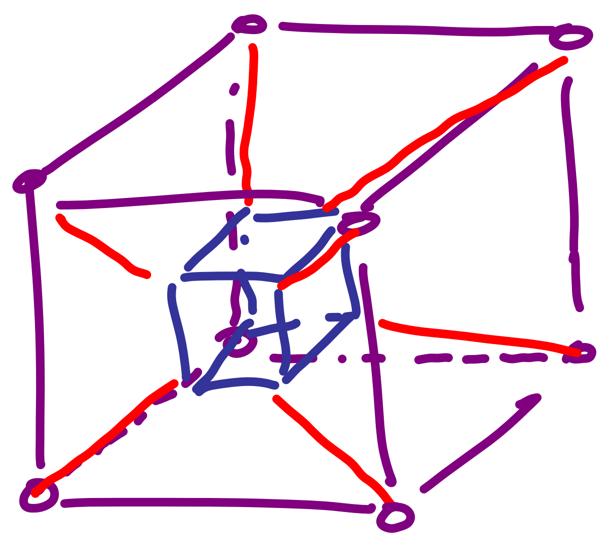
- We can also think of simplicial complexes abstractly, letting a face be a collection of the vertices in it, requiring for S a face & $T \subseteq S$ then T must also be a face.

Higher Dimensional Simplicial Complexes (& Cell Complexes)

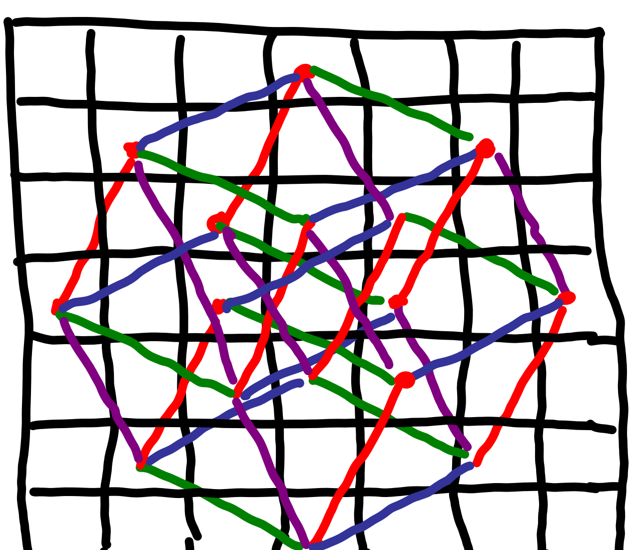
4-simplex
(needs 4-dim'l space to fit in)



4-dim'l cube:
(not a simplicial complex)



OR



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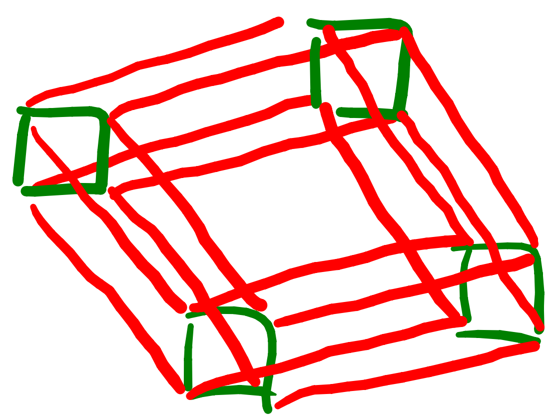
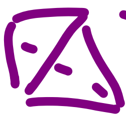


(Reduced) Euler Characteristic

- The **reduced Euler characteristic** of K, denoted $\tilde{\chi}(K) = -1 + \#\text{vertices} - \#\text{edges} + \#\text{triangles} - \ldots$

e.g. $\tilde{\chi}($  $) = -1+4-6+4=1$

$\tilde{\chi}($  $) = -1+5-9+6=1$

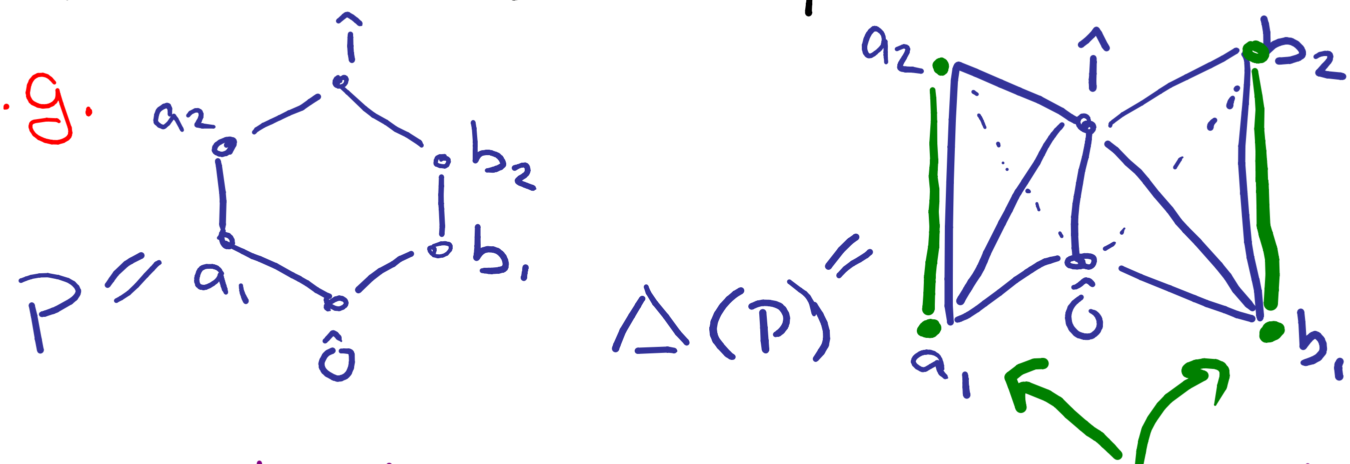
$\Big\}\ \tilde{\chi}(\text{2-sphere})$

Adding faces without changing "Topology" won't change $\tilde{\chi}$!

$$\tilde{\chi} = \underbrace{-1+3-3+1}_{=0} = \underbrace{-1+4-5+2}_{=0} = \underbrace{-1+5-8+4}_{=0}$$

Def'n: The order complex of a poset P is the abstract simplicial complex, **denoted** $\Delta(P)$, whose i-dimensional faces are the $(i+1)$-"chains" $v_0 < \dots < v_i$ in P

e.g.



Key Property (due to Hall; popularized by Rota):

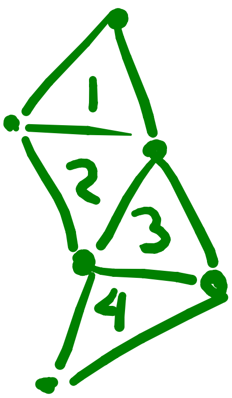
$$M_P(x,y) = \tilde{\chi}(\Delta_P(x,y)) = -1 + \#\text{vertices} - \#\text{edges} + \#\text{2-faces} \dots$$

$$= -1 + \beta_0 - \beta_1 + \beta_2 - \dots$$

- β_i = # i-dim'l hole boundaries
- $(u,v) = \{z \in P \mid u < z < v\}$
- $\Delta_P(u,v) = \Delta(\{z \in P \mid u < z < v\})$

Techniques Yielding Möbius Functions (& Poset Topology)

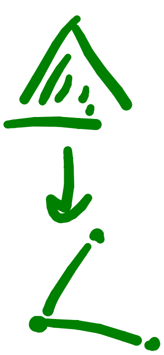
- (Lexicographic) shellability



- EL-labelings (Anders Björner)
- CL-labelings (Anders Björner & Michelle Wachs)

$\Rightarrow \Delta(P) \simeq$ 

(telling us $\tilde{\chi}$ hence μ)



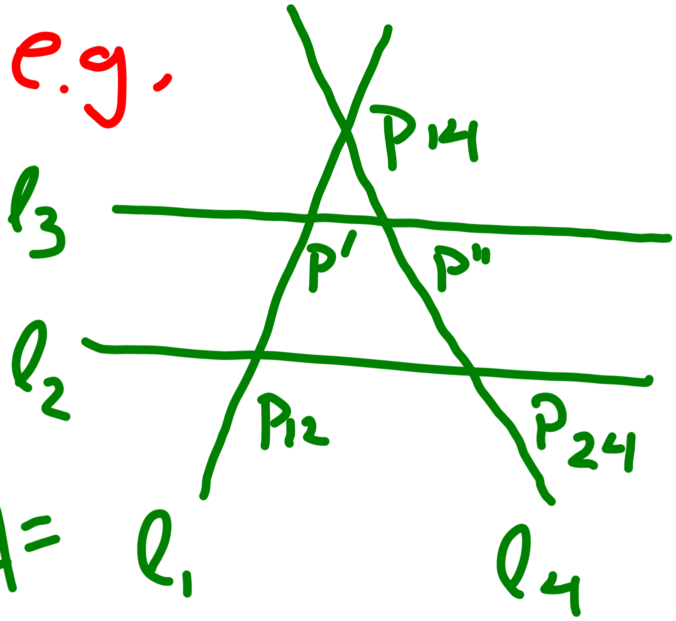
- Lexicographic discrete Morse functions (Babson-H.)

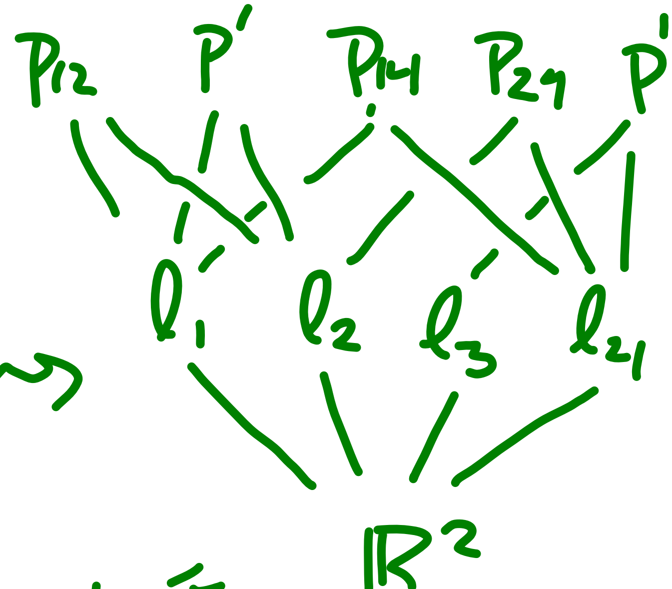
(for other topol. types)

Intersection Posets

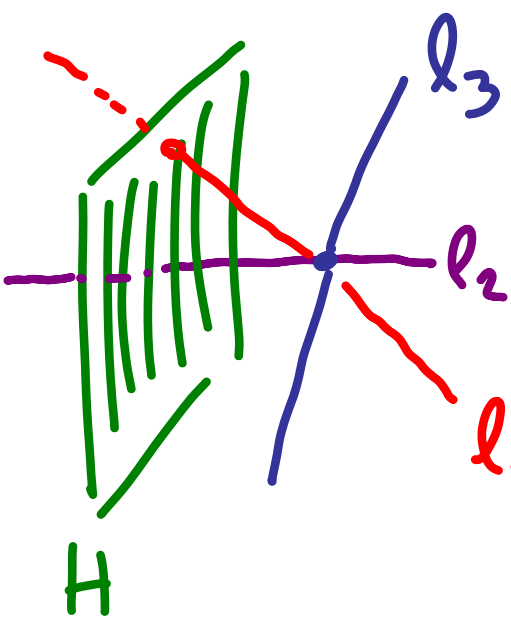
e.g.

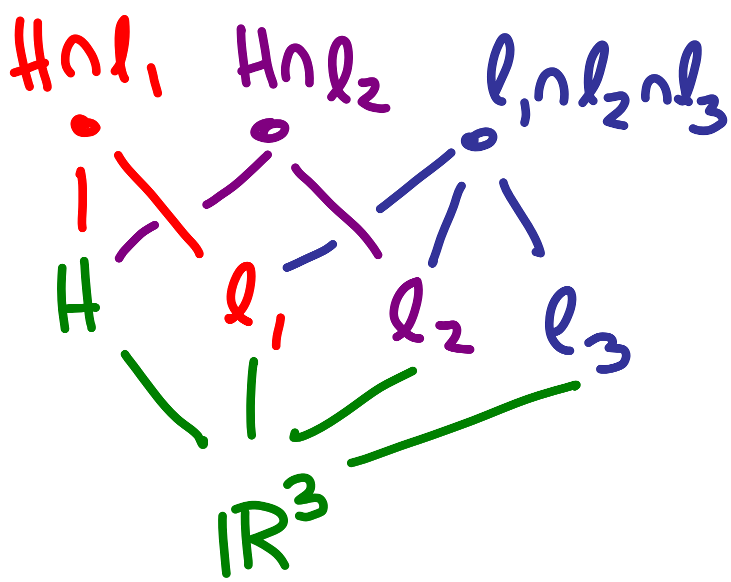
$A =$



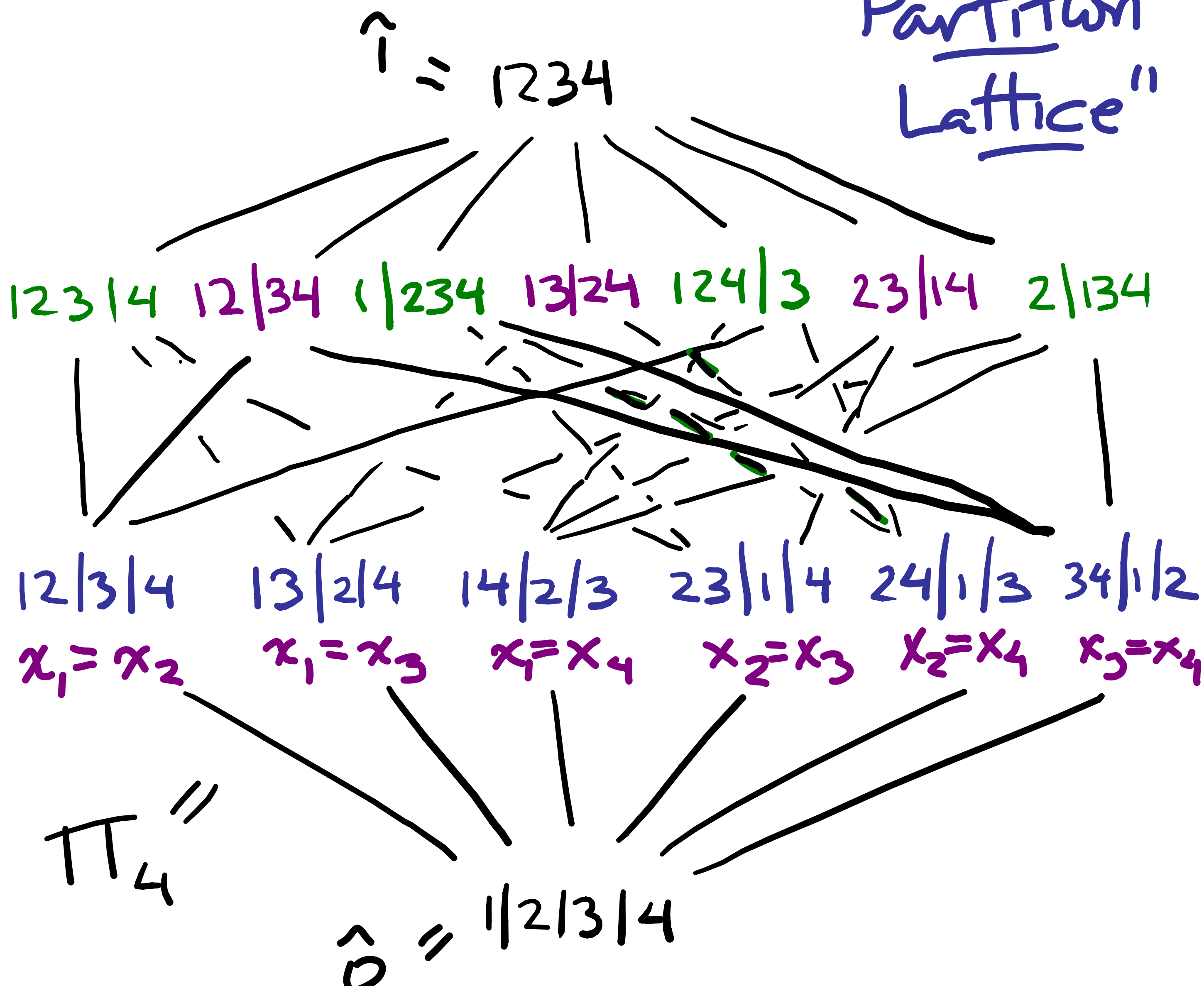


$= L_A$ (intersection poset)





Intersection Poset L_A for $A = \{x_i = x_j \mid 1 \le i < j \le n\}$ the "Partition Lattice"



$\mu_{\Pi_4}(\hat{0}, \hat{1}) = -6$

Some Applications of Möbius Functions & "Shellability"

1. Shellability of intersection posets of hyperplane arrangements due to shellability of "geometric lattices"

(Anders Björner) & "geometric semilattices"

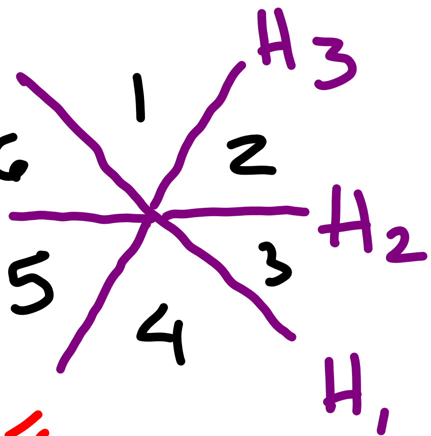
(Michelle Wachs James Walker), yielding Möbius fns of "intersection posets" of hyperplane arrangements

} useful e.g. for ...

↓

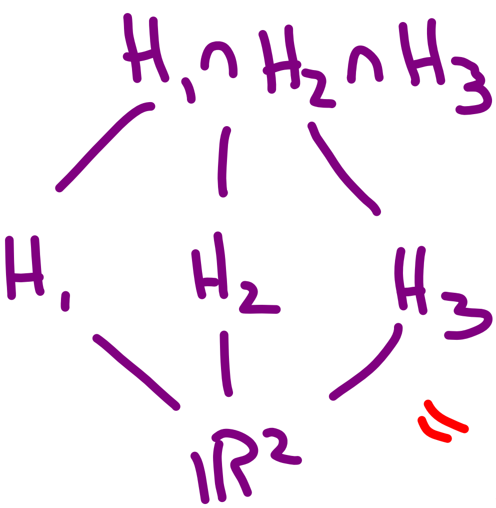
2. Tom Zaslavsky: region counting formulas for the complement of $\mathbb{R}$-hyperplane arr't A

$A =$



$$\#\text{regions} = \sum_{u \in L_A} |\mu(\vec{0}, u)|$$

$$\#\text{bdd regions} = \left| \sum_{u \in L_A} \mu(\vec{0}, u) \right|$$



$= L_A =$ "intersection poset"

e.g. $\#\text{regions} = 1 + 3 + 2$

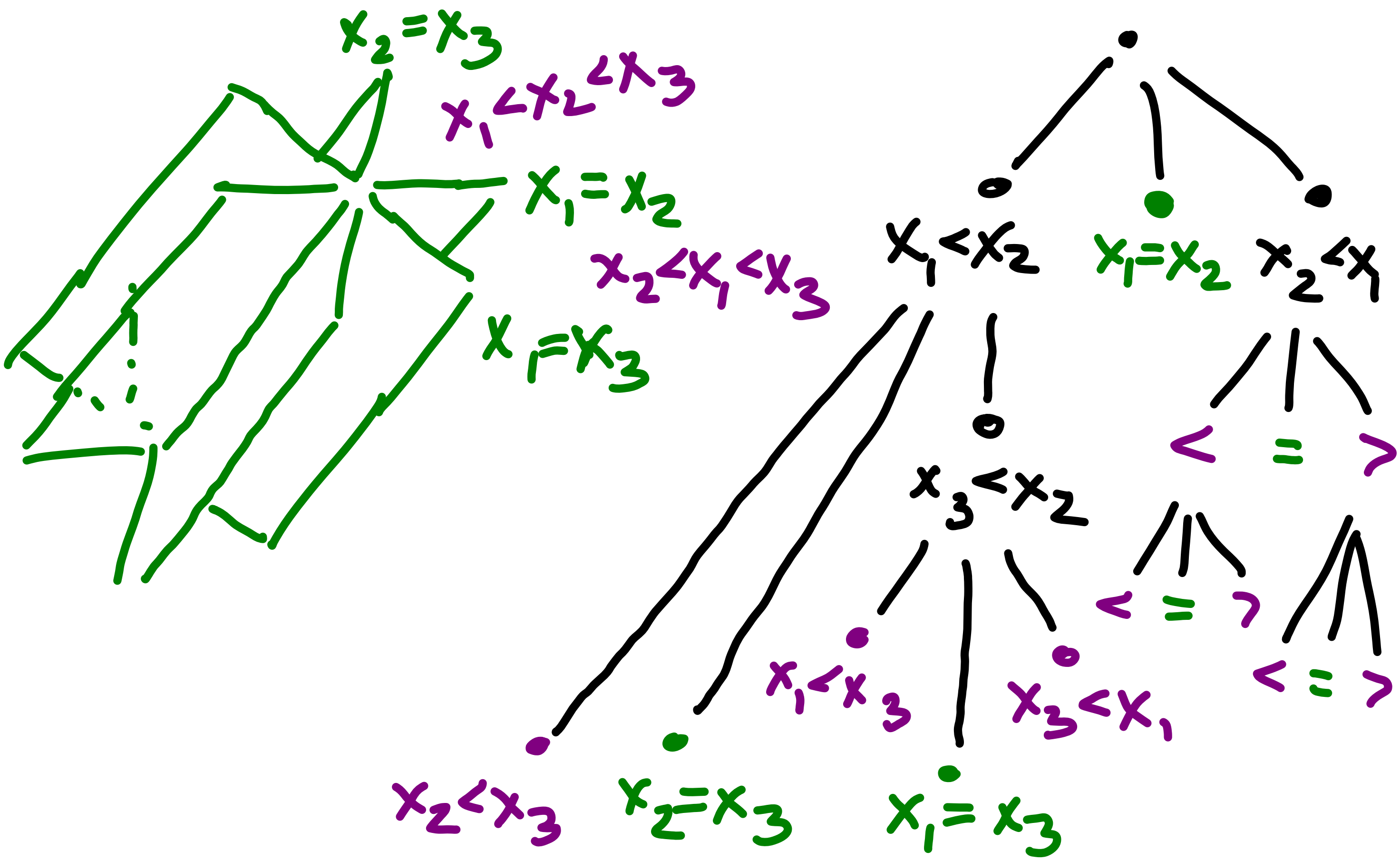
$\#\text{bdd regions} = 1 - 3 + 2$

$\mu(\mathbb{R}^2, \mathbb{R}^2) = 1$

$\mu(\mathbb{R}^2, H_i) = -1$ for $i = 1, 2, 3$

$\mu(\mathbb{R}^2, H_1 \cap H_2 \cap H_3) = 2$

3. Björner-Lovász-Yao: lower bound via Möbius fns for deciding if there are k equal coordinates in $\vec{a}=(a_1,a_2,\dots,a_n)\in\mathbb{R}^n$ by pairwise coord. comparisons, i.e. deciding whether $\vec{a}$ lies on "k-equal arr't" of subspaces $x_{i_1}=\dots=x_{i_k}$



- lower bd on # leaves (and hence on $\log_3$(depth)) was given in terms of betti #'s (i.e. # holes in each dimension) in topological space $\mathbb{R}^n -$ k-equal subspace arrangement

  subspaces like $x_1 = x_2 = x_4$ for $k = 3$

- Mark Goresky & Robert MacPherson showed how to compute these betti #'s from poset order complexes

- Björner & Wachs found shellings for these poset order complexes, namely intersection posets for "k-equal arrangement"

Appendix: Some Additional Slides Giving Further Details & Touching Upon Some Further Topics . . .

Technique: Shellability

- simplicial complex is **pure** of dim. d if all maximal faces ("facets") are d-dimensional
- simplicial complex is **shellable** if there is total order $F_1, F_2, \ldots, F_k$, a **shelling**, on facets satisfying conditions guaranteeing we can build up the complex by attaching facets in this order so each step either leaves topology (homology) unchanged or closes off a sphere S^i, increasing β_i by 1.

OR

Technique 1*: Lexicographic Shellability

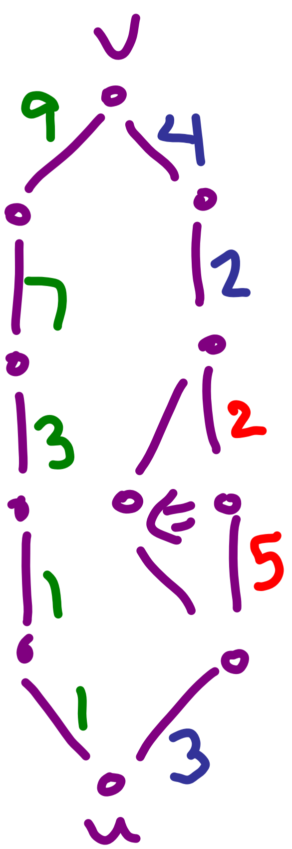
(Anders Björner & Michelle Wachs)

A poset P is **EL-shellable** if it admits labeling λ of its cover relations $x \lessdot y$ w/ integers (called an **EL-labeling**) s.t. $u < v$ implies:

(1) there is unique saturated chain $u \lessdot u_1 \lessdot \cdots \lessdot u_k \lessdot v$ s.t. $\lambda(u, u_1) \le \lambda(u_1, u_2) \le \cdots \le \lambda(u_k, v)$

and

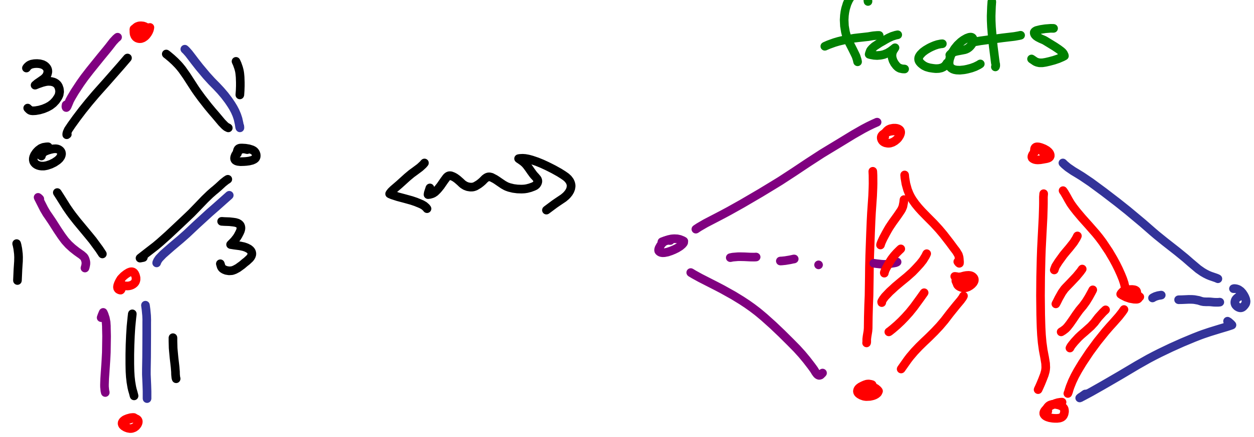
(2) $(\lambda(u,u_1), \lambda(u_1,u_2), \ldots, \lambda(u_k,v))$ is lexicographically smaller than the label sequences on all other saturated chains from u to v.



Thm (Björner): EL-labeling ⇒ Shelling

Idea: Lexicographic order on maximal chains (breaking ties arbitrarily) induces shelling order on corresponding facets of $\Delta(P)$.

- "descents in labeling" ⇝ codim. one overlap of facets

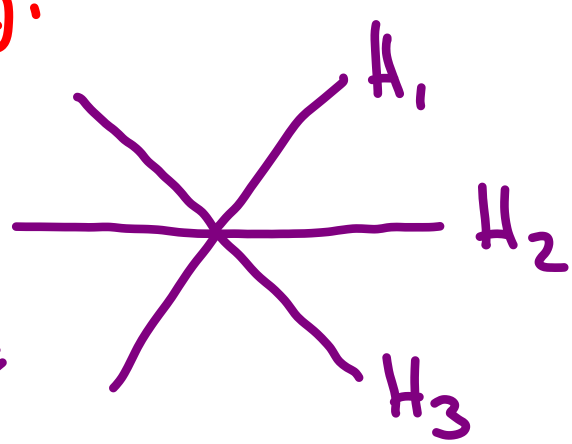


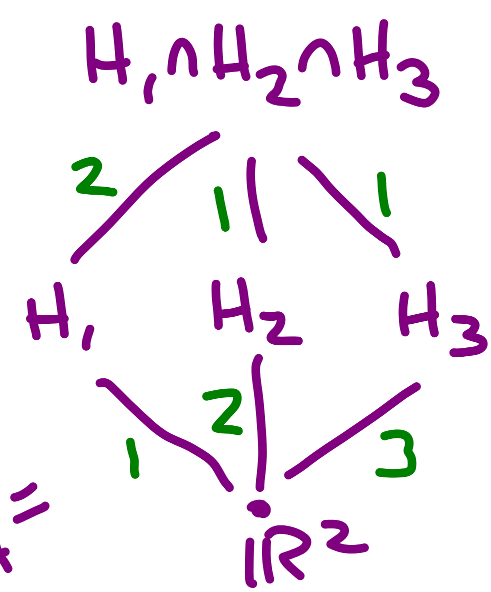
- "descending" chains ⇝ facets attaching along entire bdry ⇝ spheres
- $\mu_P(u,v) = \pm \#$ descending chains u to v (for P graded)

Example: Intersection Posets of Hyperplane Arrangements

- Choose any total order $H_1, H_2, \ldots, H_k$ on hyperplanes (resp. "atoms")
- Label $u \lessdot v$ with
  $$\min\{ i \mid H_i \not\leq u \text{ and } H_i \leq v \}$$

e.g.

$A =$ 

$L_A =$ 

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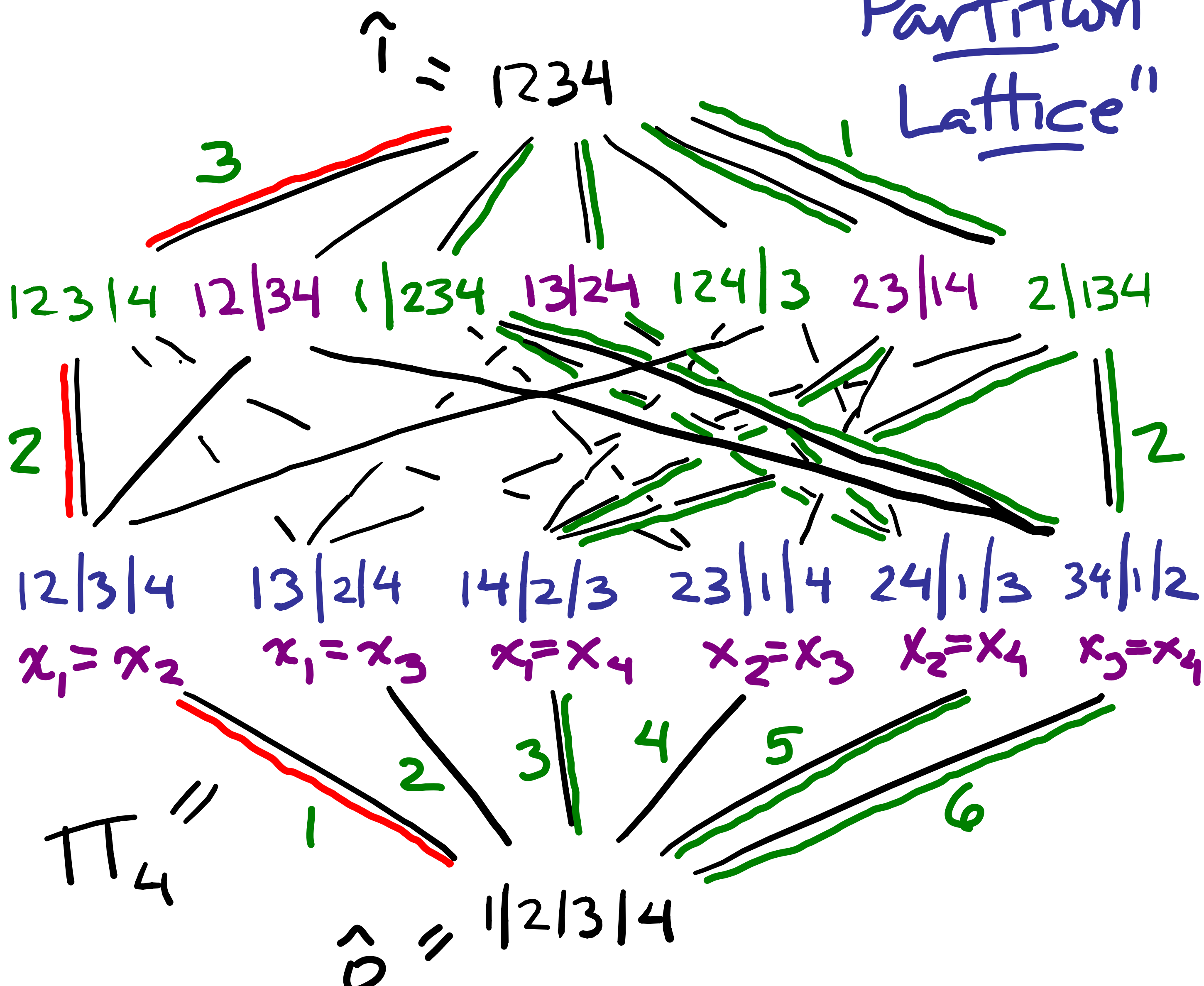
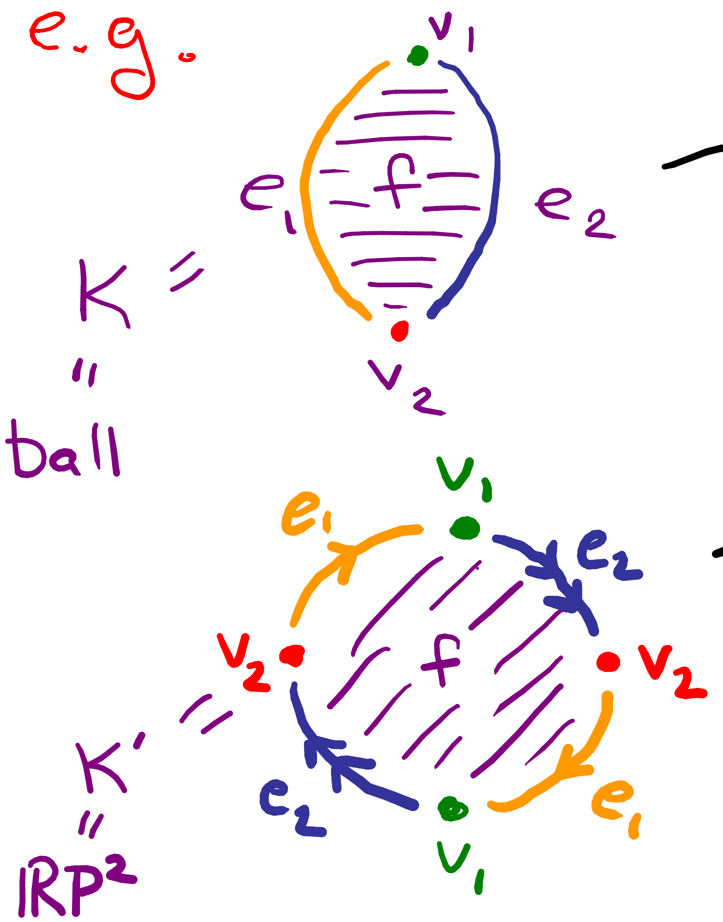
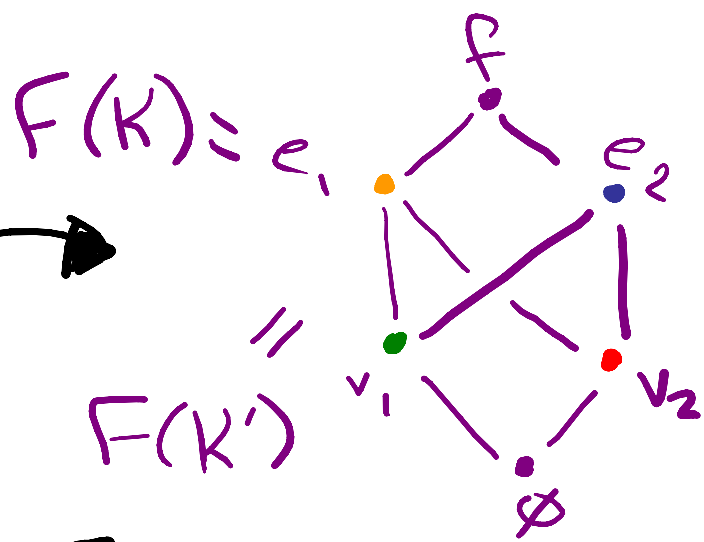


$\mu_{\Pi_4}(\hat{0},\hat{1}) = -6$

Cell Complexes & their Face Posets

e.g.





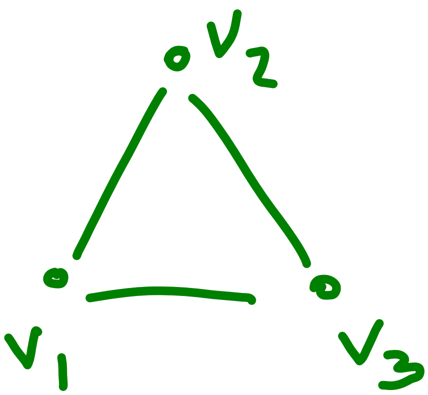
"closure poset"
or "face poset"

$(u \leq v \iff u \leq \bar{v})$

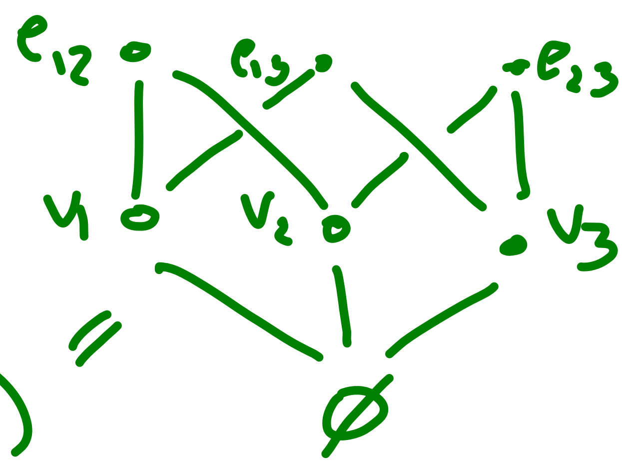
- $\mathbb{R}P^2$ has different 1st homology group than ball

A Goal of Mine: Use combinatorics of $F(K)$ + limited topological info to understand K

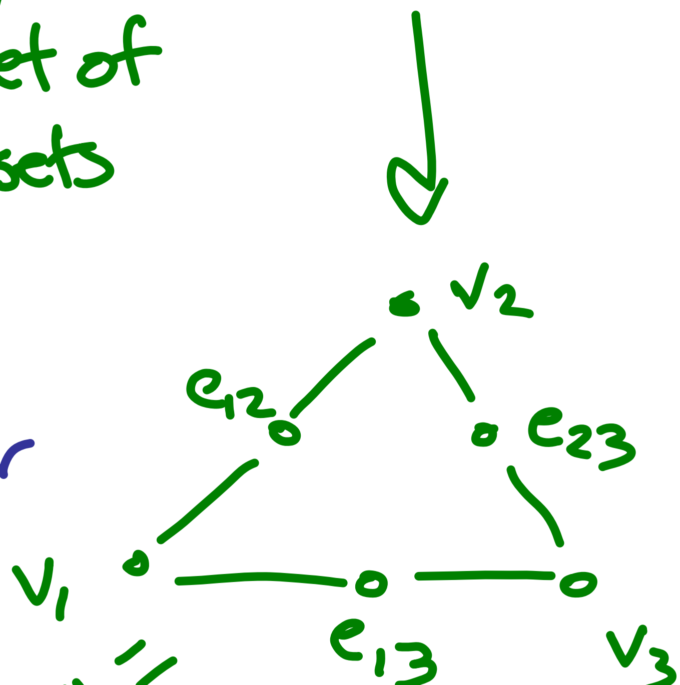
"Topological Proof" of Möbius Function for Poset of Subsets



$K =$



$= F(K) =$ poset of subsets



$-1+\beta_0-\beta_1+\beta_2-\dots$

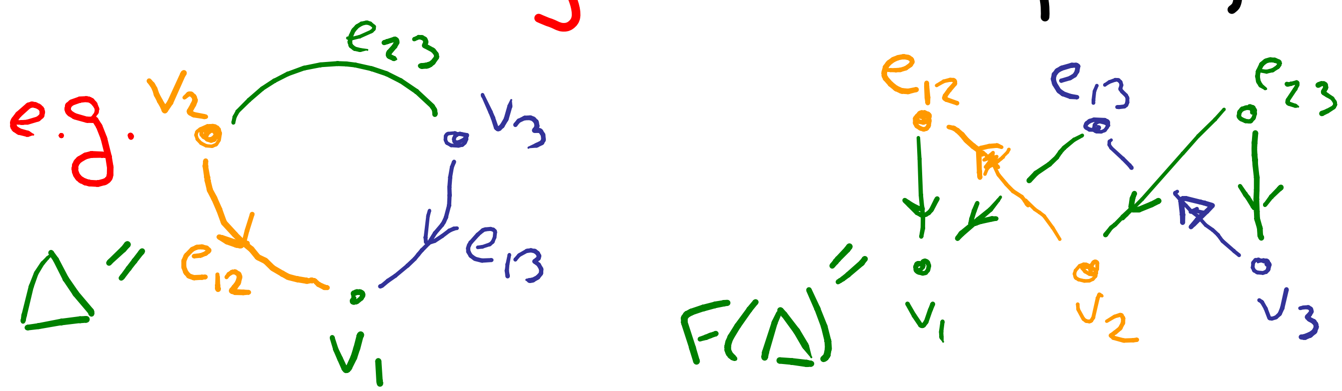
with $\beta_0=1=\beta_1$ (& for B_n then $\beta_0=\beta_{n-2}=1$)

$K \simeq sd(K) = \Delta(K) =$

Discrete Morse Theory

(due to Forman & reformulated by Chari)

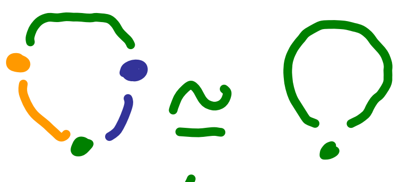
Given any regular CW complex Δ, construct an acyclic matching a.k.a. Morse matching on its face poset, i.e.,



an edge orientation s.t. "up edges" give a matching and directed graph has no cycles.

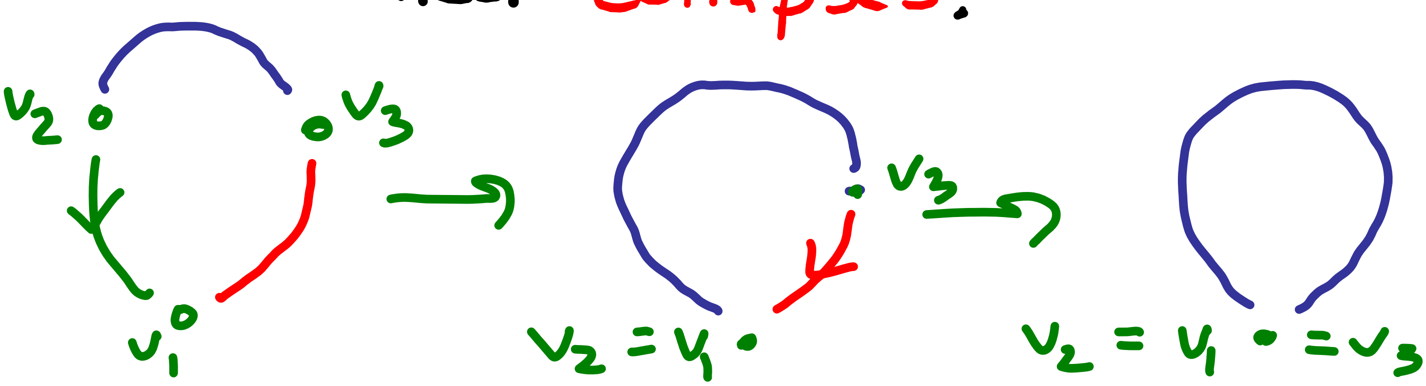
(A matching is a collection of graph edges s.t. no vertex is in more than one edge)

Theorem (Forman): $\Delta \simeq \Delta^M$ a CW complex comprised of the unmatched cells, called critical cells.

e.g. 

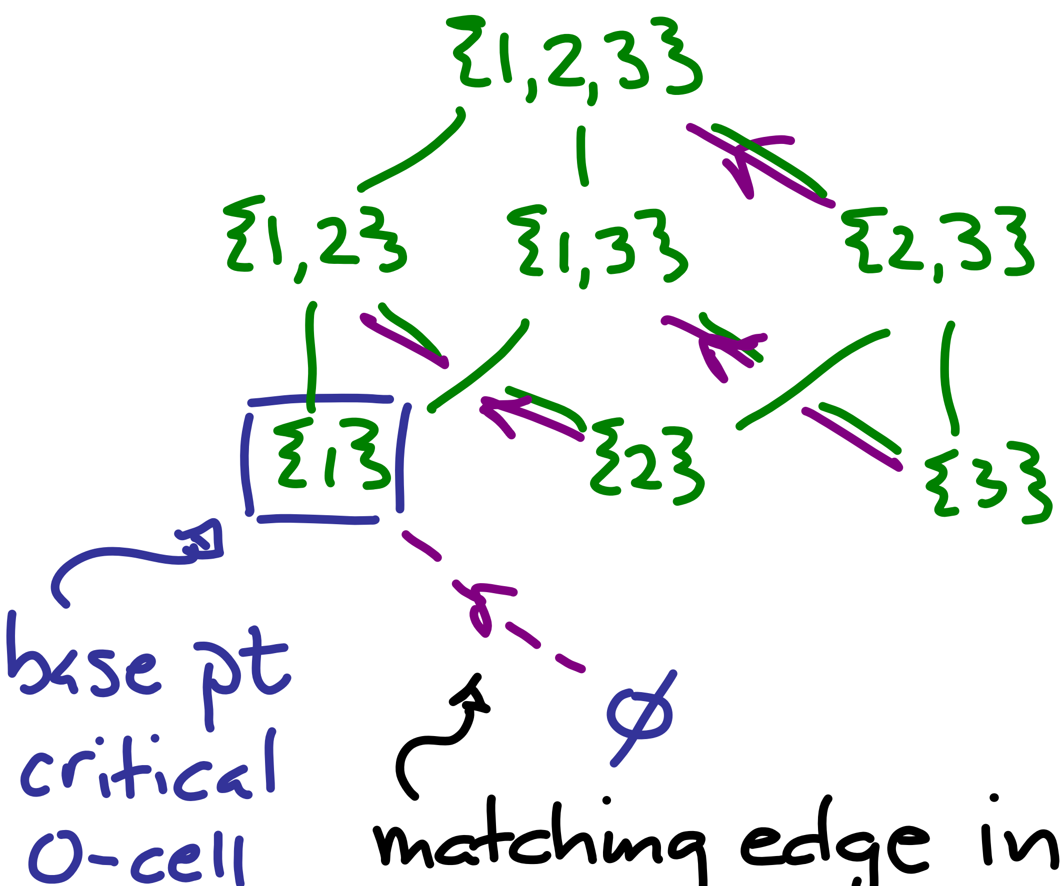
↑ same topological structure (same homology groups + more!)

Idea: Find pairs of faces where one can be "pulled across" other eliminating both without changing topology, via moves called "collapses".



First Examples

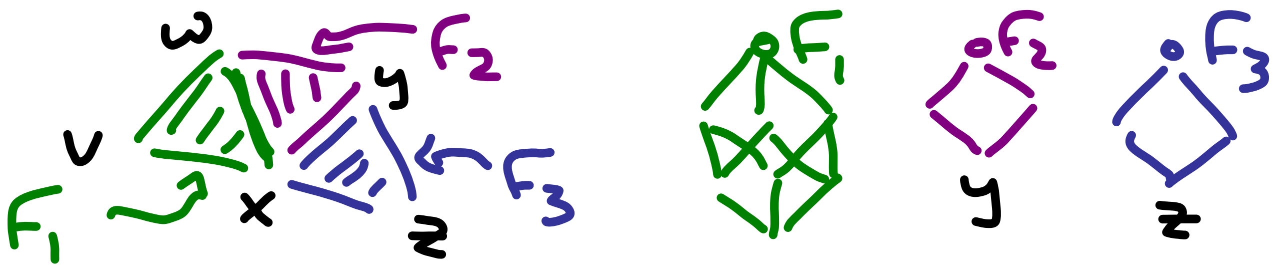
1. Boolean algebra of subsets of $\{1,2,\ldots,n\}$, face poset of simplex, matching $S \setminus \{1\}$ with $S \cup \{1\}$ $\forall S$



matching edge in "reduced homology" version of discrete Morse theory

2. Any union of acyclic matchings on $F(\Delta_2 \setminus \Delta_1), F(\Delta_3 \setminus \Delta_2), \ldots$ for $\Delta_1 \subseteq \Delta_2 \subseteq \ldots \subseteq \Delta_k = \Delta$ a filtration of subcomplexes is an acyclic matching for Δ

e.g. $\overline{F_1} \subseteq \overline{F_1 \cup F_2} \subseteq \overline{F_1 \cup F_2 \cup F_3}$



3. Shelling ⇒ Discrete Morse fn whose critical cells are the maximal faces attaching along their entire boundary

Explanation for $\Delta \simeq \Delta^M$: Matching edges specify (internal) elementary collapses preserving homotopy type

Some Consequences of $\Delta \simeq \Delta^M$:

1. If $F(\Delta)$ has complete acyclic matching (w/ $\emptyset \in F(\Delta)$) then Δ is collapsible.

Recall: Some contractible complexes are not collapsible.

e.g. dunce cap 

2. $\tilde{\chi}(\Delta) = \tilde{\chi}(\Delta^M)$

$= -1 + \#\,0\text{-cells} - \#\,1\text{-cells} + \#\,2\text{-cells} - \cdots$

$= -1 + \beta_0 - \beta_1 + \beta_2 - \cdots$

For Posets: $\mu_P(x,y) = \tilde{\chi}(\Delta(x,y)) = \tilde{\chi}(\Delta^M(x,y))$

3. Morse Inequalities:

1. $\tilde{\beta}_i(\Delta) \leq \tilde{m}_i(\Delta) = \#$ i-dim'l critical cells

2. $\sum_{i \leq j} (-1)^{j-i} \tilde{\beta}_i(\Delta) \leq \sum_{i \leq j} (-1)^{j-i} \tilde{m}_i(\Delta)$ (for each $j \leq \dim(\Delta)$)

Rk: "Greedy" matchings tend to satisfy acyclicity requirement.

Question (H.): Is there a good way to "complete the square":

lexicographic shelling $\Longrightarrow$??

$\Downarrow$ $\Downarrow$

shelling $\Longrightarrow$ discrete Morse function

to understand posets that fail to be shellable (e.g. not wedge of spheres)?

Proposed Answer (Eric Babson & P.H.)

"lexicographic discrete Morse fn's"

Research on "f-vectors"

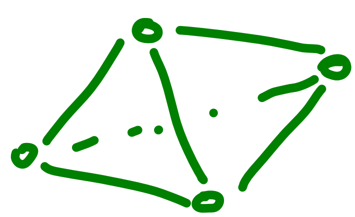
If $f_i(\Delta)$ = # i-dimensional faces in Δ

then which vectors arise as

$(f_0(\Delta), f_1(\Delta), f_2(\Delta), \ldots, f_{\dim(\Delta)}(\Delta))$

for some Δ?

e.g. $f_0(\Delta) = 4 \Rightarrow f_1(\Delta) \le \binom{4}{2} = 6$



- "Kruskal-Katona Thm" for simplicial complexes
- Richard Stanley used "commutative algebra" for spheres
- Isabella Novik for "homology spheres"

...

Topological "Pathologies"

e.g. Alexander horned ball: