Topological Combinatorics of Posets & Stratified Spaces

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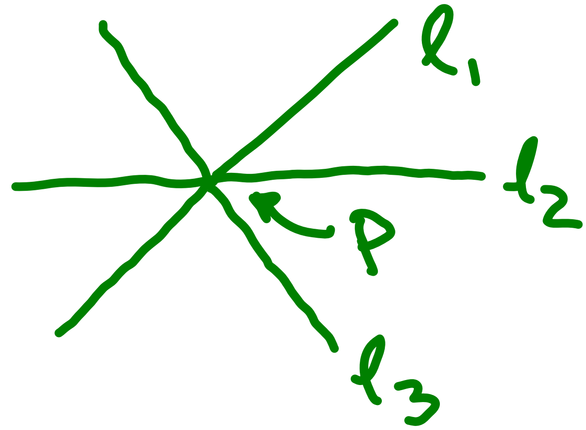
Lecture 1: Möbius fns & Shellability
(Monday Nov 7, 11am)

Lecture 2: Discrete Morse theory
(Friday Nov 11, 2pm)

Lecture 3: Stratified Spaces & face Posets
(Monday Nov 14, 11am)

Counting Topologically

e.g. "counting" points in the $\mathbb{R}^2$ complement of ⇝ yields:



$$\mathbb{R}^2 - \ell_1 - \ell_2 - \ell_3 + 2p$$

- Coefficients $1, -1, -1, -1, 2$ in such inclusion-exclusion counting formula given by "Möbius function" (upcoming)

Working over $\mathbb{F}_q$: $\#\text{pts} = q^2 - q - q - q + 2$

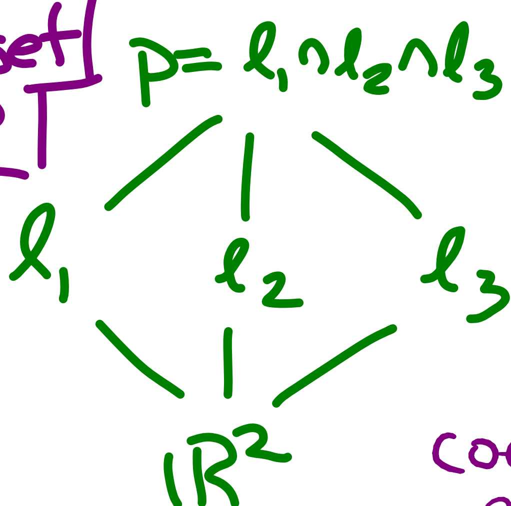
$$\| $$

$$\sum_{u \in L_A} \mu(\hat{0}, u)\, q^{\dim V - \mathrm{rk}(u)} =: \text{characteristic poly. of the arrangement}$$

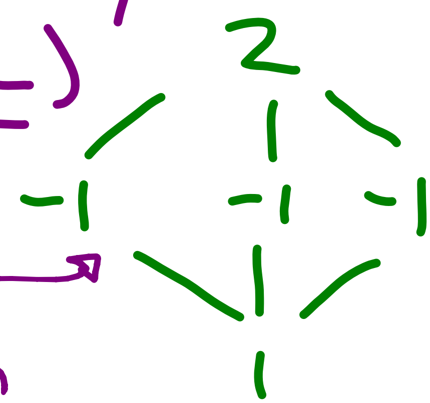
Defn: Möbius function $\mu_P(x,y)$ of partially ordered set (poset) P is defined recursively: $\mu_P(x,x)=1$

and $\mu_P(x,y) = -\sum_{x\le z<y} \mu_P(x,z)$ (for $x\neq y$) (so $\sum_{x\le z\le y} \mu_P(x,z)=0$)

Poset P



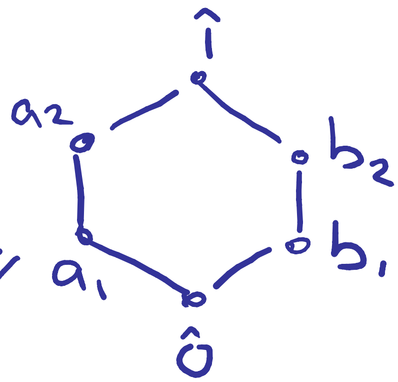
Its Mobius fn $\mu_P(\mathbb{R}^2, -)$

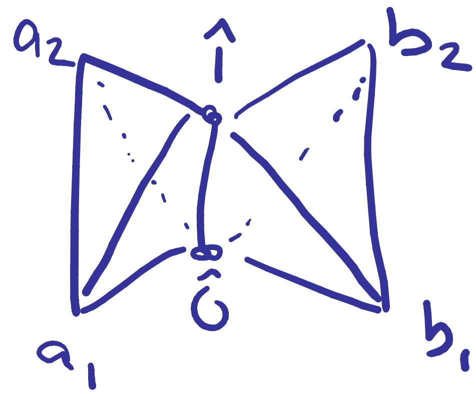


coefficients in $\mathbb{R}^2-\ell_1-\ell_2-\ell_3+2p$

- Good for detecting poset complexities & topol. structure; easy to compute!

Def'n: The order complex (or nerve) of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-"chains" $v_0 < \dots < v_i$ in P

e.g. $P =$ 

$\Delta(P) =$ 

Key Property (Hall; popularized by Rota):

$$\mu_P(x,y) = \tilde{\chi}(\Delta_P(x,y)) = -1 + \#\text{vertices} - \#\text{edges} + \#\text{2-faces} \dots$$

$$= -1 + \beta_0 - \beta_1 + \beta_2 - \dots$$

- $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$
- $(u,v) = \{z \in P \mid u < z < v\}$
- $u \lessdot v$ means $u < v$ & $\nexists z$ s.t. $u < z < v$
- saturated chains u to v := $u \lessdot \dots \lessdot v$

Techniques Yielding Möbius Functions (& Poset Topology)

- R-labeling (Stanley)

$\Rightarrow \mu_P(u,v) = \pm$ # "descending chains" u to v

- (Lexicographic) shellability
  - EL-labelings (Björner)
  - CL-labelings (Björner & Wachs)

$\Rightarrow \Delta(P)$ homotopy equiv. to wedge of spheres

- Lexicographic discrete Morse functions (Babson-H.)

(generalizes lex. shell. to more general topol. type - topic in 2nd lecture)

Some Applications (in Different Areas)

1. Finite group theory:

Thm (Shareshian): G solvable <=> subgroup lattice $L(G)$ shellable

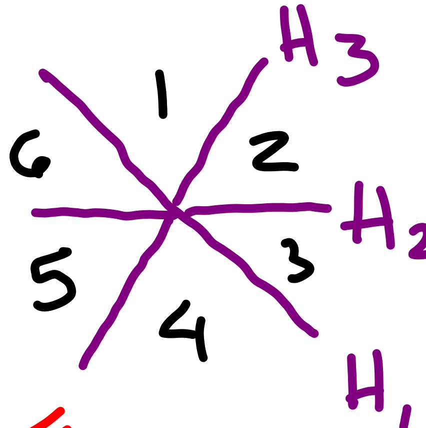
Thm (Stanley): G supersolvable <=> $L(G)$ "supersolvable" => shellable
=> char poly. factors linearly

2. Shellability of "geometric lattices" (Björner) & "geometric semilattices" (Wachs-Walker), yielding Möbius fns of "intersection posets" of hyperplane arrangements

} useful e.g. for ...

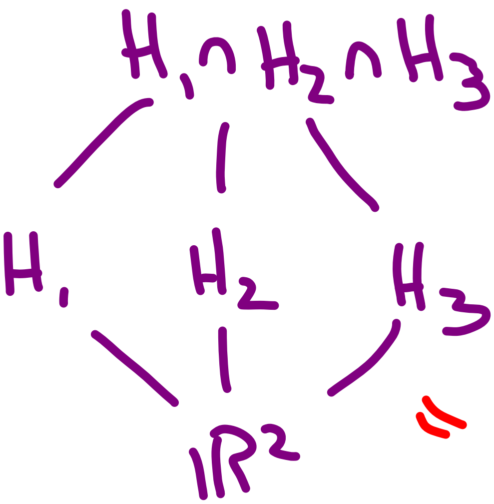
3. **Zaslavsky:** region counting formulas for the complement of $\mathbb{R}$-hyperplane arr't A

$A =$



$$\#\text{regions} = \sum_{u \in L_A} |\mu(\vec{0},u)|$$

$$\#\text{bdd regions} = \Big|\sum_{u \in L_A} \mu(\vec{0},u)\Big|$$



$= L_A =$ "intersection poset"

e.g. $\#\text{regions} = 1+3+2$

$\#\text{bdd regions} = 1-3+2$

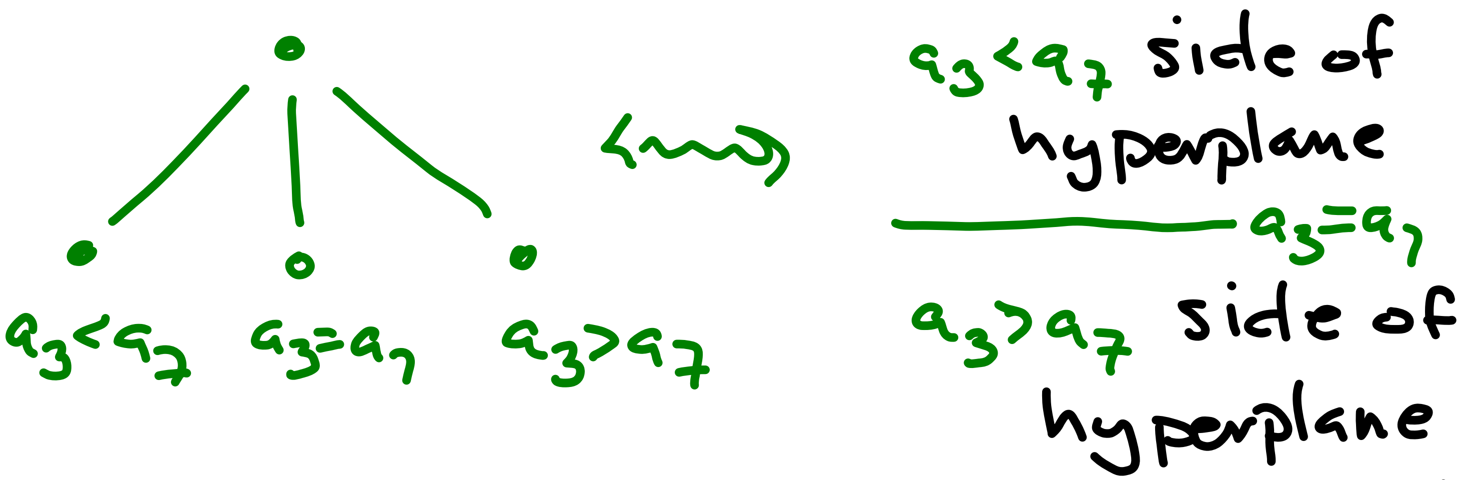
$\mu(\mathbb{R}^2,\mathbb{R}^2) = 1$

$\mu(\mathbb{R}^2, H_i) = -1$ for $i=1,2,3$

$\mu(\mathbb{R}^2, H_1 \cap H_2 \cap H_3) = 2$

(Foreshadows Goresky-MacPherson formula for subspace arr'ts)

4. Björner-Lovász: complexity theory lower bd of $O(n\log(\frac{2n}{k}))$ via Betti #'s for deciding if there are k equal coordinates in $\vec{a}=(a_1,a_2,\dots,a_n)\in\mathbb{R}^n$ by pairwise coord. comparisons



- lower bd on # leaves (hence depth) in "linear decision tree" via betti #'s of k-equal arr't complement

Remark: Topology well-suited for non-existence results (e.g. of algorithms)

Goresky-MacPherson formula

$$\tilde{H}^i(M_{\mathcal{A}}) \cong \bigoplus_{x \in L_{\mathcal{A}}^{>\hat{0}}} \tilde{H}_{\operatorname{codim}(x)-2-i}(\hat{0}, x)$$

subspace arr't complement

as groups

intersection semi-lattice

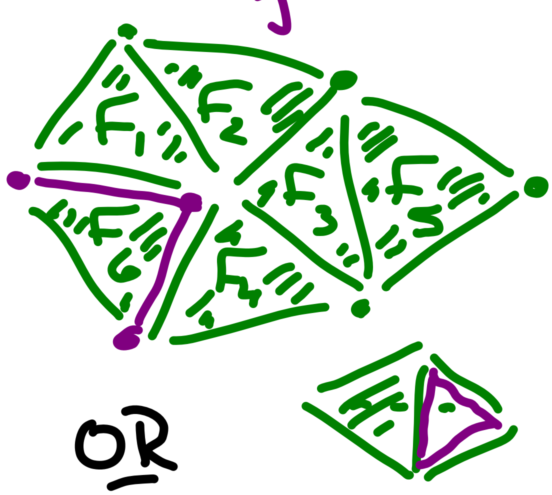
Pf: Stratified Morse theory

Usage by Björner-Lovász-Yao:

- The k-equal arrangement given by subspaces $x_{i_1} = \cdots = x_{i_k}$ for $1 \le i_1 < i_2 < \cdots < i_k \le n$ in $\mathbb{R}^n$ has Intersection lattice that is "nonpure shellable" w/ known Betti #'s
- Deciding if pt on arr't or not

Technique #1: Shellability

- simplicial complex is pure of dim. d if all maximal faces ("facets") are d-dimensional
- simplicial complex is shellable if there is total order $F_1, F_2, \ldots, F_k$, a shelling, on facets s.t. $\overline{F_j} \cap \left(\bigcup_{i<j} \overline{F_i} \right)$ is pure, codimension one subcomplex of $\overline{F_j}$ for each $j > 1$ (hence is $\partial \overline{F_j}$ or has a cone point).



- Each facet attachment preserves homotopy type or closes off a new sphere

Topological Consequences of Shellability

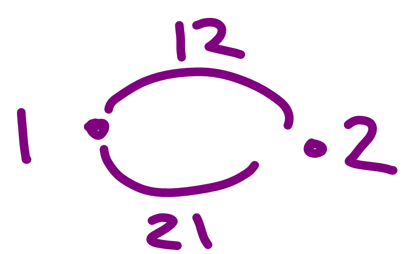
- homotopy type wedge of spheres w/ # i-dim'l spheres = # i-dim'l facets F_j s.t. $\overline{F_j} \cap \left(\bigcup_{i<j} \overline{F_i}\right) = \partial F_j$
- shellable + pure $\Rightarrow$ "homotopy Cohen-Macauley", i.e. each face F has $\text{link}_\Delta(F) \simeq \vee S^{\dim(\text{lk}_\Delta F)}$
- Δ shelling induces $\text{lk}_\Delta F$ shelling, so intervals $[u,v]$ of shellable posets are shellable

Example: Complex of Injective Words (Shelling by Björner & Wachs)

- cells $\longleftrightarrow$ injective words in alphabet $\{1,2,\ldots,n\}$
- $u \le v \iff u$ subword of v

Note: not a simplicial complex, but a **boolean cell complex**: regular + cell closures having combinatorial type of simplices.

e.g.



$F_1 = 12$
$F_2 = 21$

- shelling order = lexicographic order = dictionary order

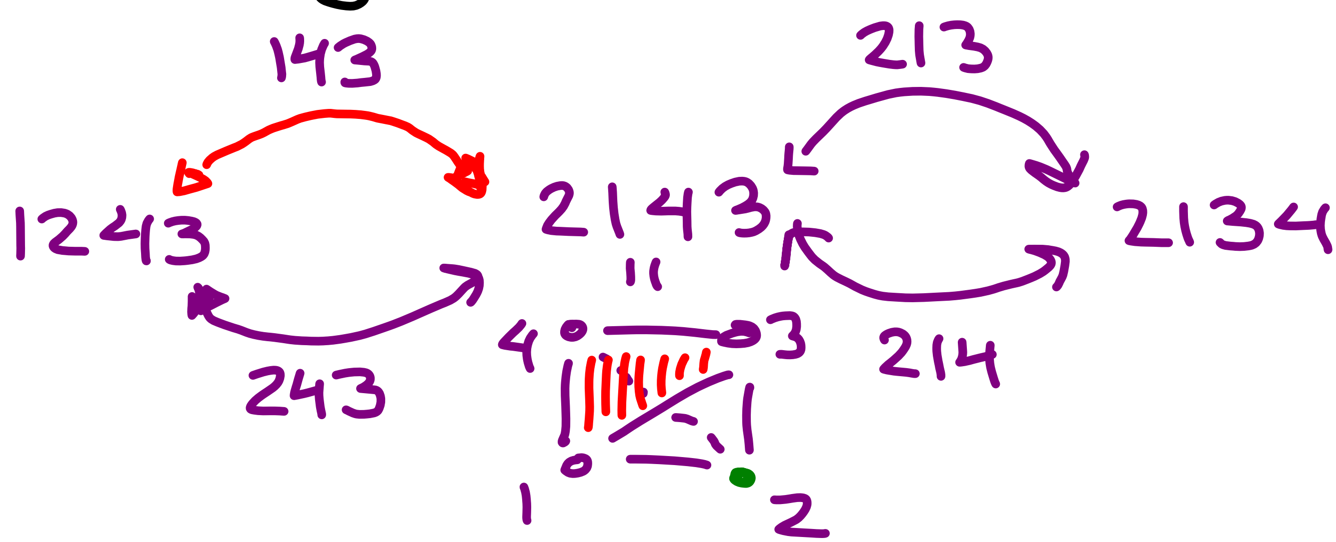
"Homology facets" $:= \left\{ F_j \;\middle|\; F_j \cap \left(\bigcup_{i<j} F_i \right) = \partial F_j \right\}$

$\Updownarrow$

length n injective words s.t. each left-to-right "max so far" immediately followed by smaller letter

$\Updownarrow$

derangements in S_n



(Arises in: Farmer, Hanlon-H., Miller-Wilson, Reiner-Webb...)

More Consequences of Shellability

- gallery connectedness
- combinatorial formula for h-numbers (which are defined by:

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{j=0}^{d} h_j t^{d-j}$$

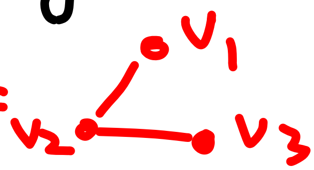
where $f_i(\Delta) = \#\, i$-dim'l faces in Δ, with $\emptyset$ as unique (-1)-dim'l face), namely $h_i = |\{F_j \mid \overline{F_j} \cap (\cup_{i<j} \overline{F_i})$ has $i+1$ max faces$\}|$

Alternate Def'n for Shelling: A total order $F_1, F_2, \ldots, F_k$ on facets of simplicial complex Δ s.t. each set of faces $\overline{F_j} \setminus (\cup_{i<j} \overline{F_i})$ has unique minimal element

Face Numbers & Commutative Algebra

h-numbers & Cohen-Macaulay property crucial to Billera-Lee-Stanley g-theorem (characterizing f-vectors of simplicial polytopes) & Stanley's upper bound thm (giving sharp upper bound on #i-faces in triang. of d-sphere with n vertices)

Key Defn: The Stanley-Reisner ring (or face ring) of simplicial complex Δ is $k[\Delta] = k[x_1, \dots, x_n]/I_\Delta$ where Δ has vertices $v_1, \dots, v_n$ and I_Δ is squarefree monomial ideal gen'd by min'l non-faces.

$\Delta =$ 

$\Rightarrow k[x_1, x_2, x_3]/(x_1 x_3)$

Technique 1*: Lexicographic Shellability

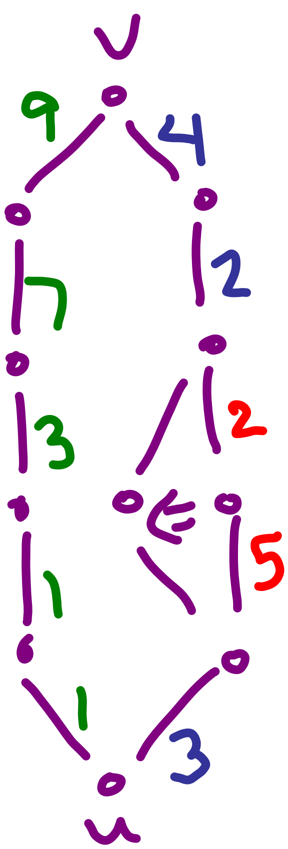
(Anders Björner & Michelle Wachs)

A poset P is EL-shellable if it admits labeling λ (called an EL-labeling) of its cover relations $x \lessdot y$ w/ integers s.t. $u < v$ implies:

(1) there is unique saturated chain $u \lessdot u_1 \lessdot \ldots \lessdot u_k \lessdot v$ s.t. $\lambda(u, u_1) \le \lambda(u_1, u_2) \le \ldots \le \lambda(u_k, v)$

and

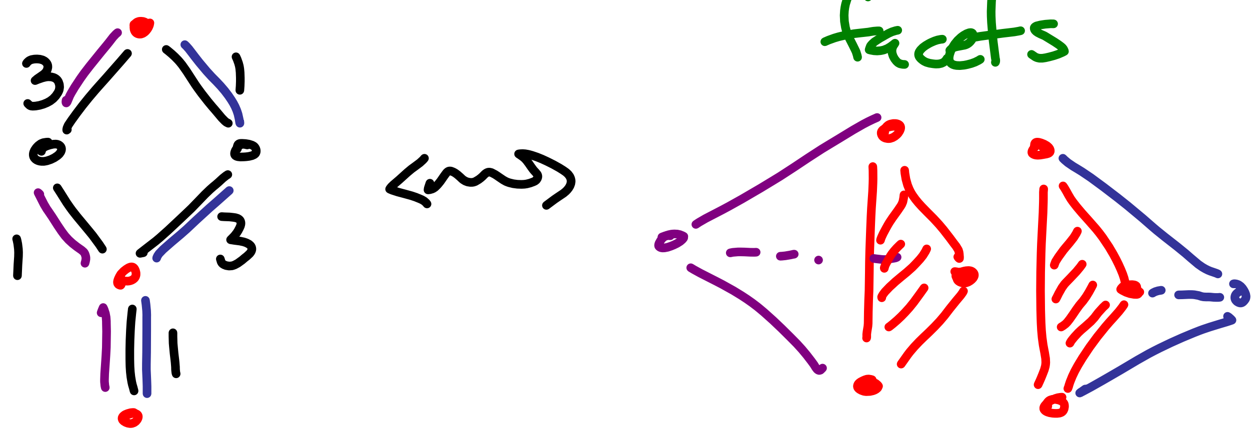
(2) $(\lambda(u,u_1), \lambda(u_1,u_2), \ldots, \lambda(u_k,v))$ is lexicographically smaller than the label sequences on all other saturated chains from u to v.



Thm (Björner): EL-labeling $\Rightarrow$ Shelling

Idea: Lexicographic order on maximal chains (breaking ties arbitrarily) induces shelling order on corresponding facets of $\Delta(P)$.

- "descents in labeling" $\leftrightsquigarrow$ codim. one overlap of facets



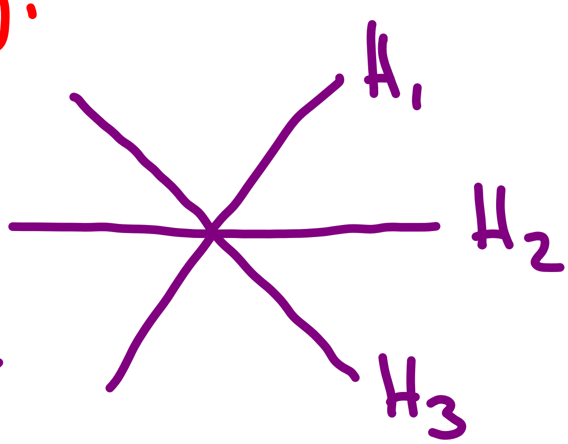
- "descending" chains $\leftrightsquigarrow$ facets attaching along entire bdry $\leftrightsquigarrow$ spheres
- $\mu_P(u,v) = \pm$ # descending chains u to v (for P graded)

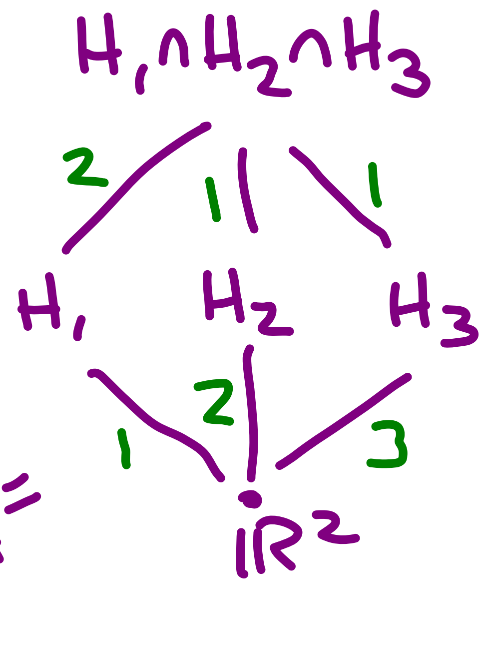
Example: Intersection Posets of Hyperplane Arrangements (as "Geometric Semi-Lattices")

- Choose any total order $H_1, H_2, \ldots, H_k$ on hyperplanes (resp. "atoms")
- Label $u \lessdot v$ with

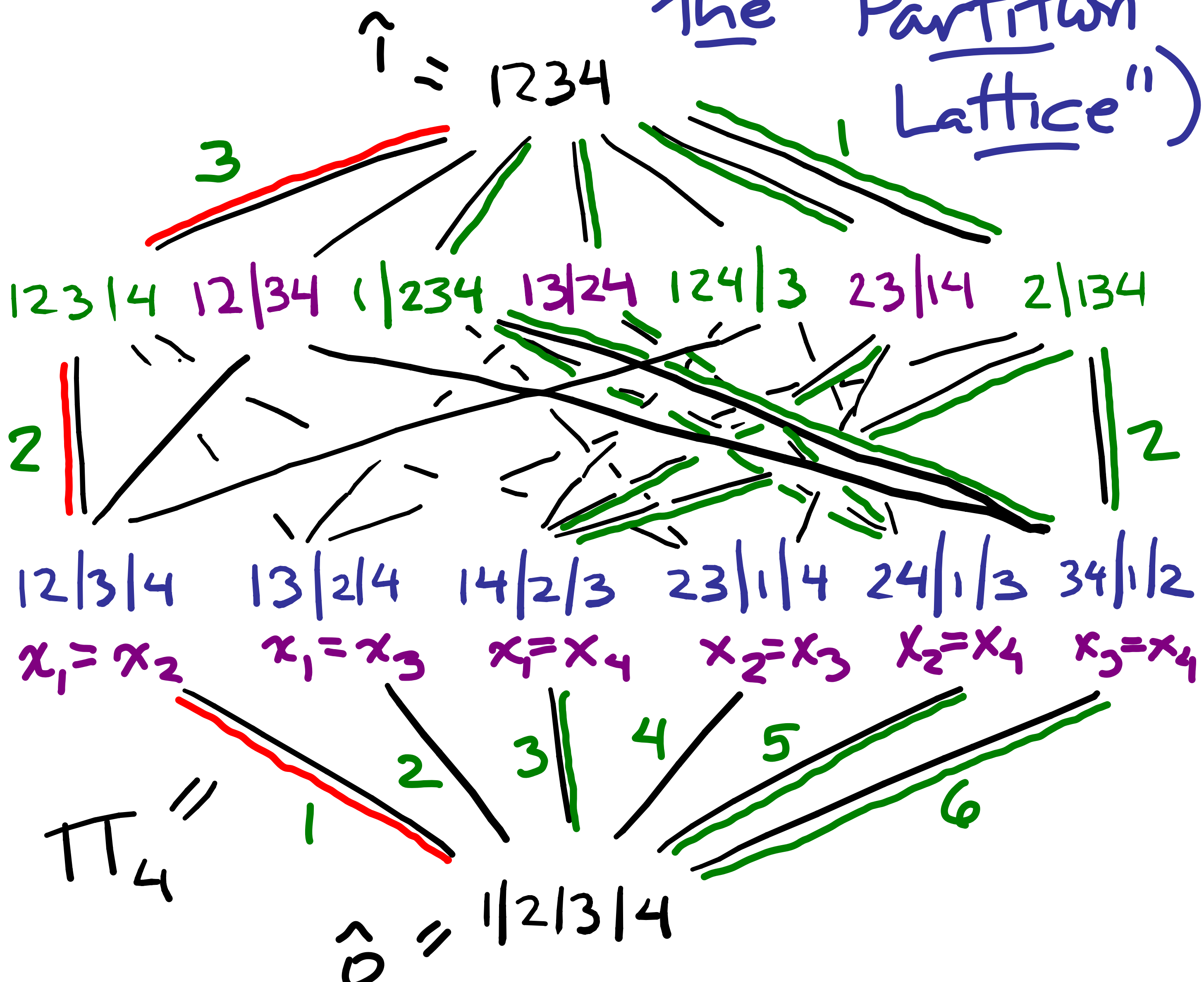
$$\min\{ i \mid H_i \not\leq u \text{ and } H_i \leq v \}$$

e.g.

$A =$


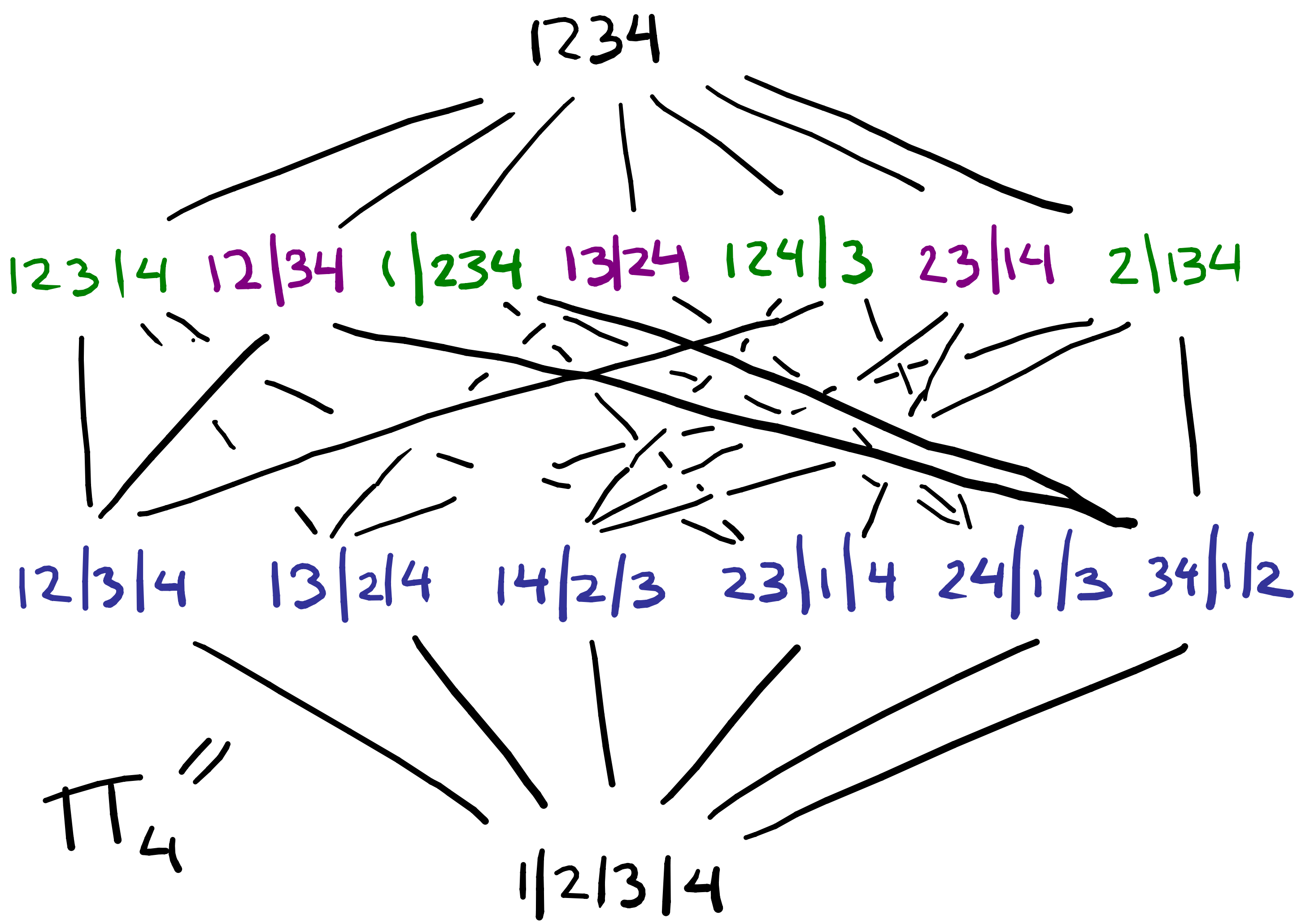
$L_A =$


Intersection Poset for Type A Coxeter Arrangement (a.k.a. Π_n the "Partition Lattice")



$$\mu_{\Pi_4}(\hat{0},\hat{1}) = -6$$

Partition Lattice Π_n & its S_n-representions



- S_n acts by permuting values

e.g. $(13)[12|3|45] = 32|1|45$

Lefschetz Characters as Möbius fns

e.g. S_n-action on partition lattice Π_n

"Lefschetz character"

$$\chi_{\Pi_n} = \bigoplus (-1)^{n-2-i} \chi_{\tilde{C}_i(\Delta(\Pi_n))}$$

by shellability of Π_n

$$= \bigoplus (-1)^{n-2-i} \chi_{\tilde{H}_i(\Delta(\Pi_n))}$$

$$= \chi_{\tilde{H}_{top}(\Delta(\Pi_n))}$$

Phil Hanlon: Determined χ_{Π_n} using:

$$\chi_{\tilde{C}_i(\Delta(\Pi_n))}(g) = \#\, i\text{-chains fixed by } g$$

$$\Downarrow$$

$$\chi_{\Pi_n}(g) = \mu_{\Pi_n^g}(\hat{0}, \hat{1})$$

Lefschetz Rep'n of Partition Lattice

Thm (Hanlon-Stanley): $\Pi_n \cong \text{sgn} \otimes \left(\zeta_n \uparrow_{C_n}^{S_n}\right)$

Thm (Joyal): $\text{lie}_n \cong \zeta_n \uparrow_{C_n}^{S_n}$

Thm (Barcelo): Explicit S_n-equivariant bijection yielding $\Pi_n \cong \text{sgn} \otimes \text{lie}_n$

Thm (Kraskiewicz & Weyman):

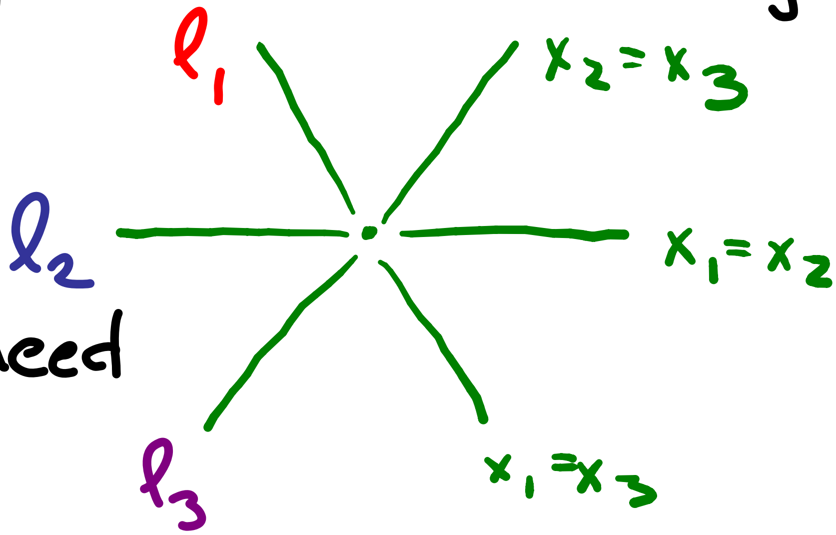
$$\text{lie}_n \cong \bigoplus_{\substack{T \text{ SYT} \\ \text{w/ } \text{maj}(T) \equiv 1 \pmod{n}}} S^{\lambda(T)}$$

Sundaram: Formula for $WH_i(\Pi_n)$ involving plethysm/wreath products & Lefschetz character Π_n for full partition lattice.

Configuration Space $PConf_n(\mathbb{R}^2)$ as Subspace Arrangement Complement

- M_n = complement of type A (complex) braid arrt $\{x_i = x_j \mid 1 \le i < j \le n\}$

Warning: figure is $\mathbb{R}$-picture, need $\mathbb{C}$-picture



(pt p_i in config space $\leftrightarrow$ $x_i \in \mathbb{C}$)

- Π_n = intersection poset $\mathcal{L}(A_{n-1})$

- i, j in same block of $\Pi_n \iff x_i = x_j$

e.g. $13|245 \leftrightarrow x_1 = x_3$
$x_2 = x_4 = x_5$

μ as Topological Shadow

- A graded poset with $\hat{0} \neq \hat{1}$ is **Eulerian** if $\mu(u,v) = (-1)^{rk(v)-rk(u)}$ for all $u \leq v$.
- A graded poset P is a **CW poset** if
  (1) $\hat{0} \in P$
  (2) P has at least one other element
  (3) $\Delta(\hat{0},u) \cong S^{rk(u)-2}$ for $u \neq \hat{0}$

Thm (Björner): P is CW poset $\iff$ there exists regular CW complex with P as poset of closure rel'ns

Cor: CW Poset $\Rightarrow$ Eulerian

Some Important CW Posets

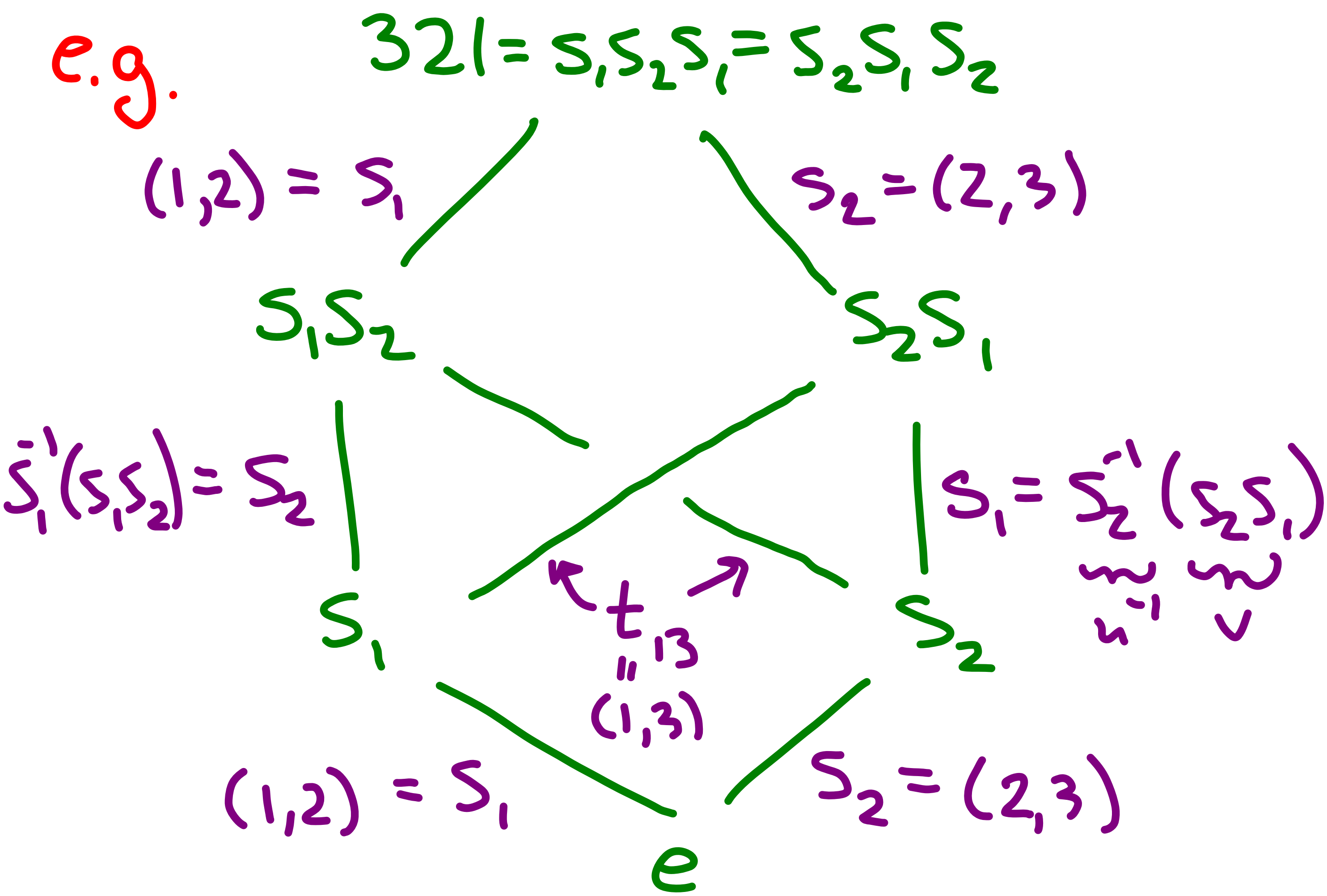
- all graded, thin, shellable posets (Danaraj-Klee)
- Bruhat order (Björner-Wachs)
- face posets of finite simplicial complexes (which are also lattices, i.e. each u & v have unique least upper bd & greatest lower bd)
- not all intervals in CW posets

A Goal of Mine: Use combinatorics of $F(K)$ + limited topological info to understand K (3rd lecture)

M. Dyer's EL-labeling for Bruhat Order via "Reflection Orders"

- Label each cover relation $u \lessdot v$ with reflection $u^{-1}v$

e.g.



Subword Complexes (introduced by Knutson & Miller)

$Q :=$ (not necessarily reduced) expression

$w :=$ Coxeter group element

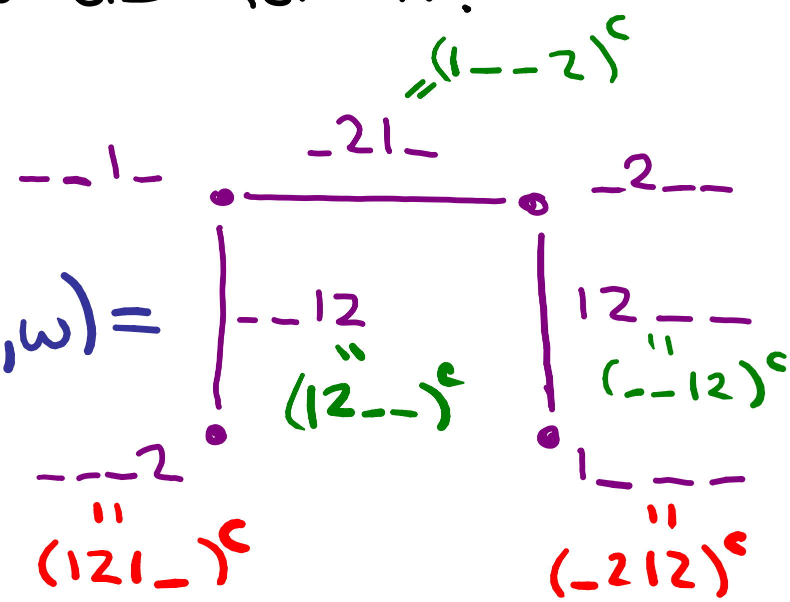
Facets of $\Delta(Q,w)$ are the subwords of Q whose complements are reduced words for w.

e.g.

$Q = (1,2,1,2)$

$w = s_1 s_2$

$\Delta(Q,w) =$



Thm (Knutson-Miller): $\Delta(Q,w)$ is "vertex decomposable" (hence shellable) ball or sphere.

(Used to study matrix Schubert varieties via "Gröbner degeneration")