Fibers of Maps to Totally Nonnegative Spaces & the Fomin-Shapiro Conjecture

Patricia Hersh
North Carolina State University

Partly joint work with Jim Davis & Ezra Miller
Today:

1. Anecdotes:
   ◦ "penalty time" for seminar speakers
   ◦ getting grilled regarding:

2. Fomin-Shapiro Conj (2014 Proof)
   ◦ $\text{im}(f_{c_i \cdots c_{ia}})$ is regular CW ball

3. New Results Regarding Fibers of $f_{c_i \cdots c_{ia}}$, including:
   ◦ cell decomposition/parametrization
   ◦ combinatorial model

(Recall: $\equiv \nsim \neq \Im \underline{\underline{\underline{\underline{\square}}}}$)
  ◦ closed ball $\sim$ homeomorphic
Topological Aspects of Total Positivity

- Lusztig (94), Fomin-M. Shapiro (00). Initiate study of totally nonneg, real part of spaces of matrices, spaces of flags (i.e. GL/n/B),...
  (i.e. having all minors nonnegative)

- Conjecturally/provably homeomorphic to closed balls

- Proving this by studying fibers (as is done in H. T. in D/H/M)
  - imposes restrictions on reln's amongst exp'd Chevalley gen's
  - reveals structure in Lusztig's canonical bases
Background on Coxeter Groups

• $s_i := (i,i+1)$ a simple reflection of type A (i.e. $W = S_n$) in $W$

• $s_{i_1} \cdots s_{i_d}$ is reduced expression for $w \in W$ if $w = s_{i_1} \cdots s_{i_d}$ for $d$ as small as possible

• length of $w$, denoted $l(w)$, is this smallest $d$.

  e.g. $s_1 s_2 s_1 = s_2 s_1 s_2$ has length 3

\[ 321 \xrightarrow{s_1} 231 \xrightarrow{s_2} 213 \xrightarrow{s_1} 123 \]
\( (i_{11},i_{1d}) \) is reduced word

**Bruhat order:** partial order

\[
\begin{align*}
    s_1s_2s_1 & = 321 \\
    231 & \sim s_2s_1 \\
    213 & \sim s_1 \\
    1 & = 123 \\
    s_1s_2 & = 312 \\
    s_2 & = 132
\end{align*}
\]

on \( \mathcal{W} \) (e.g. \( S_n \)) with \( u \leq v \) for \( u,v \in \mathcal{W} \) \( \iff \) any reduced word for \( v \) has subword that is reduced word for \( u \).
**Running Example: Totally Nonnegative Part of a Space of Matrices**

- \( \chi_i(t) = I + tE_{i,i+1} \)
  - \( \exp(t e_i) \) (type A)
  - (general finite type, exposed Chevalley generator)

- \( f_{(i_1, \ldots, i_d)} : \mathbb{R}^d_\geq \rightarrow M_{n \times n} \in \mathbb{R}^{n^2} \)
  - Reduced word \( (t_{i_1}, \ldots, t_{i_d}) \)
  - \( \chi_{i_1}(t_{i_1}) \cdots \chi_{i_d}(t_{i_d}) \)

\( \text{e.g.} \quad f_{(1,2,1)}(t_1, t_2, t_3) = x_1(t_1)x_2(t_2)x_1(t_3) \)

- We case:
  - \( \prod (1 + t^i)^{\text{tot.}} \)
  - \( \text{nonneg} \)

\( \left( \begin{array}{ccc}
1 & t_1 + t_3 & t_1t_2 \\
1 & t_3 & t_2 \\
0 & 1 & t_2 \\
0 & 0 & 1
\end{array} \right) \)
"Picture" of Map $f_{(i,2,1)}$

$\mathbb{R}^3_{>0} \cap (\Sigma t_i = K \text{ hyperplane})$

Positive constant

$f_{(1,2,1)}(t_1, t_2, t_3) = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$t_2 = 0$

$f_{(1,2,1)}(t_1, 0, t_3) = \begin{pmatrix} 1 & t_1 \\ 1 & t_3 \\ 1 & 0 \end{pmatrix}$

$\chi(t_1) \circ \chi(t_3)$

$\chi(t_1) \circ \chi(t_3)$

$= \begin{pmatrix} 1 & t_1 + t_3 \\ 1 \\ 1 \end{pmatrix} = \chi(t_1 + t_3)$

Simplex faces with same image $\chi_1^2 = \chi_1$

e.g. $\exists x_1(t) | t > 0^3 = \exists x_1(t_1)x_1(t_2) | t_1, t_2 > 0$
**CW Complexes and their Face Posets (Partially Ordered Sets)**

*Example:* $F(K) \rightarrow F(K')$

- $K = \mathbb{S}^2$ ball
- $K' = \mathbb{R}P^2$

"face poset" ($u \leq v \Rightarrow u \leq \bar{v}$)

**Recall:** A CW complex: cells $e_{\alpha} \approx \mathbb{R}^{\dim(e_{\alpha})}$, characteristic maps $f_{\alpha} : B^{\dim(e_{\alpha})} \rightarrow \nu \epsilon \beta^{\dim(e_{\alpha})}$, attaching maps $f_{\alpha} \mid \partial B^{\dim(e_{\alpha})}$.
Recall: CW complex is regular if each $f_a$ is homeomorphism.

- $\Delta(P) =$ "nerve" or "order complex" of $P$
- $K$ regular $\Rightarrow K \equiv \Delta(F(K) \setminus \emptyset) = \text{sd}(K)$

- a CW poset is any "nerve" face poset of regular CW complex
- "Shellable" + "thin" $\Rightarrow$ CW poset
  "Graded" e.g. not
  e.g. Bruhat order

(Björner-Wachs; Matthew Dyer)
**Fomin-Shapiro Conjecture:** The Bruhat stratification of $lk(id)$ in totally nonneg. real part of unipotent radical in Borel in algebraic group (e.g. $im(f_{ci,\cdots, id})$) is regular CW complex homeom. to closed ball (with Bruhat order as face poset).

$$\gamma_\omega = \left[ B^{-\omega} B^{-} \cap \text{(unipotent subgp of } G) \right]$$

- lower triangular opposite Borel $B^{-}$
- permutation
- 2 totally nonneg. part
- upper triangular, w/ 1's on diagonal
**Bruhat Order**

Closure poset \( F(L) \) for Schubert cell decomposition \( L \) of flag variety \( Fl_n = GL_n/B \) of "Schubert varieties" (over \( \mathbb{C} \)), namely for cell closures. Likewise for \( G/B \) in other types.

**Example:**

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & * & * & 1 \\
0 & * & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \geq
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & * & * & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\( IR^c = C^3 \) cell indexed by \( 1432 \in S_4 \) cell \( \cong C^2 = IR^4 \)

Flags \( \langle (\frac{1}{0}) \rangle \subseteq \langle (\frac{1}{0}), (\frac{2}{0}) \rangle \subseteq \cdot \cdot \cdot \)
A Motivation Stated by Fomin & M. Shapiro

"It should be mentioned that one of our 'hidden motivations' has been the desire to better understand the combinatorics of Kazhdan-Lusztig polynomials."

Their Results: face poset, homological results, fascinating projection map & much more!
Theorem (H., 2014, Invent.): Fomin-Shapiro Conjecture holds.

Special Case (Running example): Space of totally nonnegative upper triangular matrices with 1’s on diagonal & entries just above diagonal summing to fixed positive constant, stratified by which minors are positive & which are 0.

Concrete Realization: Products $x_i(t_i)\cdots x_{id}(td)$ of elementary matrices, by results of Whitney, Lascoux & (beyond type A) Lusztig.
**Conjecture (Davis-H-Miller):**

$f_{(i_1, \ldots, i_d)}^{-1}(P)$ is regular CW complex homeomorphic to interior dual block complex of subword complex $\Delta((i_1, \ldots, i_d), w)$ for $p \in \mathbb{P}_{w}^\circ$.

**Thm (DHM):** $f_{(i_1, \ldots, i_d)}^{-1}(P)$ has cell decomposition & "correct" face poset.

**Thm (DHM):** Interior dual block complex of $\Delta((i_1, \ldots, i_d), w)$ is contractible.
A Motivation to Study Fibers: Relations Among (Exponentiated) Chevalley Generators

Lie algebra

\[ \exp(t e_i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} t \\ t^2/2 \\ t^3/6 \end{pmatrix} + \cdots \]

Lie group

\[ \exp(t e_1) \exp(t_2 e_2) = x_i(t_1) x_j(t_2) \]

\[ f_{(i,j)}(t_1, t_2) = x_i(t_1) x_j(t_2) \]

Reln’s <--> Elts in same fiber of \( f_{(i_1, i_2, \ldots, i_n)} \)
Subword Complexes
(introduced by Allen Knutson
and Ezra Miller)

\[ \Delta(\tilde{Q}, \tilde{\omega}) = \text{(abstract) simplicial complex whose faces are}\]
reduced or nonreduced word

Coxeter group element subwords \( Q' \) of \( Q \) whose complement \( Q \setminus Q' \) contains a reduced word for \( \omega \)

Aside:
1st arose as Stanley-Reisner complexes of initial ideals of coordinate rings of matrix Schubert varieties.
Thm (Knutson-Miller): $\Delta(Q, \omega)$ is homeomorphic to ball or sphere.

Example of Subword Complex & its **Interior Dual Block Complex**

$\Delta(Q, \omega) =$

$Q = 132132$

$\omega = s_1s_3s_2$

$\Rightarrow$ its

**Interior dual block complex**

$\Rightarrow f^{-1}_{(1,3,2,1,3,2)}(M)$ for $M \in Y^0_{(1,3,2)}$
Examples of Fibers:
(with realizations as suggested by various results towards proof of DHM conjecture)

e.g.

\[(0, \frac{7}{2}, 12, \frac{5}{2}, 0) \sim (0, 0, 5, 1, 7) \sim (5, 1, 7, 0, 0) \sim (5, 0, 0, 1, 7) \]

\[f^{-1}(M) \text{ for } M = x_1(5)x_2(1)x_1(7) \]
\[ f_{(1,2,1,2,1,2)}^{-1}(M) \quad \text{for } M \in \mathcal{V}_{(1,2)} \]

\[ 2 \Rightarrow 2 \]

\[ f_{(1,2,1,2,1,2)}^{-1}(M) \quad \text{for } M \in \mathcal{V}_{(1,2)} \]
Role of $U$-Hecke Algebra in Stratification for $\text{im}(f_{(i, \ldots, i_d)})$

1. $x_i(t_1)x_i(t_2) = x_i(t_1 + t_2)$
   - $\text{supress parameters}$
   - $x_ix_i = x_i$

2. $x_i(t_1)x_{i_1}(t_2)x_i(t_3) = x_{i_1}(\frac{t_2+t_3}{t_1 t_2})x_i(t_1t_3)x_{i_1}(\frac{t_1+t_3}{t_1 t_2})$
   - (type A)
   - for $t_1, t_2, t_3 > 0$
   - $x_ix_{i_1}x_i = x_{i_1}x_ix_{i_1}$

(Analogous relations outside type A)

Upshot: $\text{im}(f_1) = \text{im}(f_2) \iff x(f_1) = x(f_2)$
- Equal as $U$-Hecke algebra elements

Thm (Lusztig): If $(i_{1}, \ldots, i_d)$ is reduced, then $f_{(i_{1}, \ldots, i_d)}$ is homeomorphism on $\mathbb{R}^d$. 
More Motivation for Nonnegative Real Part of Unipotent Radical

- Given quantized env. alg. $U=U_{\text{gr.}} \otimes U_{\text{aff.}} \otimes U_{\text{qgr.}}$ of Kac-Moody alg. (e.g. affine Lie alg.), then canonical basis is a basis $B$ for $u^-$ such that highest weight module with highest weight vector $v_\lambda$ has basis $\sum v_\lambda b (b \in B, v_\lambda b \neq 0)$ for each $\lambda$.

- $f^{-1}(i,-i) \rightarrow f^{-1}(j,-i)$ coordinate change

  $(t_1, t_2, t_3) \mapsto \left( \frac{t_1 t_3}{t_1 + t_3}, \frac{t_1 t_2}{t_1 + t_3}, \frac{t_1 t_2}{t_1 + t_3} \right)$

  tropicalizes to coordinate change:

  $(a, b, c) \mapsto (b + c - \min(a, c), \min(a, c), a + b - \min(a, c))$

  for canonical bases w/ same braid move
Faces of (Preimage) Simplex as Subexpressions in O-Hedee Algebra

- let $Y_\omega^0 = \text{open cell in } \text{im}(f_{\epsilon_1...\epsilon_n})$ indexed by $\omega \in \mathcal{W}$

- let $\delta(x_{i_1}...x_{i_d})$ denote (unsigned) O-Hedee algebra product (a.k.a. "Demazure product")

  e.g., $\delta(x_1x_2x_1x_2x_1) = \delta(x_2x_1x_2x_1) = s_1s_2s_1x_2x_1$ (since $x_2x_2 = x_2$)
Map of Face Posets

Induced by Map $f_{(i_1 \cdots i_d)}$ of Spaces

Boolean lattice $B_n$

Face poset of simplex $\text{FClm}(f_{(i_1 \cdots i_d)})$

$f : x_{i_{j_0}} \cdots x_{i_{j_r}} \mapsto S(x_{i_{j_0}} \cdots x_{i_{j_r}})$
Proof of Fomin-Shapiro Conjecture (H., 2014, Invent.)

**Set-up:** Surjective fn \( f_{(i_1, \ldots, i_d)} : \Delta_{d-1} \rightarrow \gamma \) s.t. \( f_{(i_1, \ldots, i_d)} \mid \text{int}(\Delta_{d-1}) \) homeom. to \( \text{int}(\gamma) \).

**Step 1:** Perform "cell collapses" on \( \partial(\Delta_{d-1}) \) designed to preserve homeom. type; regularity via cont. surjective fn's \( g_i : \Delta_{d-1} \rightarrow \Delta_{d-1} \) yielding \( (\Delta_{d-1})^\sim \) s.t.

\[
x_1 \sim_\Delta x_2 \Rightarrow f_{(i_1, \ldots, i_d)}(x_1) = f_{(i_1, \ldots, i_d)}(x_2)
\]

(eliminating faces given by nonreduced words)

**Step 2:** Prove \( \overline{f}_{(i_1, \ldots, i_d)} : (\Delta_{d-1})^\sim \rightarrow \gamma \) is a homeomorphism via new regularity criterion for finite cw complexes.

**Corollary of Proof:** Contractibility of fibers
Combinatorics of Fibers

\[ f_{(1,2,1)} \]

Induced map of face posets:

Thm (Armstrong-H., 2011): For each \( u \in W \), \( f^{-1}(u) = \{ x \in B_n | f(x) \geq u \} \) is dual (i.e. upside-down) to face poset for subword complex \( \Delta((i_1, \ldots, i_k), u) \).
Thm (DTHM, 2018): \( f_1^{-1}(u) \) is face poset of interior dual block complex for subword complex \( \Delta(i_{i-1},i_1),u \)

Thm (DTHM, 2018): Interior dual block complex of \( \Delta((i_{i-1},i_1),u) \) is contractible.

Pf: Discrete Morse theory

Combining: DTHM Conjecture would imply \( f_{\langle i_{i-1},-id \rangle}(p) \leq \text{interior dual block complex of } \Delta((i_{i-1},i_1),u) \) for \( p \in Y_u \), hence \( f_{\langle i_{i-1},-id \rangle}(p) \) contractible.
Topology of fibers

Thm (Davis-H-Miller): Each fiber $f_{i_1,\ldots,i_d}^{-1}(p)$ admits a cell decomposition induced by the natural cell decomposition of the simplex $\Delta_{d-1}$.

Proof: Parametrization + continuity lemmars
Topology of Fibers: Key Lemmas

**Definition (Davis-H-Miller):** The letter \( x_i \) is redundant in \( x_{i_1} = x_{i_d} \) if \( S(i_1, \ldots, i_d) = S(i_2, \ldots, i_d) \)

**Lemma (Davis-H-Miller):** \( x_i \) is non-redundant in \( x_{i_1} = x_{i_d} \)

\[
\iff \ f_{{i_1, \ldots, i_d}}^{-1}(p) \text{ for } p \in \gamma_{S(i_1, \ldots, i_d)} \\exists (t_{i_1}, \ldots, t_{i_d}) | x_i(t_i) - x_{i_d}(t_d) = p^3 \\
\text{has unique value } k_i \text{ for } t_i.
\]

\[
\iff \ f_{{i_1, \ldots, i_d}}^{-1}(p) \cong f_{{i_2, \ldots, i_d}}^{-1}(x_i(t-k_i)p)
\]
Lemma (Davis-H-Miller): Given 
\((t_1, \ldots, t_d) \in f_{(i_1, \ldots, i_d)}^{-1}(p)\) with \(t_i > 0\) and \(x_{i_1}\) redundant, then 
\(\exists (t'_1, \ldots, t'_d) \in f_{(i_1, \ldots, i_d)}^{-1}(p)\) for every \(t'_1 \in [0, t_i]\).

E.g. \(M = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)\) then 
\(f_{(1,2,1,2)}^{-1}(M) \iff \{(t_1, t_2, t_3, t_4) | x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4) = M^3\}\) achieves every \(t_1 \in [0, e^{1/4}]\).
**Parametrization:** Given \((i_1, \ldots, i_d)\), consider rightmost subword that is reduced word for \(\omega = S(i_1, \ldots, i_d)\)

**e.g.** \((1,2,1,2,1)\)

**Parametrize** \(f^{-1}(1,2,1,2,1,2,1)\) in \(T_\infty T_\infty\)

**"free" parameters**

**"dependant" parameters**

**Thm (Davis-H-Miller):**

\[ \text{dim}(\sigma) = \bigcup \mathbb{Z} \]

\[ \forall \bar{z} \leq \bar{\sigma} \]

z source vertex of \(\bar{\sigma}\)
Totally Nonnegative Spaces w/ Seemingly Analogous Structure

1. Totally nonnegative real part of Grassmannian: $\text{Gr}_{20}(k,n) = (\text{Gr}(k/n))_\mathbb{R}$

Postnikov: polytope of "plabic graphs" w/ "measurement map" to $\text{Gr}_{20}(k,n)$ 
+ theory of (reduced) plabic graphs

Postnikov-Speyer-Williams: $\text{Gr}_{20}(k,n)$ is CW complex (via attaching maps that are not homeomorphisms)

Galkin- Karp-Lam 2017 preprint: $\text{Gr}_2(k,n)$ is homeom. to closed ball.
2. Totally nonnegative real part of flag variety $\mathcal{Fl}_{20} = (Gl/B)_{20}$

**Rietsch**: poset of closure reln's, cells $R^0_{uv}$ given by $u \leq v$ in Bruhat order

**Marsh-Rietsch**: parametrization for $R^0_{uv}$

**Williams**: poset is CW poset

**Rietsch-Williams**: (1) CW complex w/ attaching maps via canonical bases.
(2) Contractibility of each cell closure

**Gekshin-Karp-Lam**: homeomorphism type for closure of "big cell"
3. Stratified Spaces of Electrical Networks (Curtis-Ingerman-Morrow, Kenyon-Wilson,..)

- Image of map $\text{Resp} : \mapsto \text{response matrix}$

$\text{Lam} : (1) E_n \twoheadrightarrow \text{Gr}_{{2n}}(n-1, 2n)$

(2) $F(E_n)$ is "Eulerian" (i.e. correct)

$H\text{-Kenyan} : F(E_n)$ shelling $\neq F(E_n)$ is CW poset (uses Dyer Bruhat shelling)

$H\text{-K Cor} :$ shelling for each $[u,v]$ in face poset for edge-product space of phylogenetic trees (so CW poset)

Galashin-Karp-Lam: Homcom. type for closure of "big cell" for graph analogue of $\omega_0$, "well-connected graph"
1. (Deconed) Totally nonneg real part unipotent radical of Borel in semisimple simply connected algebraic group
(a) Vertex “link” (nbhd) in $(Fl_n)_{\geq 0}$
   via Marsh-Rietsch parameters $\to 0$
(b) Image $f_{c(i_{rr},-id)}$ for $\omega_0 = \omega(i_{rr},id)$
(c) Bnhat order as poset of closure rel’ns
(d) Each cell closure homeom. to closed ball.
- known as Fomin-Shapiro Conjecture
- special case of $\omega_0$ in type $A$
  (shorter proof) GKL, 2017
Happy Birthday, Sergey!

\[
\begin{align*}
(0, \frac{7}{12}, 12, \frac{5}{12}, 0) & \cong (5,1,7,0,0) \\
(5,0,0,1,7) & \cong B_2 \\
\end{align*}
\]

- and thanks for making so many interesting mathematical associations!
Appendix: Further Remarks & Details
Thm (DHM): DHM Conjecture → 2nd (or 3rd?) proof of F5-Conjecture
Idea: Shellability of Brualdi Order + (Well Known) Topological Relationship
Fibers to Image: Let $g: B \to Z$ be continuous surjection from ball $B$ to Hausdorff space $Z$ whose restriction to $\text{int}(B)$ is an embedding. Suppose also:

1. $g([2B]) \simeq [2B] = S^n$
2. $g([2B]) \cap g(\text{int}(B)) = \emptyset$
3. $g^{-1}(p)$ is contractible $\forall p \in g([2B])$

Then $Z \simeq B$. 
Ingredients in this Relationship Between Fibers and Image

- CE-Approximation Theorem (Kirby-Siebenmann (all dim's); Quinn (dim 4); Armentrout (dim 3))
- $g : EB \rightarrow EB$ as above may approximated by homeomorphisms

- Local Contractibility of $\text{Home}^r(S^n, S^n)$
- any two homeomorphisms “close enough” to each other connected by path of homeomorphisms
Step 1: Cell Collapses Preserving Homeom. Type, Regularity & Topol. Manifold

- Cover each non-reduced \( \sigma \) with "nice" curves each in single fiber of \( f_{(i, \ldots , i)} \)
- Collapse \( \overline{\sigma} \) onto \( \overline{\rho} \leq \partial \sigma \) by stretching curve extensions in "collar" for \( \overline{\partial \sigma} \setminus \sigma \) to cover \( \overline{\sigma} \setminus \overline{\rho} \)

- Heavily uses combinatorics of reduced \& non-reduced words in \( \mathbf{O} \)-Hecke algebra
Step 2 via: New Regularity Criterion

Preparatory Lemma (H.): Let \( K \) be a finite CW complex with characteristic maps \( \{f_j\} \). Suppose:

1. \( \forall \alpha, f_\alpha (2B^{\dim \alpha}) \) is a union of open cells (surjectivity)

   Non-Example: \( \emptyset \)

2. \( \forall f_\alpha \), the preimages of the open cells of codim. one in \( \bar{e}_\alpha \) are dense in \( 2(B^{\dim \alpha}) \)

   Non-Example:

Then \( F(K) \) is graded by cell dimension.

Insightful feedback (Quinn): Next theorem "spreads around" injectivity requirement.
Thm (H.) Let \( K \) be finite CW complex w.r.t. characteristic maps \( f_0, f_3 \). Then \( K \) is regular w.r.t. \( f_0, f_3 \) \( \iff \)

1. \( K \) meets requirements of prop in for \( F(K) \) to be graded by cell dim.
2. \( F(K) \) is thin and each open interval \( (u, v) \) for \( \dim(v) - \dim(u) \geq 2 \) is connected (as graph)

(Combinatorial condition)
(3) For each $\alpha$, the restriction of $f_\alpha$ to preimages of codim. one cells in $\overline{e}_\alpha$ is injective.

(topological condition)

**Non-Example:**

(4) $\forall e_\epsilon \subseteq e_\alpha$, $f_\epsilon$ factors as continuous inclusion $i : B^{\dim \sigma} \to B^{\dim \alpha}$ followed by $f_\alpha$.

**Non-Example:**

Notably **Absent**: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one.

**Proof**: Induction on difference in dim.
Confirming Codimension One Injectivity via Exchange Axiom for Coxeter Groups

- Reduced subexpressions of reduced expression obtained by deleting one letter give distinct u.e.w.

- Need codim one! e.g. $s_1s_2s_1$, $s_1$, $-s_1$
(Mainly Combinatorial) Conditions Allowing Such Face Collapses Across Curves

\[ K_0 \overset{g_1}{\longrightarrow} K_1 \overset{g_2}{\longrightarrow} K_2 \overset{\ldots}{\longrightarrow} K_i \]

polytope!

• Collapse face in \( K_i \) across images of parallel line segments in \( K_0 \) satisfying:
  • Distinct endpoints condition (DE): 
    \[ x \neq x' \]
  • Distinct initial points condition (DIP): 
    \[ x \neq x' \]
  • Least upper bound condition (LU3) to preclude 
    \[ \text{(conditions checkable via O-Hecke algebra)} \]
Additional Key Challenges

1. $\mathcal{O}$-Hecke algebra lacks inverses; lacks cancellation law.
   **Key Idea:** find ways to transfer properties from Coxeter gp

2. Need change-of-coords for braid moves as homeomorphisms on closed cells
   **Key Idea:** induction by embedding smaller instance

3. Need maps to extend to full complex

4. Tricky combinatorics to verify LUB at each collapsing step,