Topological Combinatorics of Bruhat Order & Total Positivity

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Topological Aspects of Total Positivity

- Lusztig initiated study of Totally nonnegative, real part of (matrix) Schubert varieties
  (i.e. part with minors all nonnegative in spaces of matrices or of flags)
- Topology (homeomorphism type) is conjecturally/provably trivial.
- Proving this puts restrictions on possible relations among Chevalley generators.
- Reveals structure in canonical bases; a motivation for cluster algebras
- Main Result of Talk: Proof of Fomin-Shapiro Conjecture via new tools exploiting interplay of combinatorial data & topological data

The Totally Nonnegative Part of a Space of Matrices

- $$x_i(t) = \underbrace{I_n + tE_{i,i+1}}_{\text{(type A)}} = \begin{array}{cc} & \text{column } i+1 \\ & \\ & \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & t & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow \text{row } i \\ \\ \\ \end{array}$$

  $$x_i(t) = \underbrace{\exp(te_i)}_{\text{(general type)}}$$

- $$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$$

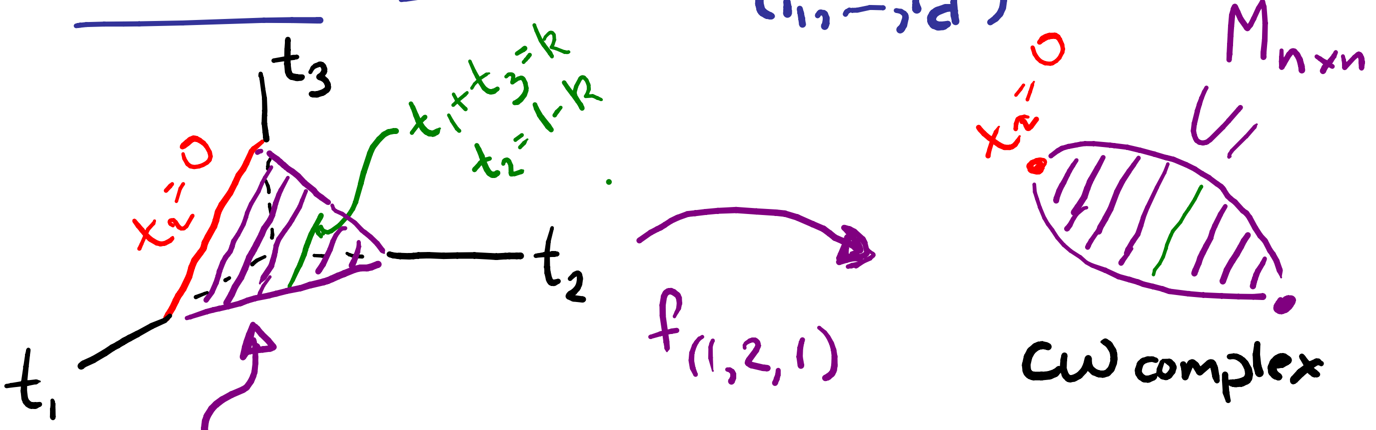
  $$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

e.g. $$f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1,\dots,i_d)}$



$\mathbb{R}^3_{\geq 0} \cap (\sum t_i = 1$ hyperplane$)$

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow t_2 = 0$$

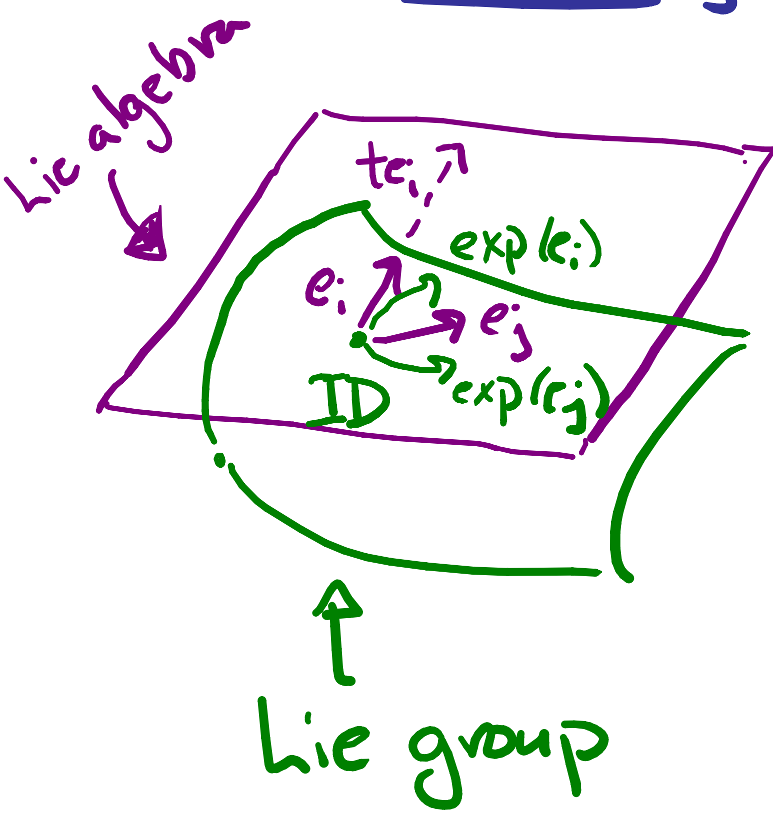
$$x_1(t_1)\circ x_1(t_3)$$

$$f_{(1,2,1)}(t_1,0,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

Non-injectivity: results from "nil-moves" $x_i(u)x_i(v) = x_i(u+v)$ & "long braid moves"

A Motivation: Understanding Relations Among (Exponentiated) Chevalley Generators



Chevalley generators

$$t_1 e_i \circ t_2 e_j$$

$$\Big\downarrow \text{exp map}$$

$$\exp(t_1 e_i \circ t_2 e_j)$$
$$\|$$
$$\exp(t_1 e_i)\exp(t_2 e_j)$$
$$\|$$
$$x_i(t_1)\, x_j(t_2)$$

$$\exp(te_i) = \boxed{ID + te_i} + \underbrace{\frac{t^2 e_i^2}{2}}_{0=} + \underbrace{\frac{t^3 e_i^3}{6}}_{0=} + \cdots$$

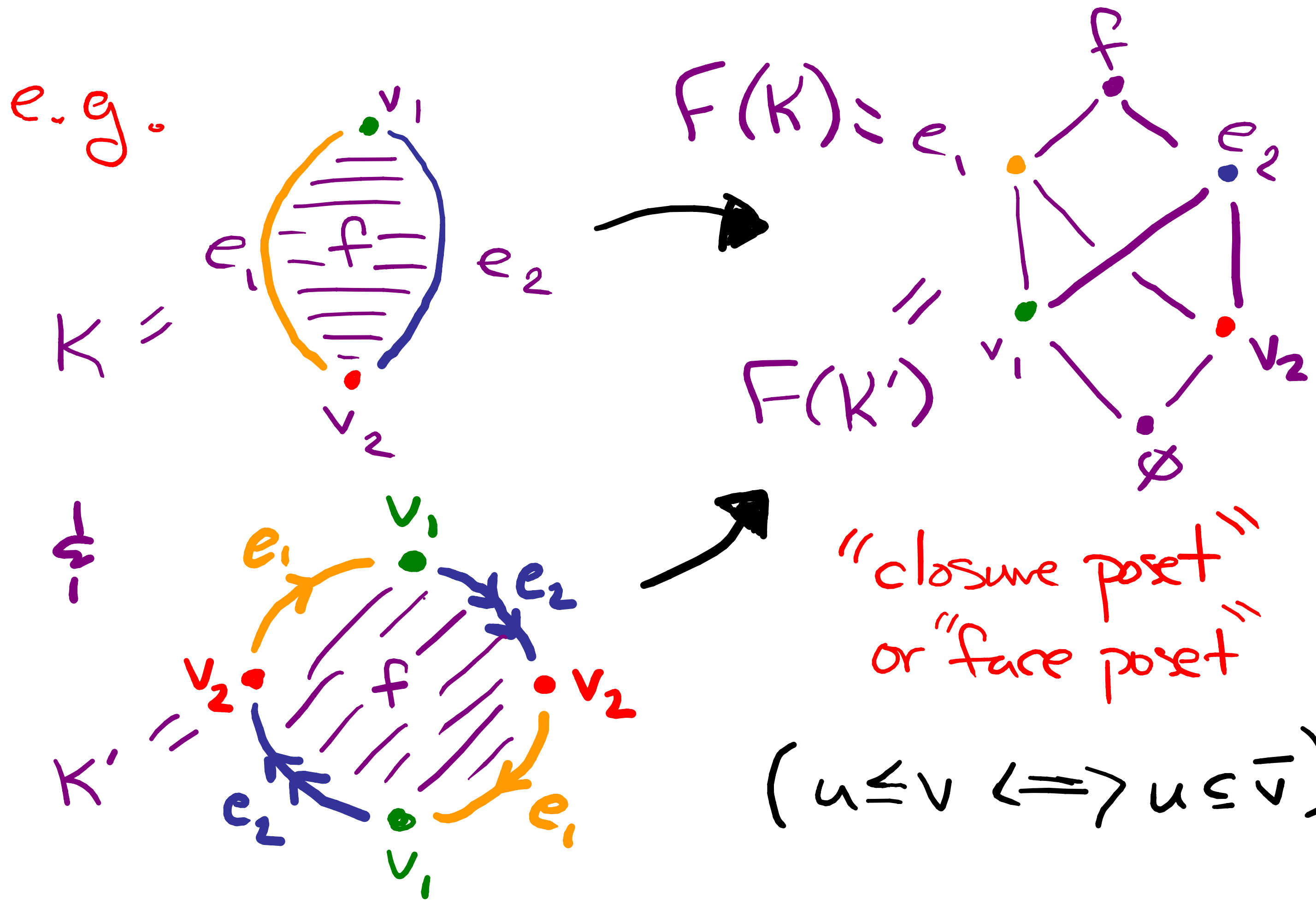
We Prove: Only the "obvious" relations occur

Connection to Schubert Varieties & Bruhat Decompositions

- Y_w° = image of $f_{(i_1,\ldots,i_d)}: \mathbb{R}^d_{>0} \to M_{n\times n}$
- $Y_w = \overline{Y_w^\circ}$ = image from $\mathbb{R}^d_{\geq 0}$
  = totally nonnegative part of $\overline{B^- w B^-} \cap \begin{pmatrix}\text{unipotent} \\ \text{radical of } B\end{pmatrix}$
- Y_{w_0} = totally nonnegative part of space of upper triangular matrices w/ 1's on diagonal
  (old result of Whitney-type A)

Fomin-Shapiro Conjecture: Link of identity within Y_w is regular CW complex homeomorphic to a ball, stratified by which minors are positive & which are zero. The Bruhat intervals serve as closure poset.

Background: CW Complexes and their Closure Posets



"closure poset" or "face poset"

$(u \le v \iff u \le \bar{v})$

CW complexes: comprised of pieces called **cells** each homeomorphic to an open ball

- higher dimensional cells glued to unions of lower dimensional ones by **attaching maps**.

Regular CW Complexes

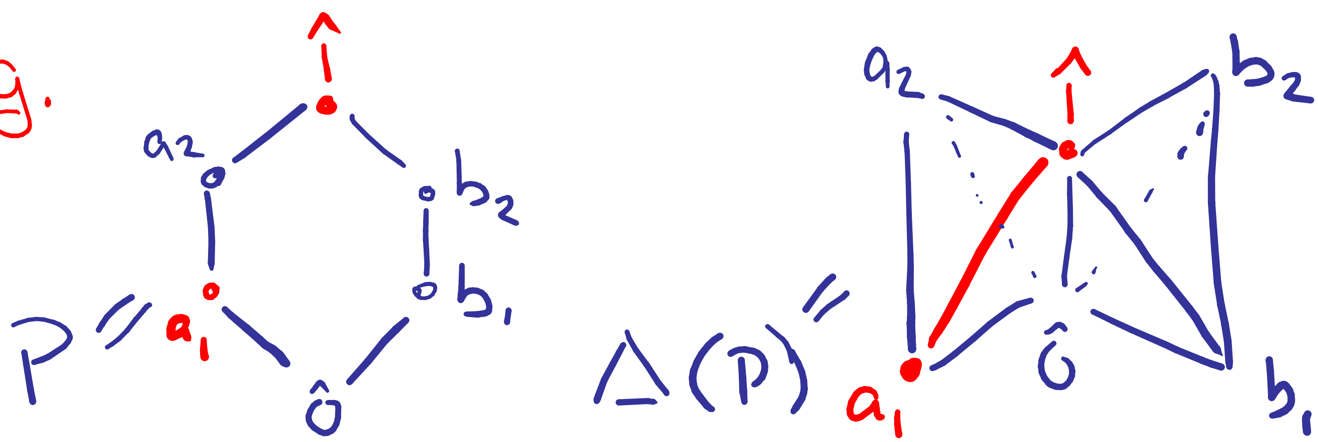
- A CW complex is regular if the attaching map for each cell is a homeomorphism (hence injective).
  e.g. all simplicial complexes & polytopes

- Why Useful level of Generality:
  - K regular $\Longrightarrow$ $K \cong \Delta(F(K) \setminus \hat{0}) = sd(K)$
    so context where combinatorics of closure poset determines topol. structure
  - seemingly encompasses spaces of interest from combinatorial rep'n theory

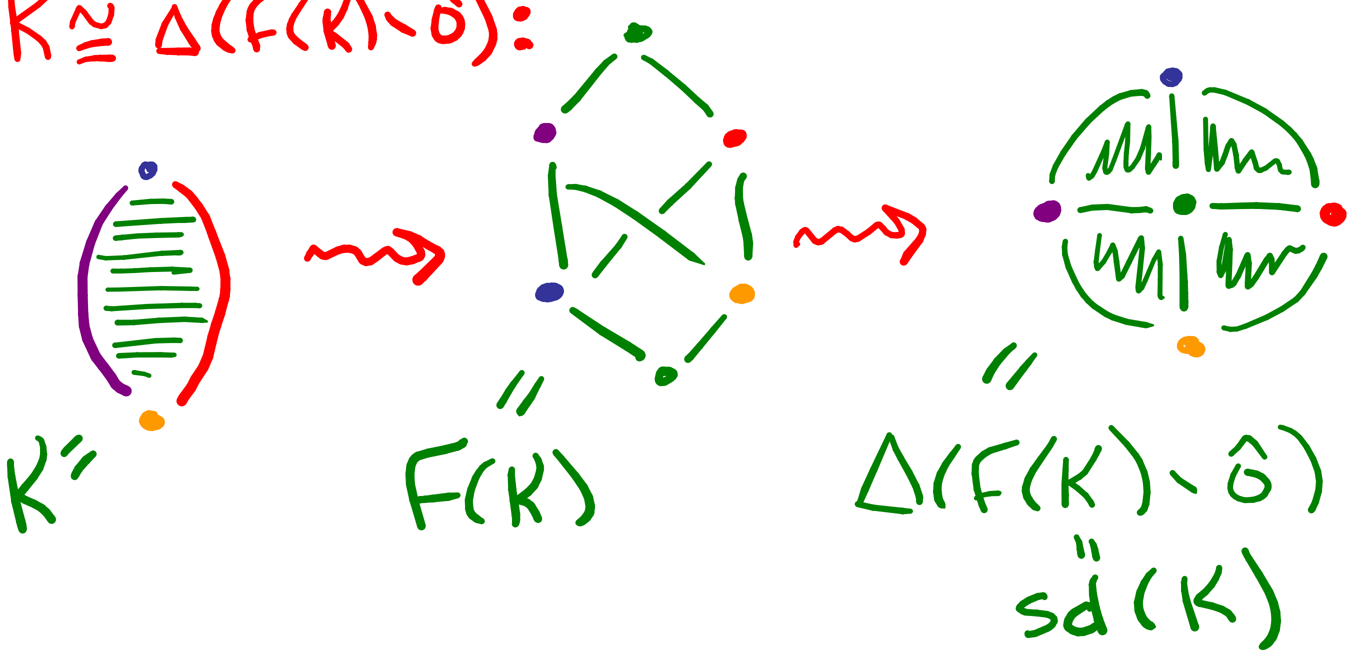
- Challenges: Methods for simplicial complexes do not apply; homeomorphism type is not purely combinatorial.

Recall: The order complex of a poset P is the simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-chains $v_0 < \cdots < v_i$ in P.

e.g.



$K \cong \Delta(F(K) \setminus \hat{0})$:



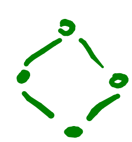
Defn (Björner): A finite, graded poset P is **CW poset** if

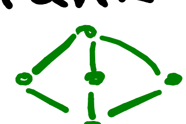
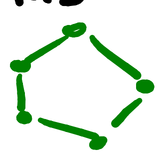
- P has unique min'l elt. $\hat{0}$
- P has additional element(s)
- $x \neq \hat{0} \Rightarrow \Delta(\hat{0},x) \cong S^{\operatorname{rk} x - 2}$

Thm (Björner): P is CW poset $\Leftrightarrow$ there exists regular CW complex K with $P = F(K)$.

Method of Proof (via Danaraj-Klee, 1974 result)

thin + shellable $\Rightarrow$ CW poset

Recall: a graded poset is **thin** if  each rank 2 interval has 4 elements

e.g.  not thin;  not graded

Some Known Examples of CW Posets

- Simplicial posets (trivially)
- Bruhat order (Björner & Wachs)
- Closure poset for double Bruhat decomp. of totally nonneg. part of flag variety (Williams)
- Closure poset of double suspension of Poincare homology 3-sphere with "big cell" glued in

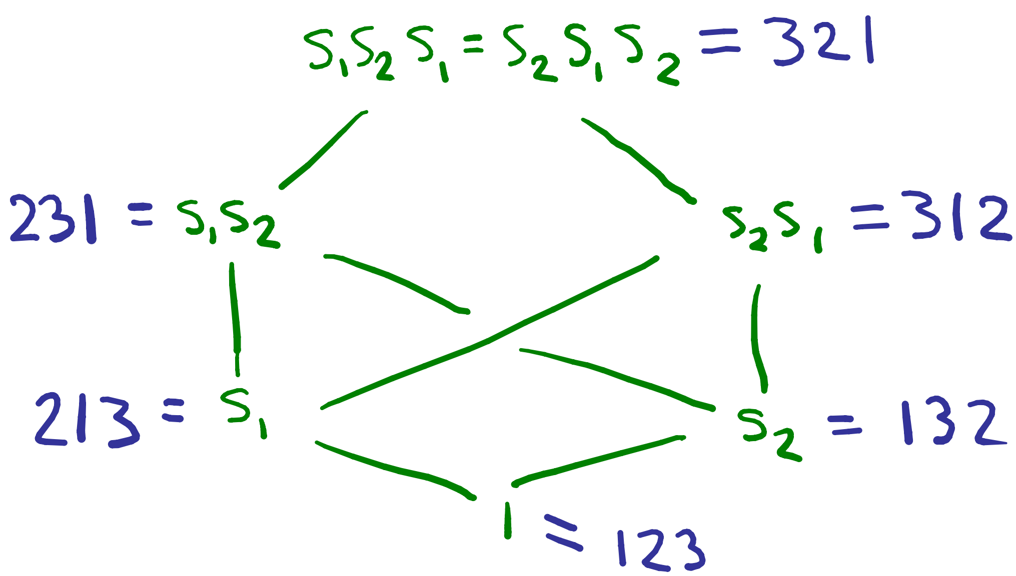
A Word of Caution: Does not imply stratified space with this closure poset is regular CW complex (e.g. $\mathbb{RP}_2$ example earlier is not)

Recall: The Bruhat order is partial order on Coxeter group W with $u \leq v$ $\Leftrightarrow$ there exists reduced expressions $r(u)$ and $r(v)$ with $r(u)$ subexpression of $r(v)$.

$$\text{e.g. } s_1 s_3 \leq_{\text{Bruhat}} s_1 s_2 s_3 s_1 s_2$$

Bruhat order for $W = S_3$:

generators $s_1 := (1,2)$ and $s_2 := (2,3)$



- Closure poset for Schubert cell decompositions & totally nonneg. parts

Question (Bernstein): Find naturally arising regular CW complexes homeomorphic to balls which have the Bruhat intervals as closure posets.

Conjectural Solution (Sergey Fomin & Michael Shapiro, 2000):

Bruhat stratification of link of origin of totally nonnegative part of unipotent radical of Borel in semisimple, simply connected algebraic group (and suitable links), i.e. $\mathrm{im}(f_{(i_1,\ldots,i_d)})$ on $\mathbb{R}^d_{\geq 0} \cap S_1^{d-1}$

Theorem (H.): Fomin-Shapiro Conjecture indeed holds.

Special Case of Type A: Space of Totally nonnegative upper triangular matrices with 1's on diagonal & entries just above diagonal summing to fixed, positive constant, i.e. im $(f_{(i_1,\ldots,i_d)})$ on $\mathbb{R}^d_{\geq 0} \cap S_1^{d-1}$.

Now to better understanding these spaces...

Using 0-Hecke Algebra to Analyze which Simplex Faces map to Same Cells

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$ "nil-move"

$$\Big\downarrow \text{ suppress parameters}$$

$x_i^2 = x_i$ (0-Hecke alg. rel'n, up to sign)

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1t_2}{t_1+t_3}\right)$

$$\Big\downarrow \text{ (type A)}$$

"long braid move" with enrichment from parameters

$x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1}$

(similar relation holds outside type A)

e.g.

$$x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4) = x_2(\underbrace{t_1'}_{\frac{t_2t_3}{t_1+t_3}})x_1(\underbrace{t_2'}_{t_1+t_3})x_2(\underbrace{t_3'}_{\frac{t_1t_2}{t_1+t_3}}+t_4)$$

$x_1x_2x_1x_2 \to x_2x_1x_2x_2 \to x_2x_1x_2$

Aside: Change of coordinates map for braid moves has many amazing properties

e.g. $(t_1, t_2, t_3) \mapsto \left(\frac{t_2 t_3}{t_1 + t_3}, t_1 + t_3, \frac{t_1 t_2}{t_1 + t_3} \right)$

in type A

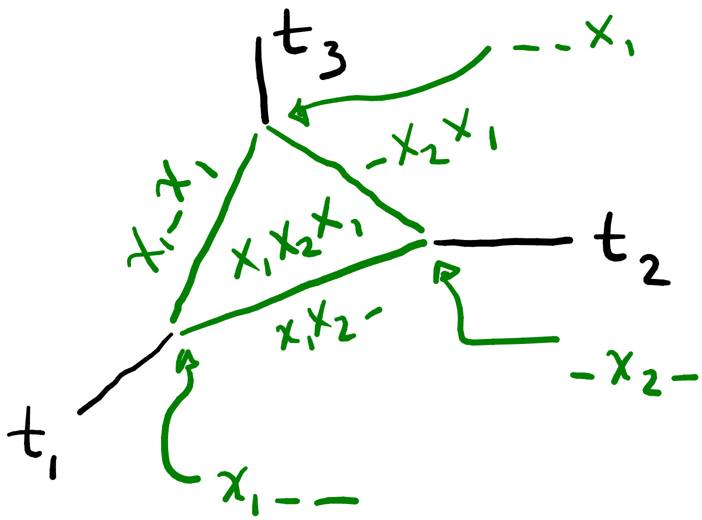
- Tropicalizes to change-of-basis map for canonical bases:

$(a, b, c) \mapsto (b + c - \min(a, c), \min(a, c), a + b - \min(a, c))$

- A motivation for development of cluster algebras (& mutation)

Suggested exercise: verify map is involution (to see hints of map's remarkable structure)

Indexing Faces of Preimage by Words in 0-Hecke Algebra



Thm (Lusztig): If $(i_1, \ldots, i_d)$ is reduced word, then $f_{(i_1, \ldots, i_d)}$ acts homeomorphically on $\mathbb{R}^d_{>0}$ with image the part of unipotent radical of Borel having certain minors (proscribed by w) positive and others zero.

Upshot: Injection on each face of simplex whose word is reduced

Key Observation About $f_{(i_1,\dots,i_d)}$:

$$\mathrm{im}(f_1)=\mathrm{im}(f_2) \iff \underbrace{x(f_1)=x(f_2)}_{\text{equal as 0-Hecke algebra elements}}$$

Idea: Combine Lusztig's result with properties of reduced & nonreduced words in 0-Hecke algebra.

Proof Overview for Fomin-Shapiro Conjecture:

1. Apply "collapses" to preimage simplex Δ_d eliminating faces indexed by nonreduced words, yielding $\Delta_d/\sim \cong \Delta_d$ regular CW complex with $x\sim y \Rightarrow f_{(i_1,\dots,i_d)}(x)=f_{(i_1,\dots,i_d)}(y)$.
2. Use new regular CW complex criterion to prove $f_{(i_1,\dots,i_d)}$ on $\Delta_d/\sim$ is homeomorphism.

Strategy to Study Images of Maps from Polytopes (e.g. Spaces of Parameters)

Set-up: Continuous, surjective fn

$$f : P \to Y$$

from convex polytope P s.t. f maps $\text{int}(P)$ homeomorphically to $\text{int}(Y)$

Step 1: Perform "collapses" on ∂P preserving regularity and homeomorphism type - via continuous, surjective collapsing functions $P \to P$ yielding $P/\sim$ with fewer cells s.t. $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$

Step 2: Prove $f : P/\sim \to Y$ is homeomorphism by new regularity criterion

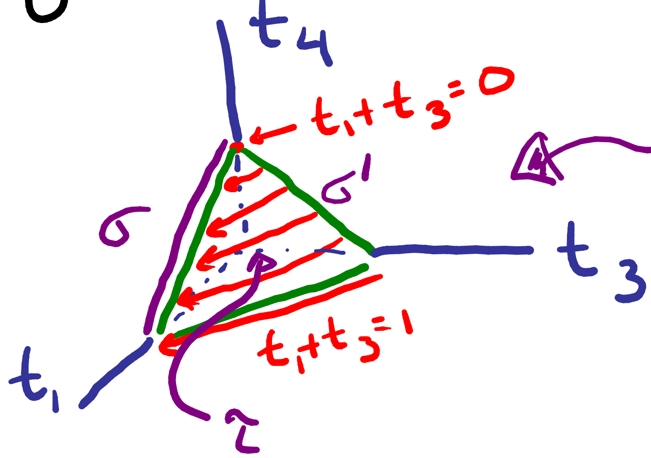
Collapsing Nonreduced Cells in Main Application:

e.g. $x_1(t_1)\,x_2(0)\,x_1(t_3)\,x_7(t_4)$
$$\| $$
$$x_1(t_1+t_3)\,x_7(t_4)$$

implying fibers of $f_{(i,\,id)}$ are parallel line segments $\left\{\begin{matrix} t_1+t_3=k \\ t_4=1-k \end{matrix}\right\}$

given by constants $0\le k\le 1$

- collapse segments onto endpts in σ, ID-ing σ w/ σ' & eliminating τ



Δ Inside bdry of higher dim'l ball

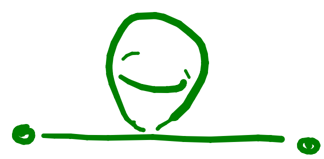
- Nonreduced faces all admit $x_i(u)\,x_i(v)=x_i(u+v)$ after long braid moves

New Regularity Criterion:

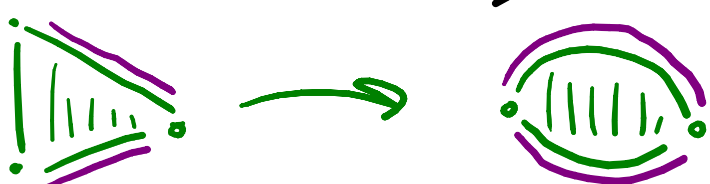
Prop'n (H.) Let K be a finite CW complex w/ characteristic maps $\{f_\alpha\}$.

Suppose

(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example: 

(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example: 

Then $F(K)$ is graded by cell dimension.

Remark: Next theorem "spreads around" injectivity requirement

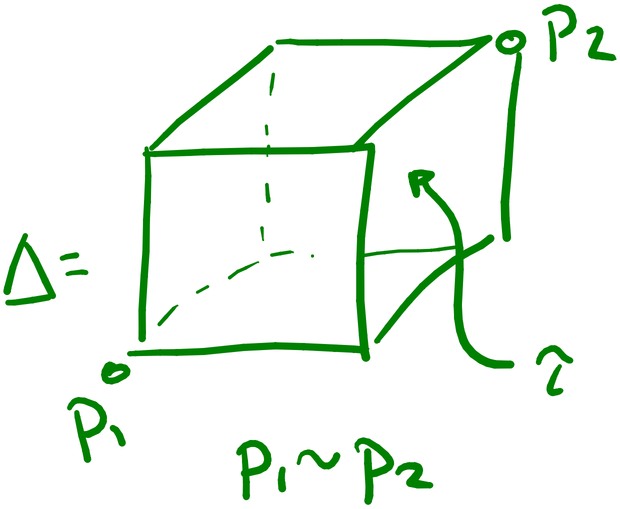
Thm (H.) Let K be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then K is regular w.r.t. $\{f_\alpha\}$ $\Leftrightarrow$

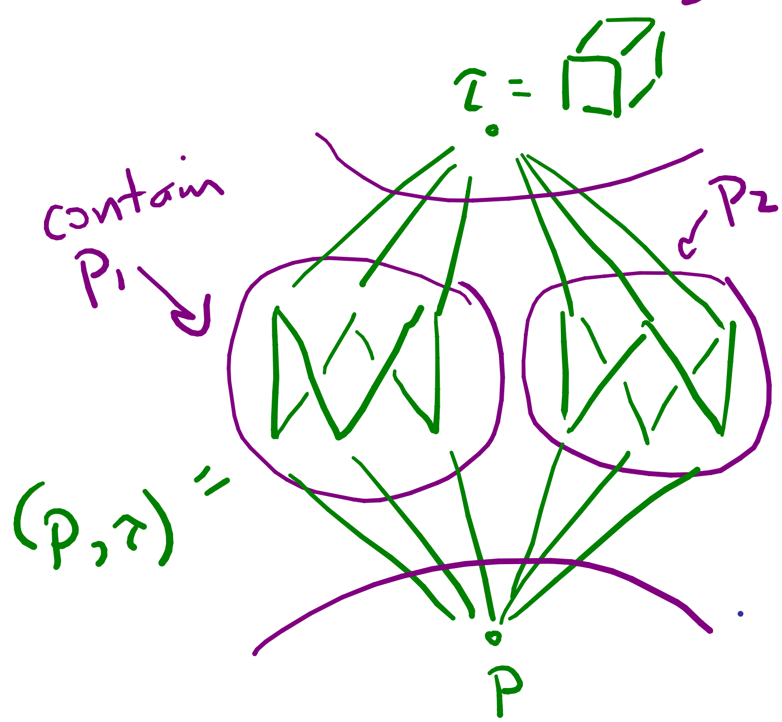
(1) K meets requirements of prop'n for $F(K)$ to be graded by cell dim.

(2) $F(K)$ is thin and each open interval (u,v) for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

Non-Example





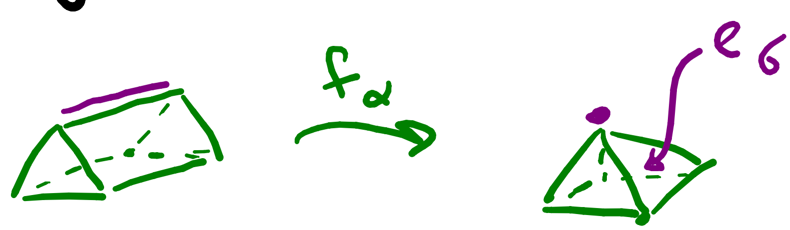
(3) For each α, the restriction of f_α to preimages of codim. one cells in $\overline{e_\alpha}$ is injective.

(topological condition)

Non-Example:

(4) $\forall\, e_\sigma \subseteq \overline{e_\alpha}$, f_σ factors as continuous inclusion $i : B^{\dim \sigma} \to B^{\dim \alpha}$ followed by f_α.

Non-Example: (due to David Speyer)



Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one

Example Where Injectivity is (Much) Easier in Codimension One

By exchange axiom for Coxeter groups

$$s_1 s_2 s_1 s_3$$

$s_1 s_2 s_1 -$	$s_1 s_2 - s_3$	$\cancel{s_1 - s_1 s_3}$	$- s_2 s_1 s_3$
$\parallel$	$\parallel$	$\cancel{\parallel}$	$\parallel$
3214	2341	$\cancel{1243}$	3142

various ways to delete a letter obtaining reduced expression gives <u>distinct</u> Coxeter group elements

Higher Codim:

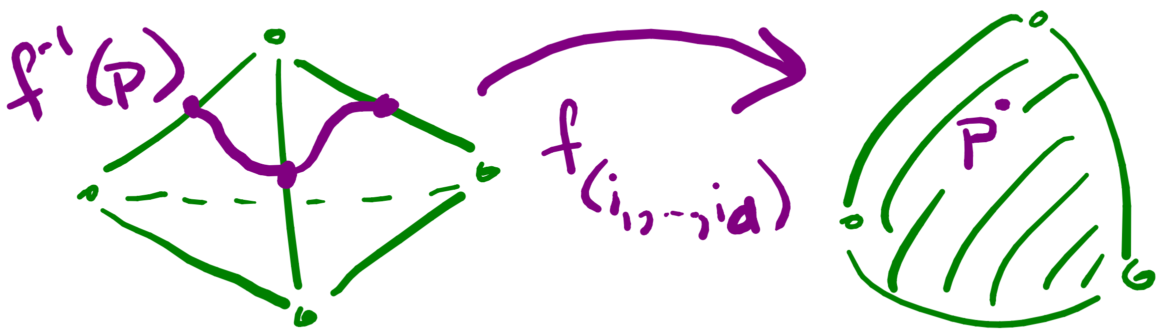
e.g.

$$s_1 s_2 s_1 s_3$$

$s_1 - - -$ $\qquad$ $- - s_1 -$

Conjectural Further Structure in the Relations Among Chevalley Generators

Conjecture (Davis-H.-Miller): $f^{-1}_{(i_1,\dots,i_d)}(p)$ for each $p \in Y_w^\circ$ is a regular CW complex homeomorphic to a ball with closure poset dual to face poset for interior of subword complex $\Delta((i_1,\dots,i_d),w)$



Intuition: "Tune" parameters up or down within fiber:

$$\underset{(23)}{\boxed{x_1(t_1)}}\;\underset{(12)}{x_2(t_2)}\;\underset{(13)}{x_1(t_3)}\;\underset{(23)}{\boxed{x_2(t_4)}}$$

$$= x_2\!\left(\frac{t_2t_3}{t_1+t_3}\right)x_1(t_1+t_3)\,x_2\!\left(\frac{t_1t_2}{t_1+t_3}+t_4\right)$$

Subword Complexes & their Role in This Story

$Q :=$ (not necessarily reduced) expression

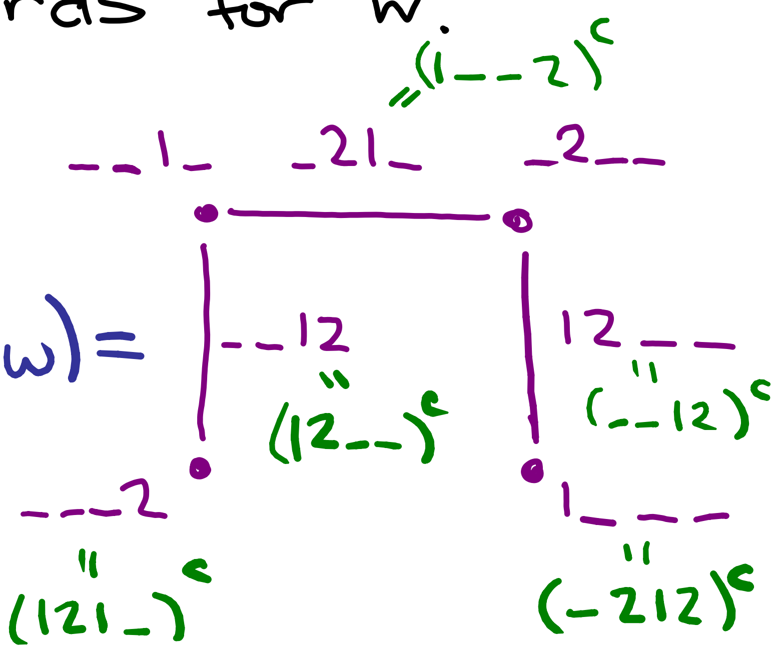
$w :=$ Coxeter group element

Facets of $\Delta(Q,w)$ are the subwords of Q whose complements are reduced words for w.

e.g.

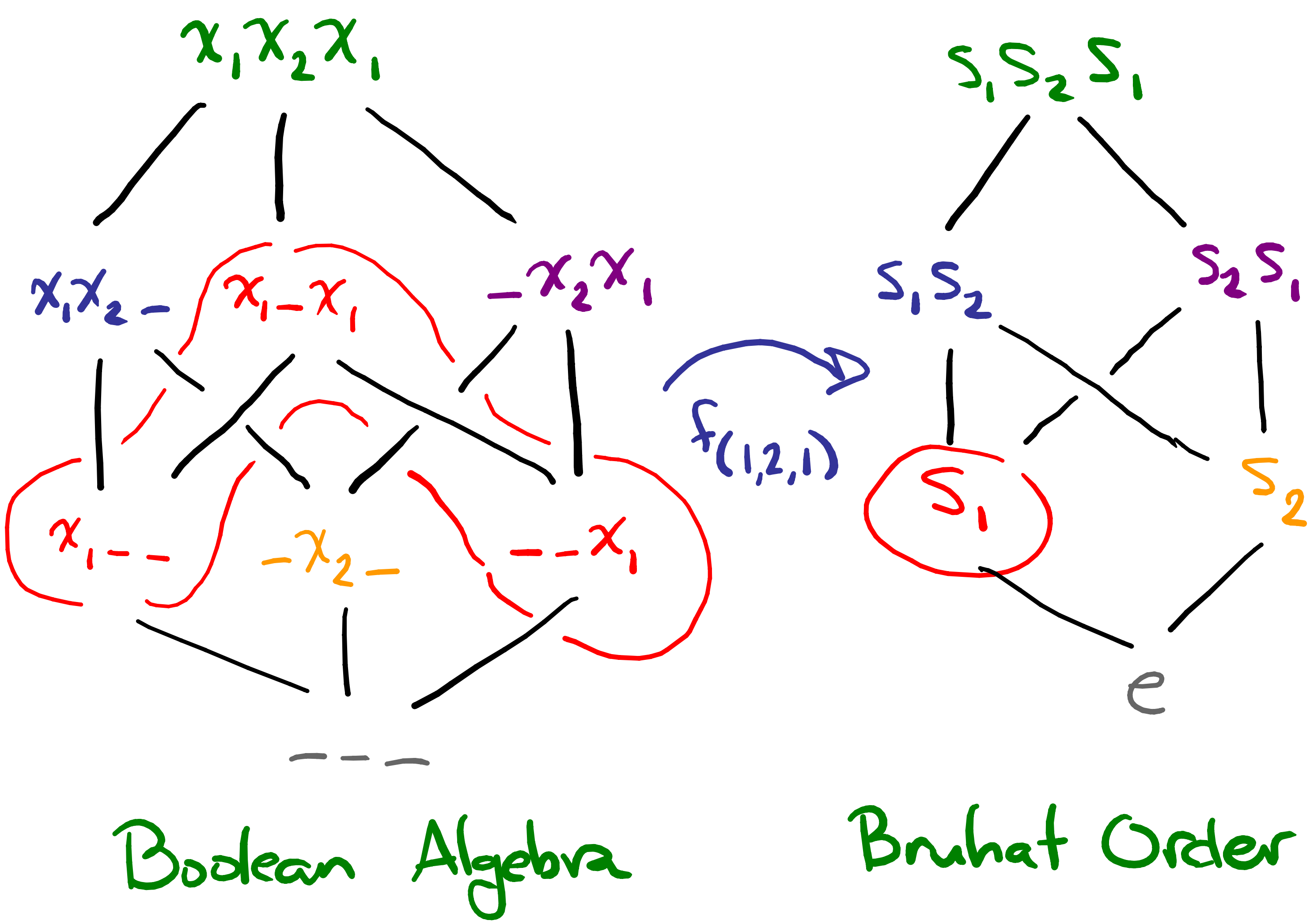
$Q = (1,2,1,2)$ $\quad \Delta(Q,w) =$

$w = s_1 s_2$



Thm (Knutson-Miller): $\Delta(Q,w)$ is vertex decomposable (hence shellable) ball or sphere.

A Poset Map (on Face Posets) induced from $f_{(i_1, \ldots, i_d)}$



- Apply braid moves & $x_i^2 \to x_i$ to get reduced expression; replace x_i's by s_i's

Homotopy Type of Bruhat Intervals:
New Proof by Quillen Fibre Lemma

Thm (Armstrong-H.): The poset map $f_{(i_1 \ldots i_d)}$ yields short proof of:

$$\Delta_{Bruhat}(u,v) \simeq S^{rk\,v - rk\,u - 2} \qquad \text{for all } u \leq v$$

Idea: • fibers $f_{\geq}^{-1}(u) = \{x \in B_n \mid f(x) \geq u\}$ are dual to face posets of subword complexes – proven to be balls by Allen Knutson & Ezra Miller.

Subword complexes previously arose as:

Stanley-Reisner complex for Gröbner degeneration of matrix Schubert variety ideal (Knutson and Miller)

Open Questions & Future Directions:

1. Analogous map, theory of "reduced expressions" & topological results for totally nonnegative part of flag variety? (see progress by Postnikov, Rietsch, Williams, Speyer, Marsh, ...)
2. Explanation why subword complexes arising in distinct, but related settings? Bigger unifying picture?

More Details (slides & papers)

- http://www4.ncsu.edu/~plhersh

Thank you!