Fibers of Maps to Totally Nonnegative Spaces & the Fomin-Shapiro Conjecture

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- partly joint work with Jim Davis & Ezra Miller

Today:

1. Anecdotes:
   - "penalty time" for seminar speakers
   - getting grilled regarding:
2. Fomin-Shapiro Conj (2014 Proof)
   - im$(f_{(i_1,\dots,i_d)})$ is regular CW ball
3. New Results Regarding Fibers of $f_{(i_1,\dots,i_d)}$, including:
   - cell decomposition/parametrization
   - combinatorial model

(Recall: 

closed ball $\cong$ homeomorphic $\not\cong$)

Topological Aspects of Total Positivity

- Lusztig (94), Fomin–M. Shapiro (00)... initiate study of: totally nonneg, real part of spaces of matrices, spaces of flags (i.e. GL_n/B)... (i.e. having all minors nonnegative)
- Conjecturably / provably homeomorphic to closed balls
- Proving this by studying fibers (as is done in H. & in DHM)
  - imposes restrictions on reln's amongst exp'd Chevalley gen's
  - reveals structure in Lusztig's canonical bases

Background on Coxeter Groups

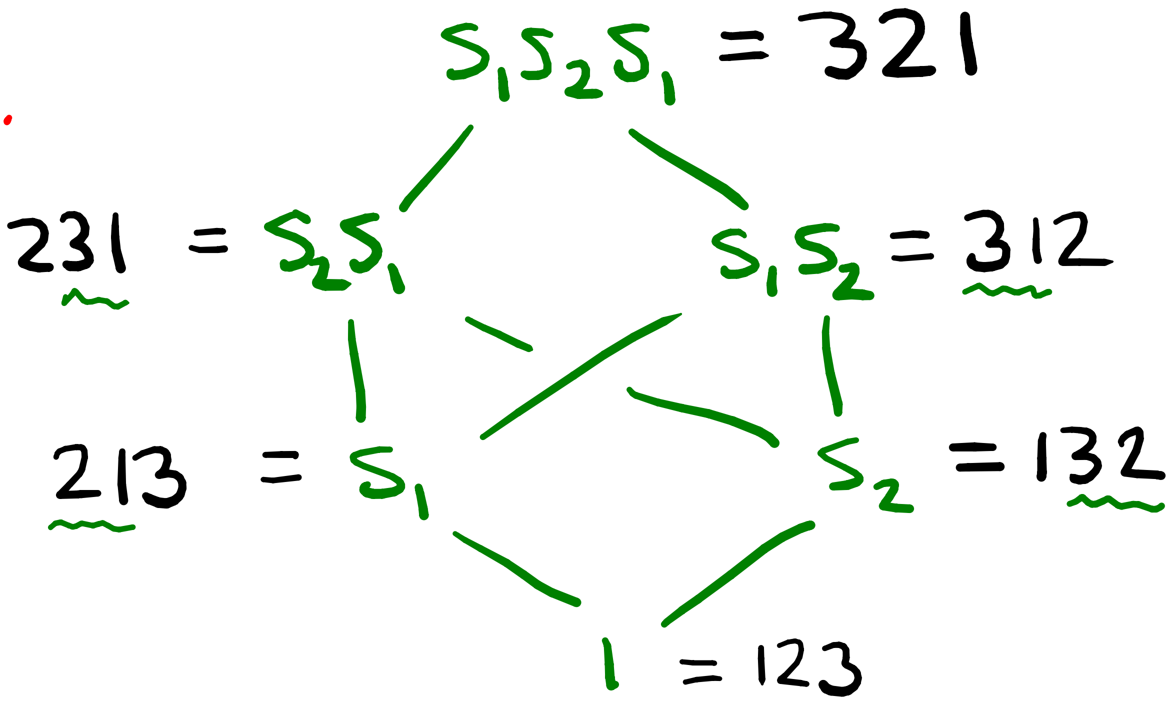
- $s_i := \underbrace{(i, i+1)}_{\text{type A (i.e. } W = S_n)}$ a **simple reflection** in W
- $s_{i_1} \cdots s_{i_d}$ is **reduced expression** for $w \in W$ if $w = s_{i_1} \cdots s_{i_d}$ for d as small as possible
- **length** of w, denoted $\ell(w)$, is this smallest d.

e.g. $s_1 s_2 s_1 = s_2 s_1 s_2$ has length 3

$$\begin{array}{c} s_1 s_2 s_1 \\ \| \\ 321 \xrightarrow{s_1} 231 \xrightarrow{s_2} 213 \xrightarrow{s_1} 123 \end{array}$$

- $(i_1, \ldots, i_d)$ is **reduced word**
- **Bruhat order:** partial order

e.g.



on W (e.g. S_n) with $u \leq v$ for $u, v \in W \iff$ any reduced word for v has subword that is reduced word for u.

Running Example: Totally Nonnegative Part of a Space of Matrices

- $x_i(t) = I_n + tE_{i,i+1} = \begin{array}{c} \text{column } i+1 \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow \text{row } i \end{array}$

  $x_i(t) = \exp(te_i)$ ← (general finite type, expon'd Chevalley generator)

  $I_n + tE_{i,i+1}$ ← (type A)

- $f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \subseteq \mathbb{R}^{n^2}$

  $(i_1,\dots,i_d)$: reduced word

  $(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$

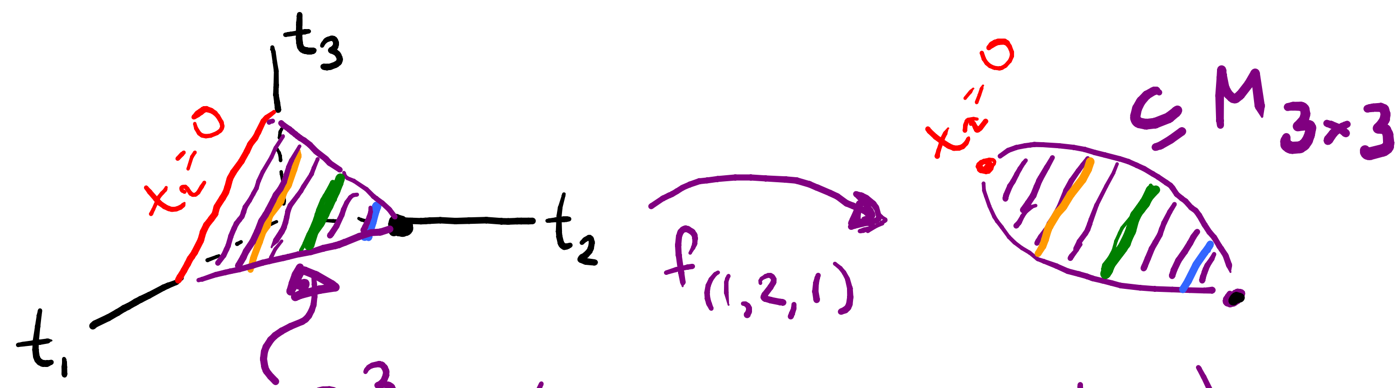
e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

w_0 case:

$$\left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \middle| \begin{array}{l} \text{tot.} \\ \text{nonneg} \end{array} \right\} = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1,\dots,i_d)}$



$\mathbb{R}^3_{\geq 0} \cap (\sum t_i = \underbrace{K}_{\text{positive constant}}$ hyperplane$)$

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$\Big\downarrow\ t_2 = 0$

$$f_{(1,2,1)}(t_1,0,t_3) = \underbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_1)}\underbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_3)}$$

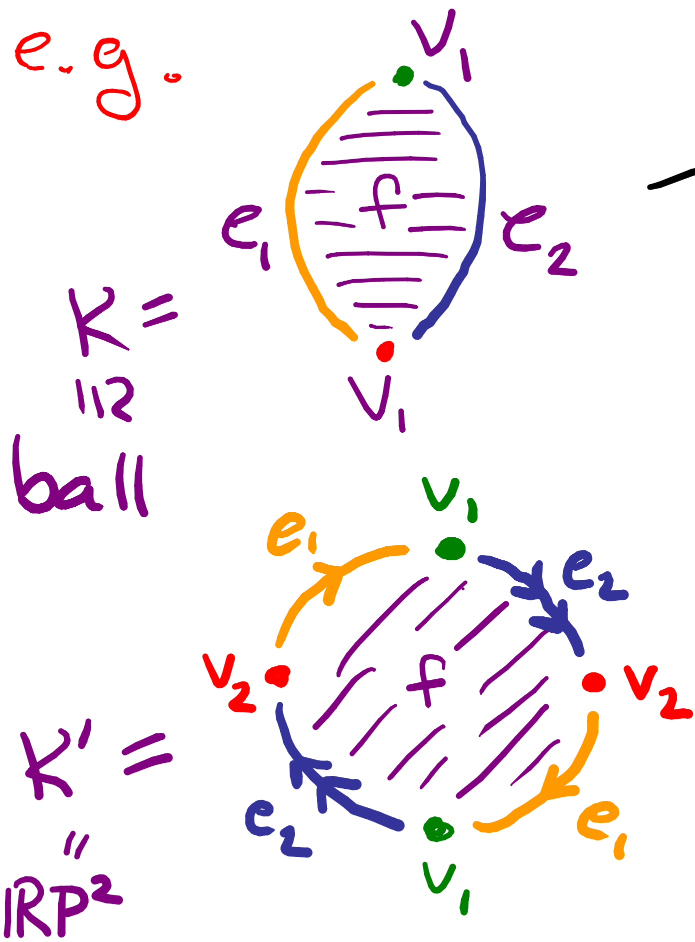
$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

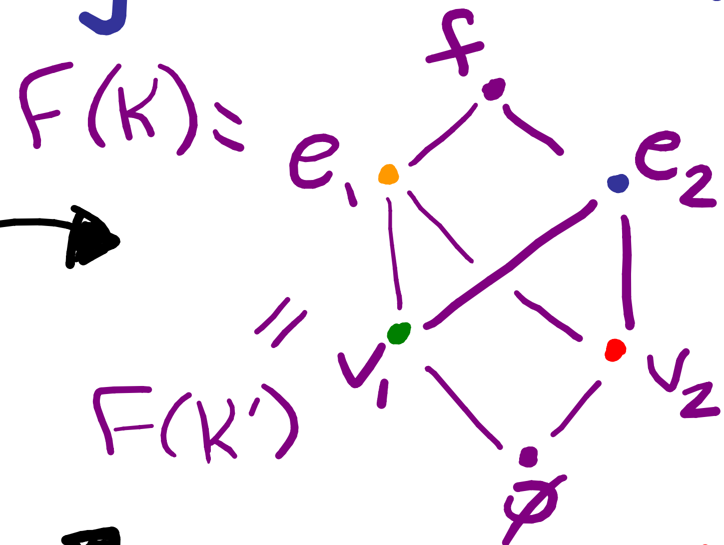
simplex faces w/ same image "$x_1^2 = x_1$"

e.g. $\{x_1(t) \mid t>0\} = \{x_1(t_1)x_1(t_2) \mid t_1,t_2>0\}$

CW Complexes & their Face Posets (Partially Ordered Sets)

e.g.





"face poset"

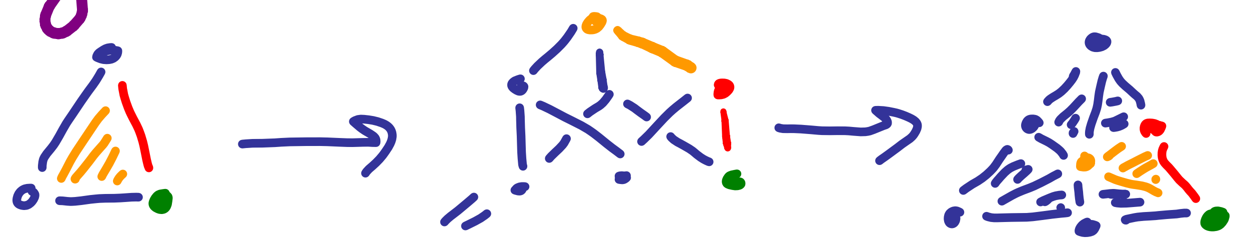
$(u \leq v \iff u \leq \bar{v})$

<u>Recall</u>: A CW complex: cells $e_\alpha \cong \mathbb{R}^{\dim(\alpha)}$, characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \to \bigcup_{\beta \subseteq \bar{\alpha}} e_\beta$

& attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

Recall: CW complex is regular if each f_α is homeomorphism.

- $\Delta(P)$ = "nerve" or "order complex" of P
- K regular $\Rightarrow K \cong \Delta(F(K) \setminus \emptyset) = sd(K)$

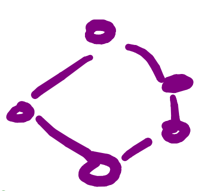


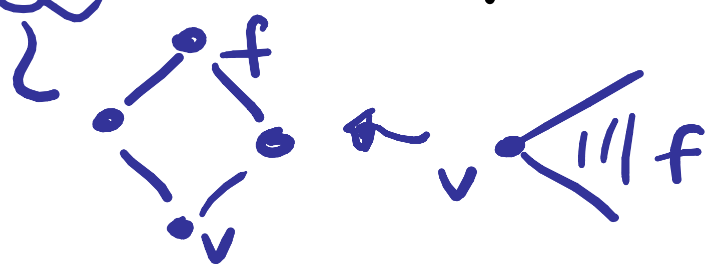
$K = \qquad F(K)\setminus\emptyset = \qquad \Delta(F(K)\setminus\emptyset)$

"nerve"

- a CW poset is any face poset of regular CW complex
- "Shellable" + "Graded" + "thin" $\Rightarrow$ CW poset

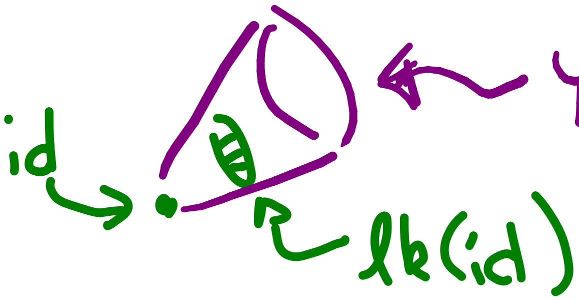
e.g. not





e.g. Bruhat order (Björner-Wachs; Matthew Dyer)

Fomin-Shapiro Conjecture: The Bruhat stratification of $lk(id)$ in totally nonneg. real part of unipotent radical in Borel in algebraic group (e.g. $im(f_{(i_1,\ldots,i_d)})$) is regular CW complex homeom. to closed ball (with Bruhat order as face poset).



$$Y_w = \left[\overline{B^- w B^-} \cap \begin{pmatrix}\text{unipotent} \\ \text{subgp of } B\end{pmatrix}\right]_{\geq 0}$$

lower triangular opposite Borel B^-

permutation w

upper triang. w/ 1's on diagonal

totally nonneg. part

Bruhat Order

Closure poset $F(L)$ for Schubert cell decomposition L of flag variety $\mathcal{F}l_n = GL_n/B$ & "Schubert varieties" (over $\mathbb{C}$), namely for cell closures.

Likewise for G/B in other types.

e.g.
$$\overline{\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \\ 0 & * & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}} \supseteq \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right\}$$

$\updownarrow$ $\qquad\qquad\qquad\qquad$ $\updownarrow$

$\mathbb{R}^6 = \mathbb{C}^3$ cell indexed by $1432 \in S_4$ $\qquad$ 1342-indexed cell $\cong \mathbb{C}^2 = \mathbb{R}^4$

$\updownarrow$

flags $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ * \\ * \\ 1 \end{pmatrix} \right\rangle \subseteq \dots$

A Motivation Stated by Fomin & M. Shapiro

"It should be mentioned that one of our 'hidden motivations' has been the desire to better understand the combinatorics of Kazhdan-Lusztig polynomials".

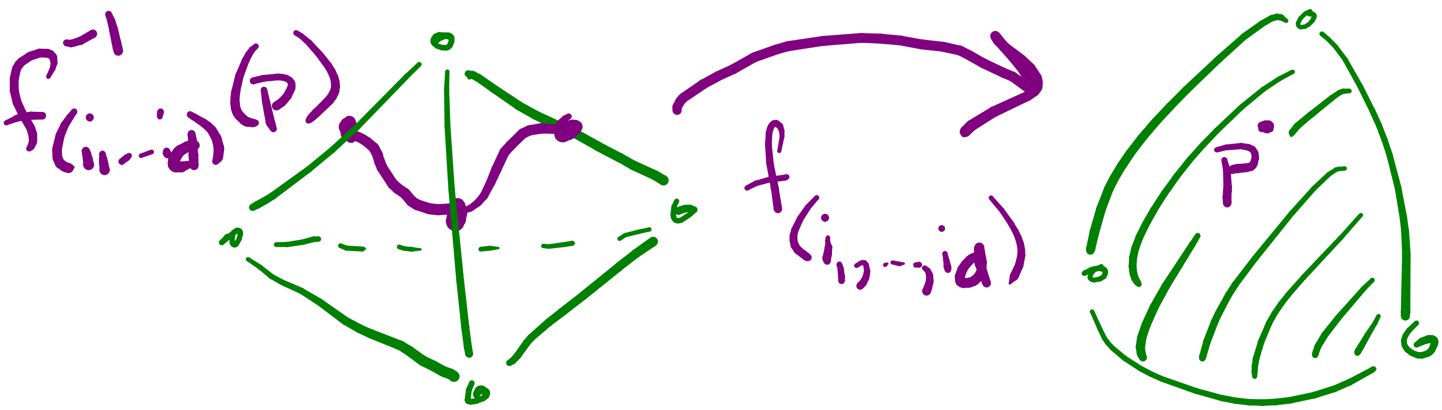
Their Results: face poset, homological results, fascinating projection map & much more!

Theorem (H., 2014, Invent.):
Fomin-Shapiro Conjecture holds.

Special Case (Running example):
Space of totally nonnegative upper triangular matrices with 1's on diagonal & entries just above diagonal summing to fixed positive constant, stratified by which minors are positive & which are 0.

Concrete Realization: Products $x_{i_1}(t_1) \cdots x_{i_d}(t_d)$ of elementary matrices, by results of Whitney, Loewner & (beyond type A) Lusztig.

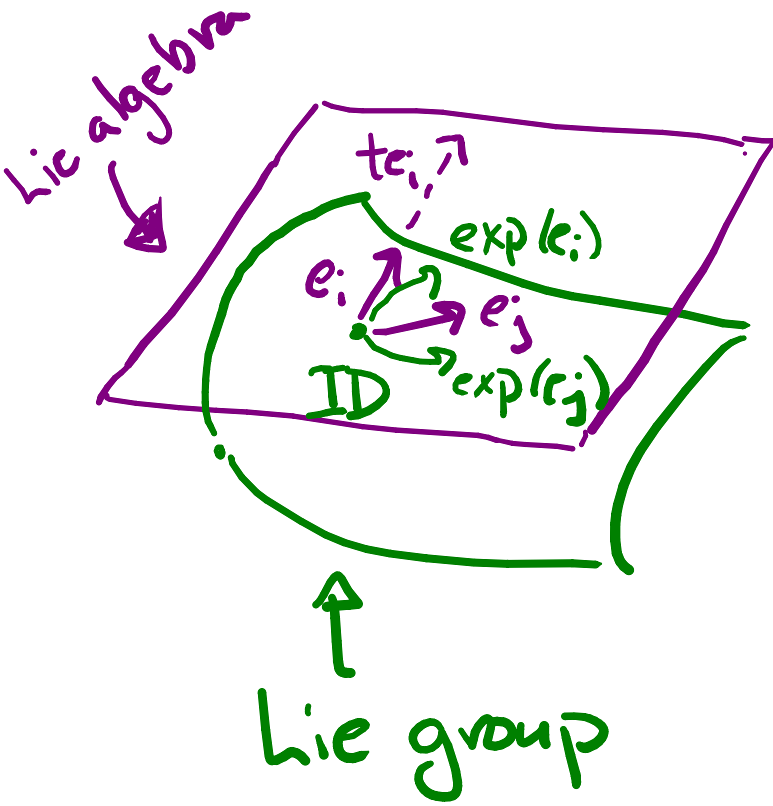
Conjecture (Davis-H-Miller): $f^{-1}_{(i_1,\dots,i_d)}(p)$ is regular CW complex homeomorphic to interior dual block complex of subword complex $\Delta((i_1,\dots,i_d),w)$ for $p \in Y_w^o$.



Thm (DHM): $f^{-1}_{(i_1,\dots,i_d)}(p)$ has cell decomposition & "correct" face poset.

Thm (DHM): Interior dual block complex of $\Delta((i_1,\dots,i_d),w)$ is contractible.

A Motivation to Study Fibers: Relations Among (Exponentiated) Chevalley Generators



$$te_i = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\exp(\_)} \begin{pmatrix} 1 & t & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\exp(t_1 e_i)\exp(t_2 e_j)$$

$$f_{(i,j)}(t_1, t_2) = x_i(t_1)\, x_j(t_2)$$

$$\exp(te_i) = \boxed{ID + te_i} + \underbrace{t^2 \frac{e_i^2}{2}}_{0=} + \underbrace{t^3 \frac{e_i^3}{6}}_{0=} + \ldots$$

Reln's $\Leftrightarrow$ Elts in same fiber of $f_{(i_1, \ldots, i_d)}$

Subword Complexes

(introduced by Allen Knutson & Ezra Miller)

$\Delta(\underset{\sim}{Q}, \underset{\sim}{w})$ = (abstract) simplicial complex whose faces are subwords Q' of Q whose complement $Q \setminus Q'$ contains a reduced word for w

Q: reduced or nonreduced word

w: Coxeter group element

Aside:

1st arose as Stanley-Reisner complexes of initial ideals of coordinate rings of matrix Schubert varieties.

Thm (Knutson-Miller): $\Delta(Q,\omega)$ is homeomorphic to ball or sphere.

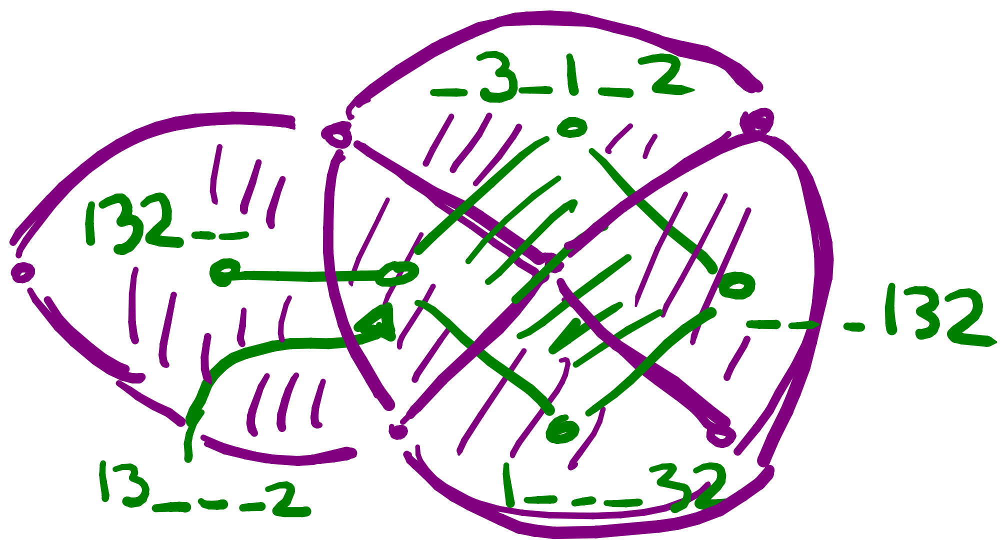
Example of Subword Complex & its Interior Dual Block Complex

$\Delta(Q,\omega) =$

$Q = 132132$

$\omega = s_1 s_3 s_2$

& its



Interior dual block complex

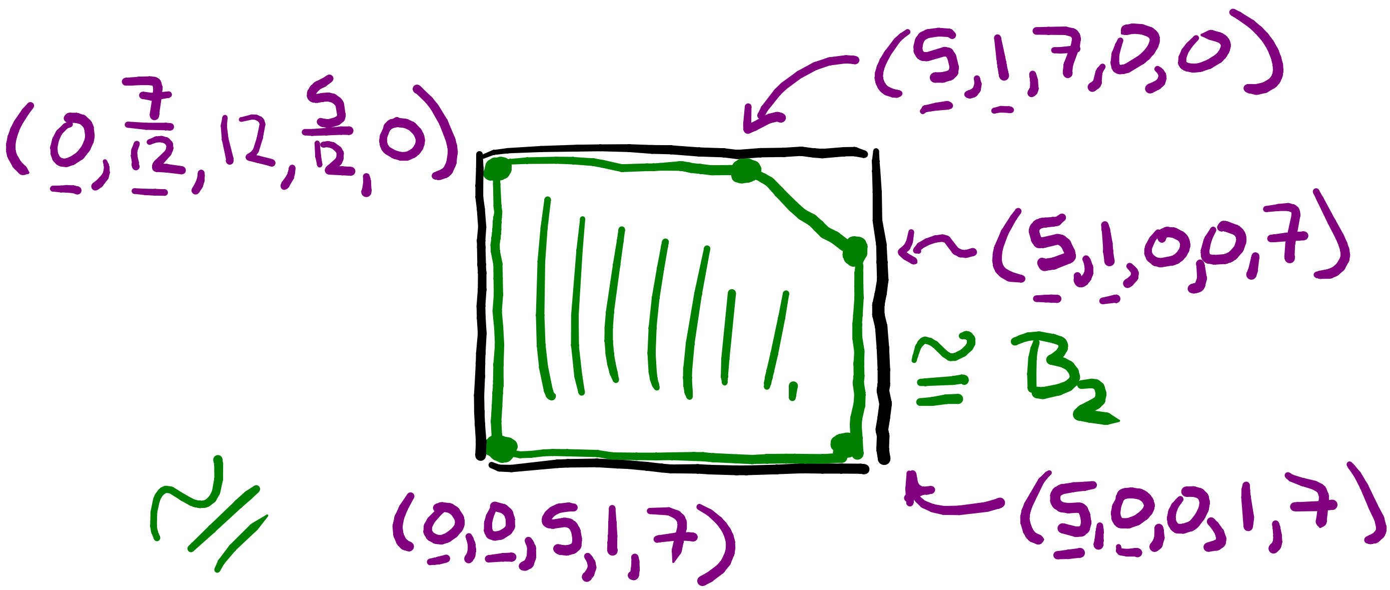
$$\| $$

$f^{-1}_{(1,3,2,1,3,2)}(M)$ for $M \in Y^{\circ}_{(1,3,2)}$

Examples of Fibers:

(with realizations as suggested by various results towards proof of DHM conjecture)

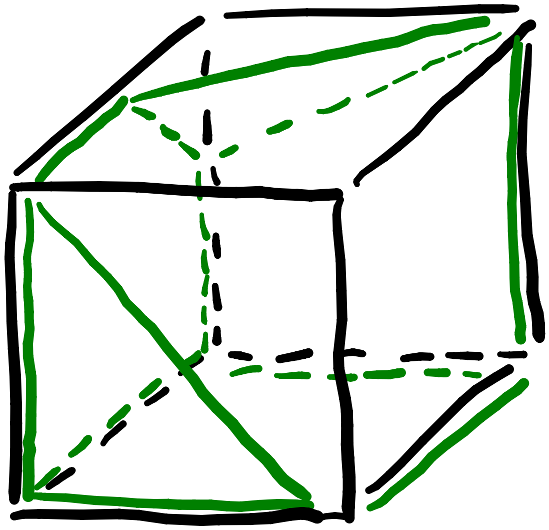
e.g.



$f^{-1}_{(1,2,1,2,1)}(M)$ for $M = x_1(5)x_2(1)x_1(7)$

$\cap$

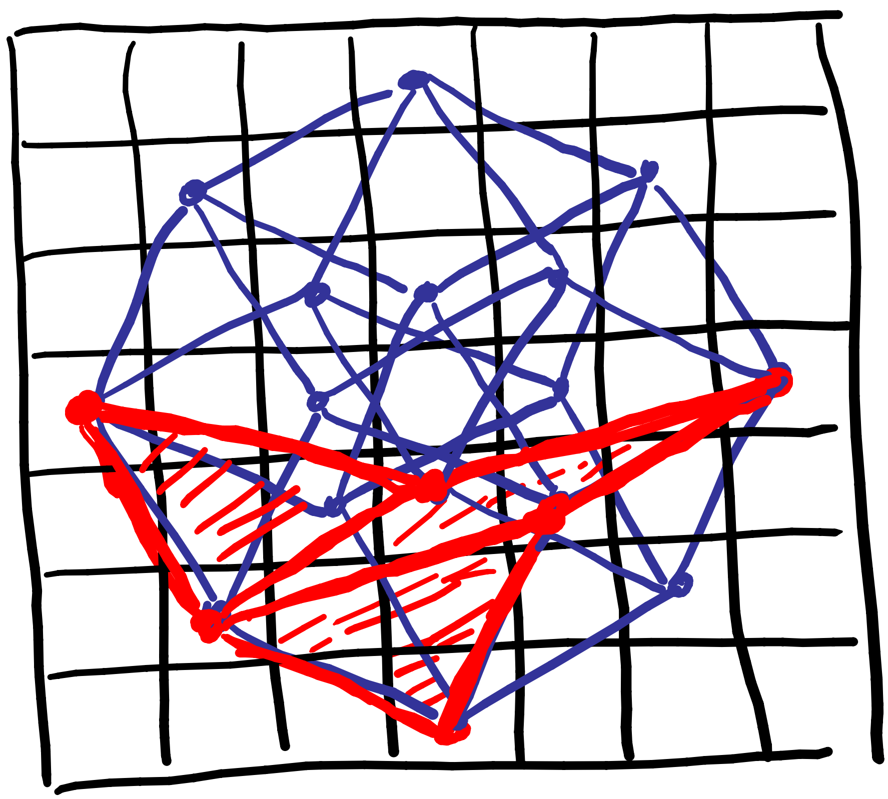
$U^{o}_{s_1s_2s_1}$



$\cong B_3$

$\cong$

$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^{\circ}_{s_1 s_2 s_1}$



$\cong$

$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^{\circ}_{(1,2)}$

Role of 0-Hecke Algebra in Stratification for $\mathrm{im}(f_{(i_1,\dots,i_d)})$

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

$\rightsquigarrow$ suppress parameters

$x_i x_i = x_i$

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2 t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1 t_2}{t_1+t_3}\right)$

for $t_1, t_2, t_3 > 0$

$\rightsquigarrow$ (type A)

$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$

(& analogous relations outside type A)

Upshot: $\mathrm{im}(f_1) = \mathrm{im}(f_2) \iff \underbrace{x(f_1) = x(f_2)}_{\text{equal as 0-Hecke algebra elements}}$

Thm (Lusztig): If $(i_1,\dots,i_d)$ is reduced, then $f_{(i_1,\dots,i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

More Motivation for Nonnegative Real Part of Unipotent Radical

- Given quantized env. alg. $U = U^- \otimes_{\mathbb{Q}(v)} U^0 \otimes_{\mathbb{Q}(v)} U^+$ of Kac-Moody alg. (e.g. affine Lie alg.), then canonical basis is a basis B for U^- such that highest weight module with highest weight vector v_λ has basis $\{v_\lambda b \mid b \in B, v_\lambda b \neq 0\}$ for each λ.

- $f^{-1}_{(i_1, \dots, i_d)}(p) \to f^{-1}_{(j_1, \dots, j_d)}(p)$ coordinate change

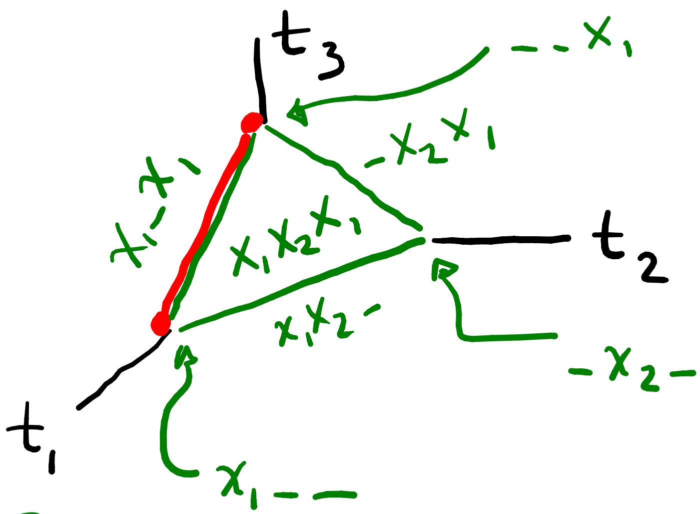
$$(t_1, t_2, t_3) \longmapsto \left(\frac{t_2 t_3}{t_1 + t_3}, t_1 + t_3, \frac{t_1 t_2}{t_1 + t_3}\right)$$

tropicalizes to coordinate change:

$$(a, b, c) \mapsto (b + c - \min(a,c), \min(a,c), a + b - \min(a,c))$$

for canonical bases w/ same braid move

Faces of (Preimage) Simplex as Subexpressions in 0-Hecke Algebra



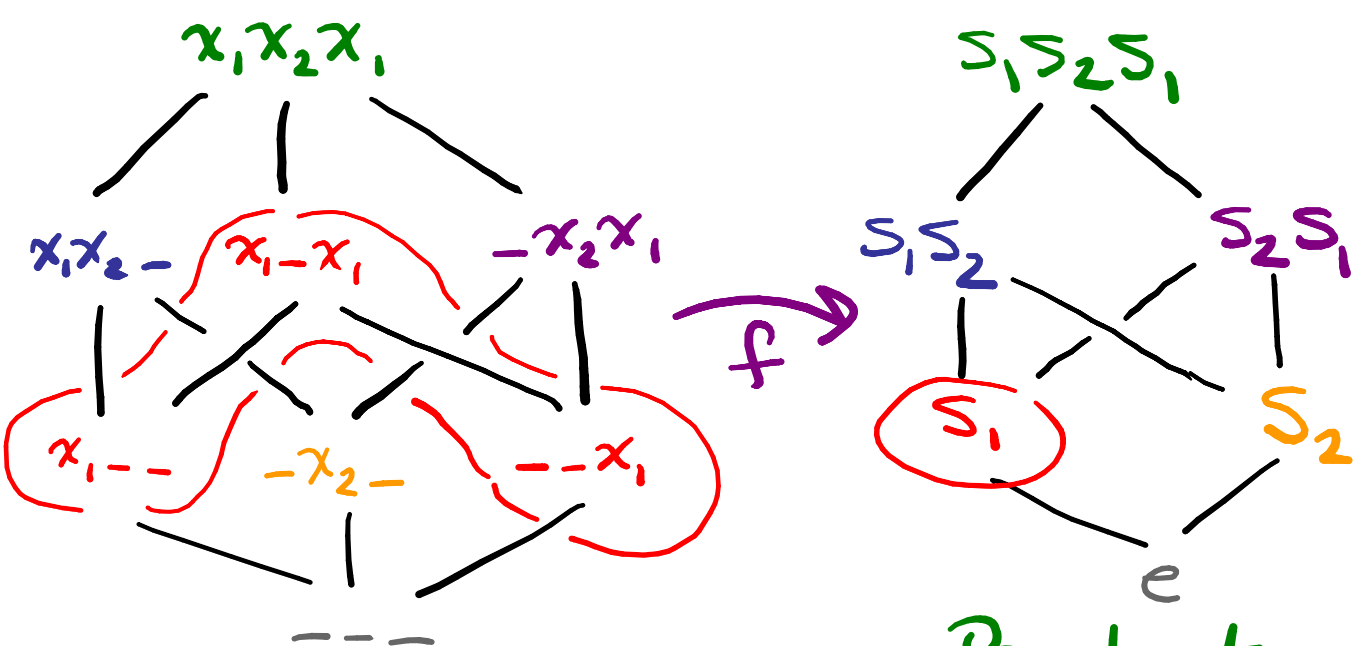
- let Y_w^o = open cell in $\mathrm{im}(f_{(i_1 \ldots i_d)})$ indexed by $w \in W$

- let $\delta(x_{i_1} \ldots x_{i_d})$ denote (unsigned) 0-Hecke algebra product (a.k.a. "Demazure product")

e.g. $\delta(\underbrace{x_1x_2x_1}_{x_2x_1x_2}x_2x_1) = \delta(x_2x_1x_2x_1) = s_1s_2s_1$

since $x_2x_2 = x_2$

Map of Face Posets Induced by Map $f_{(i_1,\dots,i_d)}$ of Spaces



Boolean lattice B_n = Face poset of simplex

Bruhat order = $F(\mathrm{im}(f_{(i_1,\dots,i_d)}))$

$$f: x_{i_{j_1}}\cdots x_{i_{j_r}} \longmapsto \delta(x_{i_{j_1}}\cdots x_{i_{j_r}})$$

Proof of Fomin-Shapiro Conjecture

(H., 2014, Invent.)

Set-up: Surjective fn $f_{(i_1,\dots,i_d)}: \Delta_{d-1} \to Y$

s.t. $f_{(i_1,\dots,i_d)}\big|_{\mathrm{int}(\Delta_{d-1})}$ homeom. to $\mathrm{int}(Y)$.

Step 1: Perform "cell collapses" on $\partial(\Delta_{d-1})$ designed to preserve homeom. type & regularity via cont. surjective fns $g_i: \Delta_{d-1} \to \Delta_{d-1}$ yielding $(\Delta_{d-1})/\sim$ s.t.

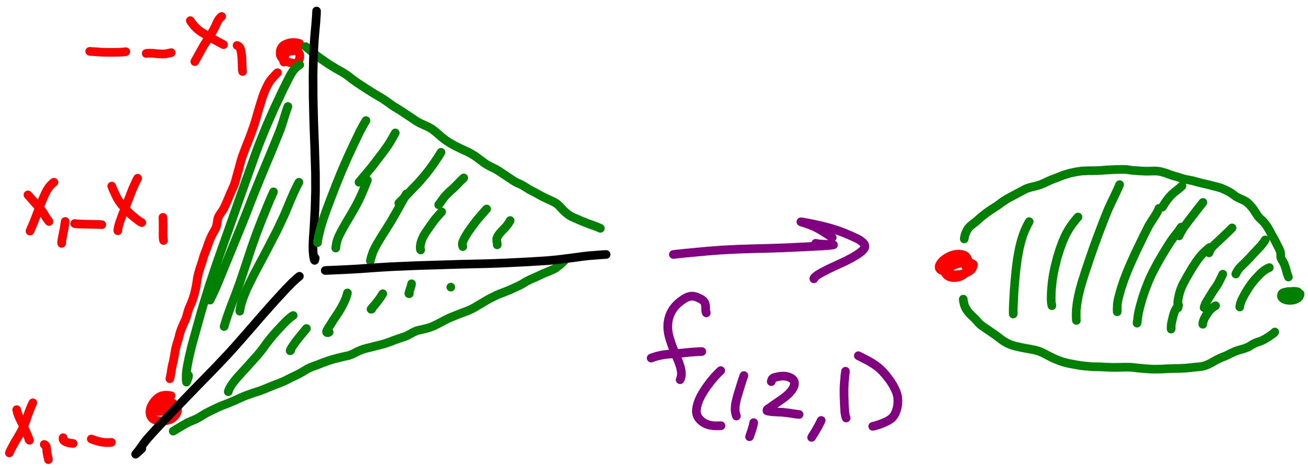
$$x_1 \sim x_2 \Rightarrow f_{(i_1,\dots,i_d)}(x_1) = f_{(i_1,\dots,i_d)}(x_2)$$

(eliminating faces given by nonreduced words)

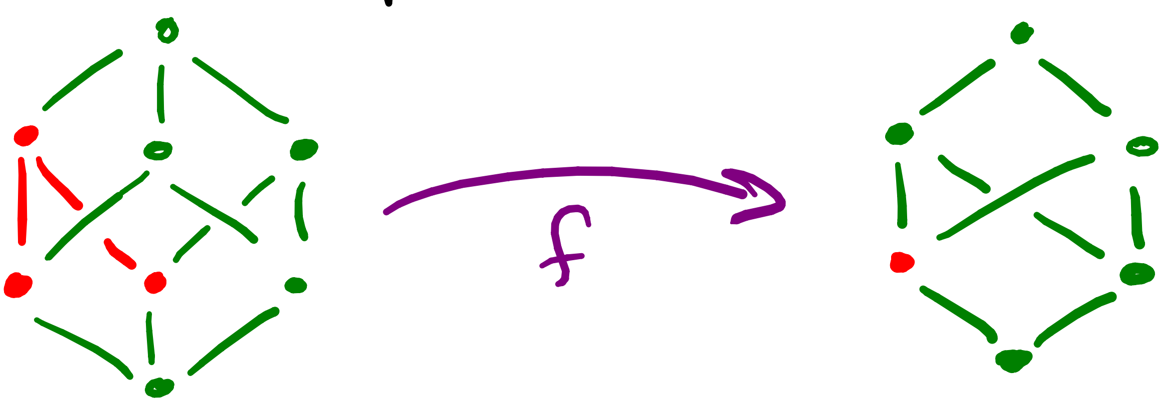
Step 2: Prove $\overline{f}_{(i_1,\dots,i_d)}: (\Delta_{d-1})/\sim \;\to Y$ is a homeomorphism via new regularity criterion for finite CW complexes.

Corollary of Proof: Contractibility of Fibers

Combinatorics of Fibers



Induced map of face posets:



Thm (Armstrong-H., 2011): For each $u \in W$, $f^{-1}_{\geq}(u) = \{x \in B_n \mid f(x) \geq u\}$ is dual (i.e. upside-down) to face poset for subword complex $\Delta((i_1, \dots, i_d), u)$.

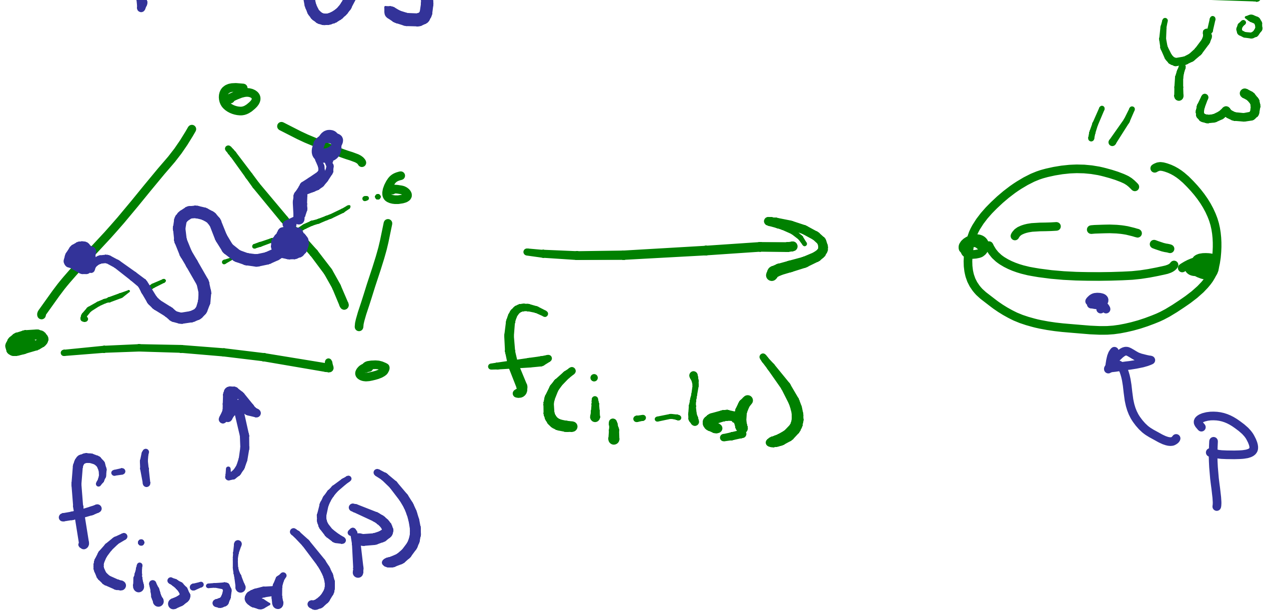
<u>Thm</u> (DHM, 2018): $f_{=}^{-1}(u)$ is face poset of interior dual block complex for subword complex $\Delta((i_1,\ldots,i_d),u)$

<u>Thm</u> (DHM, 2018): Interior dual block complex of $\Delta((i_1,\ldots,i_d),u)$ is contractible.

<u>Pf</u>: Discrete Morse theory

<u>Combining</u>: DHM Conjecture would imply $f^{-1}_{(i_1,\ldots,i_d)}(p) \cong$ interior dual block complex of $\Delta((i_1,\ldots,i_d),u)$ for $p \in Y_u^{o}$, hence $f^{-1}_{(i_1,\ldots,i_d)}(p)$ contractible.

Topology of Fibers



Thm (Davis-H-Miller): Each fiber $f^{-1}_{(i_1,\ldots,i_d)}(p)$ admits a cell decomposition induced by the natural cell decomposition of the simplex Δ_{d-1}.

Proof: Parametrization + continuity lemmas

Topology of Fibers: Key Lemmas

Defn (Davis-H-Miller): The letter x_{i_1} is **redundant** in $x_{i_1} \cdots x_{i_d}$ if $\delta(i_1, \dots, i_d) = \delta(i_2, \dots, i_d)$

Lemma (Davis-H-Miller): x_{i_1} is non-redundant in $x_{i_1} \cdots x_{i_d}$

$\Leftrightarrow$ $f^{-1}_{(i_1 \dots i_d)}(p)$ for $p \in Y^{\circ}_{\delta(i_1 \cdots i_d)}$

$$\{(t_1, \dots, t_d) \mid x_{i_1}(t_1) \cdots x_{i_d}(t_d) = p\}$$

has unique value k_1 for t_1.

$\Leftrightarrow$ $f^{-1}_{(i_1 \cdots i_d)}(p) \cong f^{-1}_{(i_2 \cdots i_d)}(x_{i_1}(-k_1) p)$

Lemma (Davis-H-Miller): Given $(t_1, \ldots, t_d) \in f^{-1}_{(i_1,\ldots,i_d)}(p)$ with $t_1 > 0$ and x_{i_1} redundant, then $\exists (t_1', \ldots, t_d') \in f^{-1}_{(i_1,\ldots,i_d)}(p)$ for every $t_1' \in [0, t_1]$.

e.g. $M = \begin{pmatrix} 1 & \pi & e \\ & 1 & 14 \\ & & 1 \end{pmatrix}$ then $f^{-1}_{(1,2,1,2)}(M)$

$= \{(t_1, t_2, t_3, t_4) \mid x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4) = M\}$

achieves every $t_1 \in [0, \frac{e}{14}]$.

Parametrization: Given $(i_1,\dots,i_d)$, consider rightmost subword that is reduced word for $w=\delta(i_1,\dots,i_d)$

e.g. $(1,2,1,2,1,2,1)$

Parametrize $f^{-1}_{(1,2,1,2,1,2,1)}(p)$ in $Y^{\circ}_{s_1 s_2 s_1}$

$$x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4)x_1(t_5)x_2(t_6)x_1(t_7)$$

"free" parameters "dependant" parameters

Thm (Davis-H-Miller):

$$[0,1)^{\dim(\sigma)} \cong \bigcup_{v \le \bar{\tau} \le \bar{\sigma}} \tau$$

source vertex of $\bar{\sigma}$

Totally Nonnegative Spaces w/ Seemingly Analogous Structure

1. Totally nonnegative real part of Grassmannian: $Gr_{\geq 0}(k,n) = (Gl_n/P)_{\geq 0}$

Postnikov: polytope of "plabic graphs" w/ "measurement map" to $Gr_{\geq 0}(k,n)$ + theory of (reduced) plabic graphs

Postnikov-Speyer-Williams: $Gr_{\geq 0}(k,n)$ is CW complex (via attaching maps that are not homeomorphisms)

Galashin-Karp-Lam 2017 preprint: $Gr_{\geq}(k,n)$ is homeom. to closed ball.

2. Totally nonnegative real part of flag variety $\mathcal{F}\ell_{\geq 0} = (GL/B)_{\geq 0}$

Rietsch: poset of closure relns. Cells $R^\circ_{u,v}$ given by $u \leq v$ in Bruhat order

Marsh-Rietsch: parametrization for $R^\circ_{u,v}$

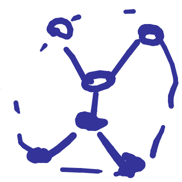
Williams: poset is CW poset

Rietsch-Williams: (1) CW complex w/ attaching maps via canonical bases.
(2) Contractibility of each cell closure

Galashin-Karp-Lam: Homeomorphism type for closure of "big cell"

3. Stratified Spaces of Electrical Networks

(Curtis-Ingerman-Morrow, Kenyon-Wilson, ...)

• Image of map Resp :  $\mapsto$ response matrix

Lam: (1) $E_n \hookrightarrow Gr_{\geq 0}(n-1, 2n)$

(2) $F(E_n)$ is "Eulerian" ("i.e. correct Euler char")

H-Kenyon: $F(E_n)$ shelling & $F(E_n)$ is CW poset (uses Dyer Bruhat shelling)

H-K Cor: shelling for each $[u,v]$ in face poset for edge-product space of phylogenetic trees (so CW poset)

Galashin-Karp-Lam: Homeom. type for closure of "big cell" for graph analogue of w_0, "well-connected graph"

4. (Deconed) Totally nonneg. real part unipotent radical of Borel in semisimple simply connected algebraic group

(a) Vertex "link" (nbhd) in $(\mathcal{F}l_n)_{\geq 0}$ via Marsh-Rietsch parameters $\to 0$

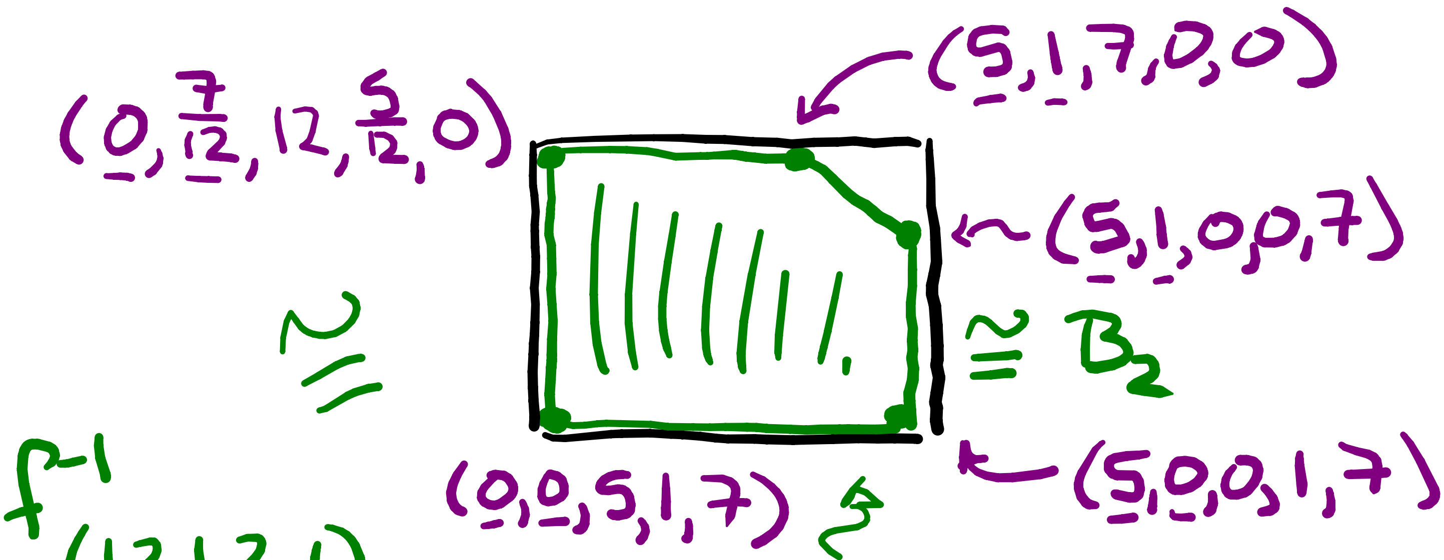
(b) Image$(f_{(i_1,\dots,i_d)})$ for $w_0 = w(i_1,\dots,i_d)$

(c) Bruhat order as poset of closure rel'ns

(d) Each cell closure homeom. to closed ball.

- known as Fomin-Shapiro Conjecture
- proved by H., 2014, Invent.
- special case of w_0 in type A (shorter proof) GKL, 2017

Happy Birthday, Sergey!



An associahedron!

– and thanks for making so many interesting mathematical associations!

Appendix: Further Remarks & Details

Thm (DHM): DHM Conjecture $\Rightarrow$ 2nd (or 3rd?) proof of FS-Conjecture

Idea: Shellability of Bruhat Order + (Well Known) Topological Relationship

Fibers to Image: Let $g: B \to Z$ be continuous surjection from ball B to Hausdorff space Z whose restriction to $\text{int}(B)$ is an embedding. Suppose also:

(1) $g(\partial B) \cong \partial B = S^n$

(2) $g(\partial B) \cap g(\text{int}(B)) = \emptyset$

(3) $g^{-1}(p)$ is contractible $\forall p \in g(\partial B)$

Then $Z \cong B$.

Ingredients in this Relationship Between fibers & Image

- "CE-Approximation Theorem (Kirby-Siebenmann (all dim's); Quinn (dim 4); Armentrout (dim 3)

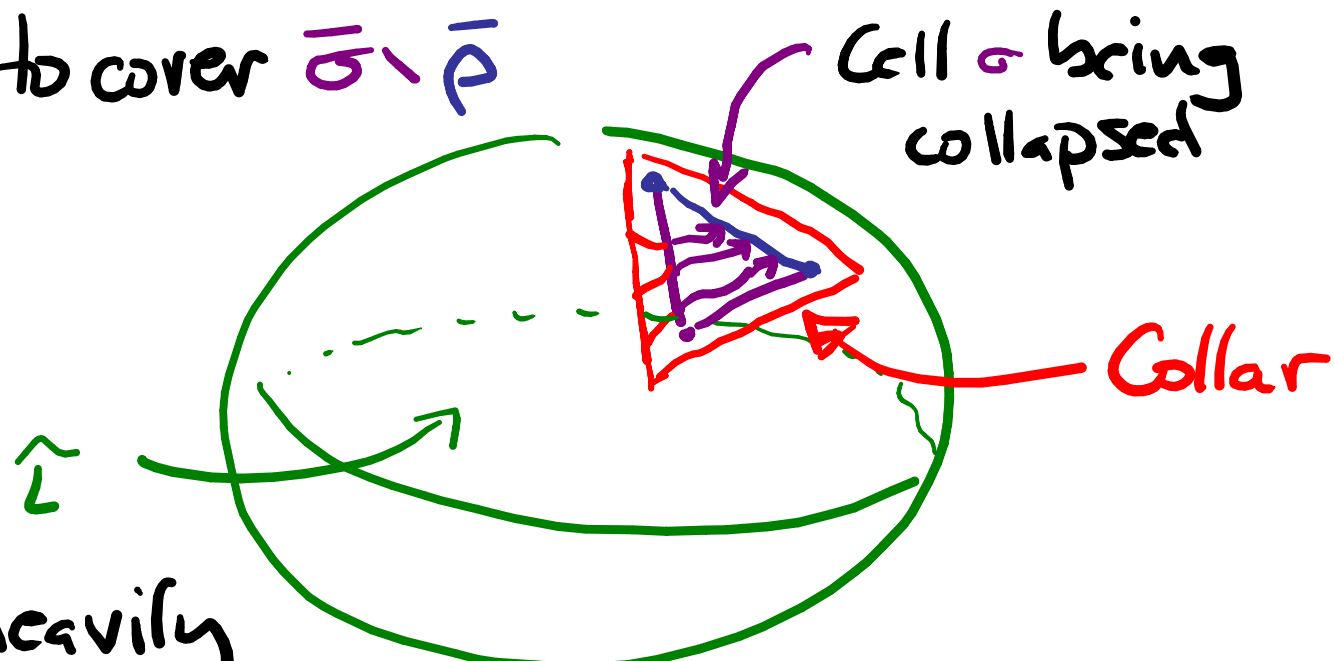
  - $g: \partial B \to \partial B$ as above may approximated by homeomorphisms

- Local Contractibility of $\mathrm{Homeo}^+(S^n, S^n)$

  - any two homeomorphisms "close enough" to each other connected by path of homeomorphisms

Step 1: Cell Collapses Preserving Homeom. Type, Regularity & Topol. Manifold

- Cover each non-reduced σ with "nice" curves each in single fiber of $f_{(i_1,\ldots,i_d)}$
- Collapse $\overline{\sigma}$ onto $\overline{\rho} \subseteq \partial\sigma$ by stretching curve extensions in "collar" for $\partial Z \setminus \sigma$ to cover $\overline{\sigma} \setminus \overline{\rho}$



◈ heavily uses combinatorics of reduced & nonreduced words in 0-Hecke algebra

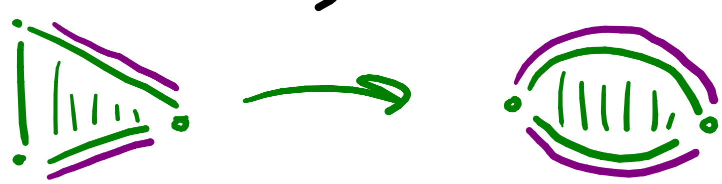
Step 2 via: New Regularity Criterion

Preparatory Lemma (H.): Let K be a finite CW complex w/ characteristic maps $\{f_\alpha\}$. Suppose:

(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example: 

(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example: 

Then $F(K)$ is graded by cell dimension.

Insightful feedback (Quinn): Next theorem "spreads around" injectivity requirement.

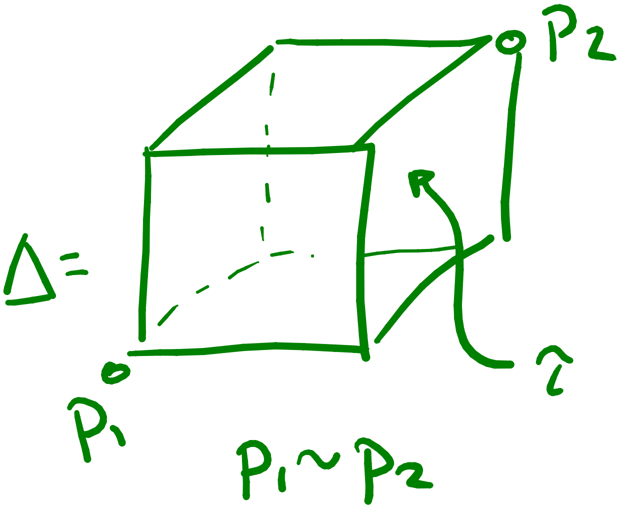
Thm (H.) Let K be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then K is regular w.r.t. $\{f_\alpha\}$ $\iff$

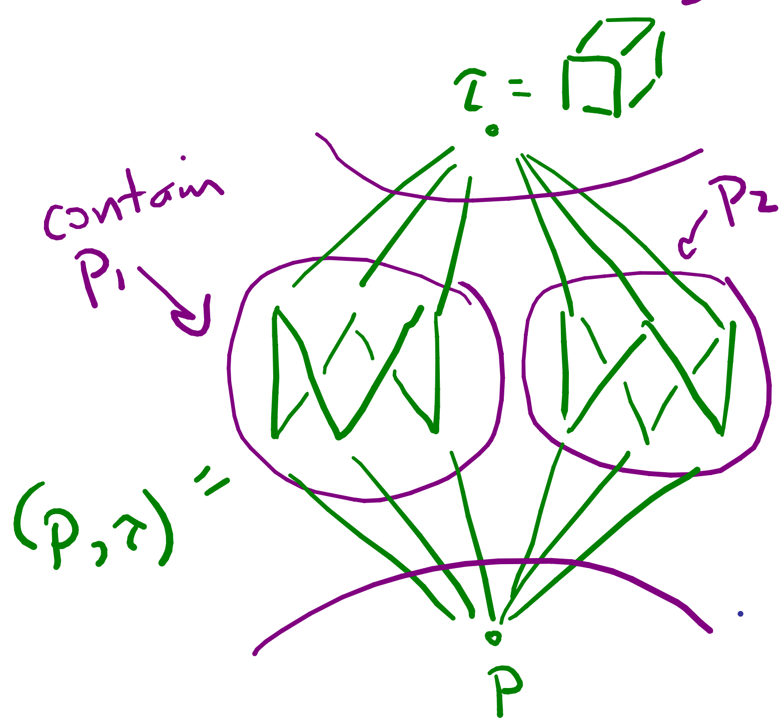
(1) K meets requirements of prop'n for $F(K)$ to be graded by cell dim.

(2) $F(K)$ is thin and each open interval (u,v) for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

Non-Example





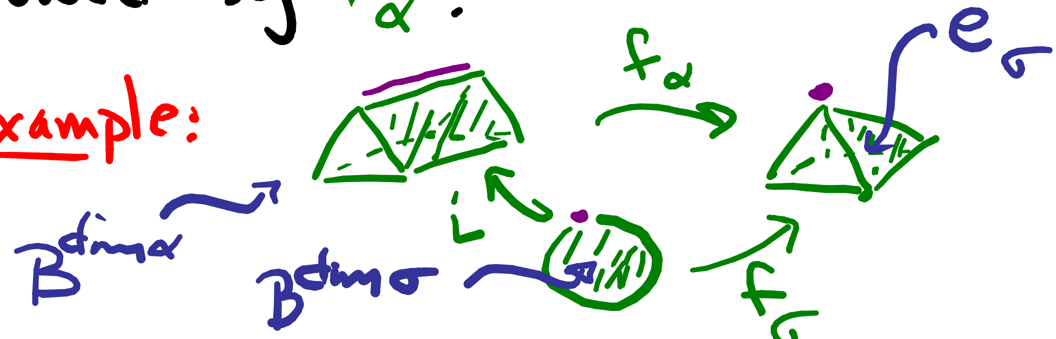
(3) For each α, the restriction of f_α to preimages of codim. one cells in $\overline{e_\alpha}$ is injective.

(topological condition)

Non-Example:

(4) $\forall\, e_\sigma \subseteq \overline{e_\alpha}$, f_σ factors as continuous inclusion $\iota : B^{\dim \sigma} \to B^{\dim \alpha}$ followed by f_α.

Non-Example:



Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one.

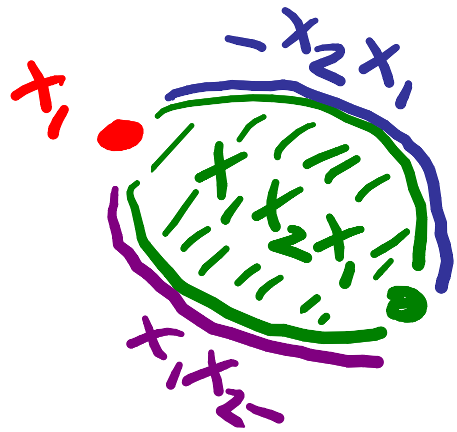
Proof: Induction on difference in dim.

Confirming Codimension One Injectivity via Exchange Axiom for Coxeter Groups

$$s_1 s_2 s_1$$

$$\_\, s_2 s_1 \qquad \cancel{s_1 \_\, s_1} \qquad s_1 s_2\, \_$$

non-reduced



- reduced subexpressions of reduced expression obtained by deleting one letter give distinct $u \in W$
- Need codim one! e.g. $s_1 s_2 s_1$

$$s_1\, \_\, \_ \qquad \_\, \_\, s_1$$

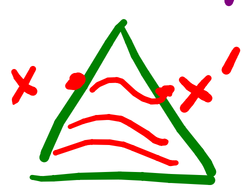
(Mainly Combinatorial) Conditions Allowing Such Face Collapses Across Curves

$$K_0 \xrightarrow{g_1} K_1 \xrightarrow{g_2} K_2 \to \cdots \to K_i$$

$$K_0 = \Delta, \quad K_1 \cong \Delta/\sim_1, \quad K_1 \cong \Delta, \quad K_2 \cong \Delta/\sim_2$$

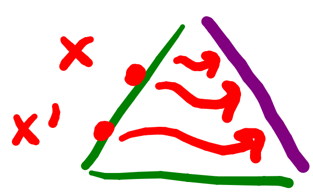
polytope!

- collapse face in K_i across images of parallel line segments in K_0 satisfying:
  - Distinct endpoints condition (DE):



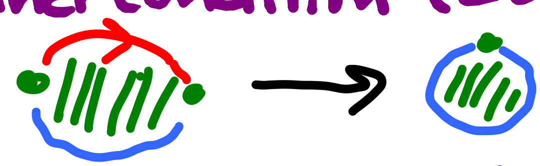
$x \neq x'$

  - Distinct initial points condition (DIP):



$x \neq x'$

  - Least upper bound condition (LUB) to preclude



(conditions checkable via 0-Hecke algebra)

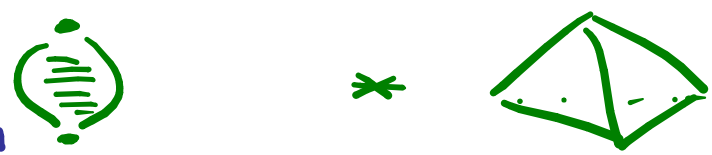
Additional Key Challenges

1. 0-Hecke algebra lacks inverses & lacks cancellation law.

   Key Idea: find ways to transfer properties from Coxeter gp

2. Need change-of-coord's for braid moves as homeom's on closed cells

   Key Idea: induction by embedding smaller instance



im$(f_{(i_1 \ldots i_{d'-1})})$

$$(\underbrace{i_1, i_2, i_3, \ldots\ldots, i_{d'-1}}_{\text{all possible collapses done}}, \underbrace{i_{d'}, \ldots, i_d}_{\text{not in any collapses yet}})$$

3. Need maps to extend to full complex

4. Tricky combinatorics to verify LUB at each collapsing step.