Combinatorics & Topology of Generically Injective Maps

Patricia Hersh
North Carolina State University

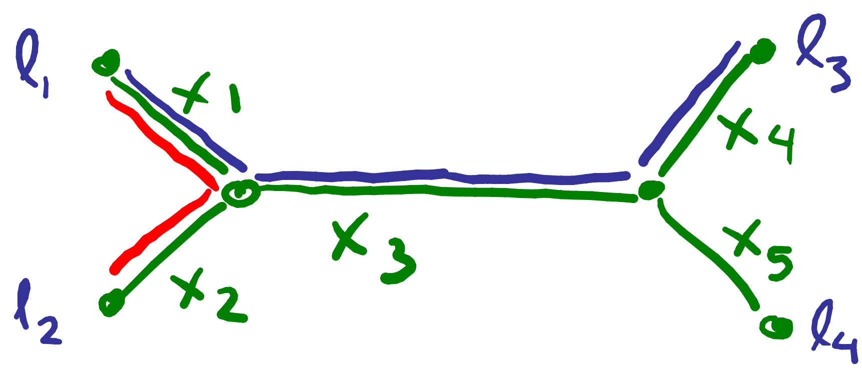
- Talk Plan:
  - (1) Examples
  - (2) Background
  - (3) Monomial Maps: CW Decomposition of Image
- As Time Permits:
  - (4) New tools - regularity criterion
    - collapses preserving homeomorphism type

I. A Motivating Example of Monomial Map

(from study of phylogenetic trees)

If each edge e_i in a tree is present with probability x_i, can we recover these edge probabilities from path probabilities p_{ij} for pairs of leaves?

e.g.



$$(x_1, x_2, x_3, x_4, x_5) \longmapsto (\underbrace{x_1 x_2}_{p_{12}}, \underbrace{x_1 x_3 x_4}_{p_{13}}, \underbrace{x_1 x_3 x_5}_{p_{14}}, \ldots)$$

Answer: yes, if...

- Our Focus: systematic framework for such questions regarding monomial (& nonnegative polynomial) maps; new combinatorial-topological tools

A Polynomial Map to Totally Nonnegative Real Part of Space of Matrices

- $$x_i(t) = \underset{\text{(type A)}}{I_n} + tE_{i,i+1} = \begin{array}{cc} & \text{column } i+1 \\ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & t & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow \text{row } i \\ \\ \\ \\ \end{array} \end{array}$$

  $$x_i(t) = \underbrace{\exp}_{\text{exponentiated Chevalley generator}}(te_i)$$

- $$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$$

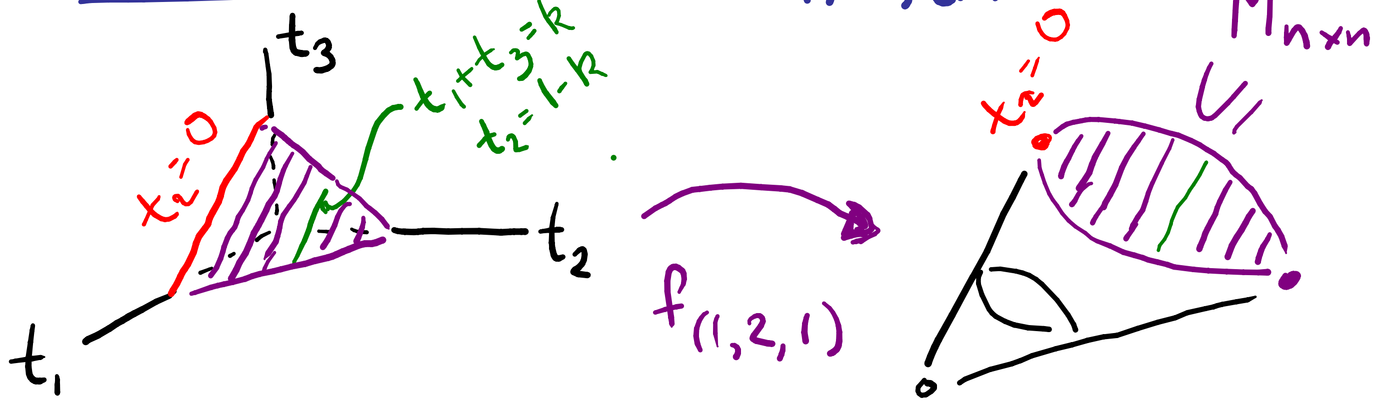
  $$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1,\ldots,i_d)}$



(injective on interior, not on boundary)

$$f_{(1,2,1)}(t_1,t_2,t_3)=\begin{pmatrix}1&t_1&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&&\\&1&t_2\\&&1\end{pmatrix}\begin{pmatrix}1&t_3&\\&1&\\&&1\end{pmatrix}$$

$$\Big\downarrow{\scriptstyle t_2=0}$$

$$x_1(t_1)\cdot x_1(t_3)$$

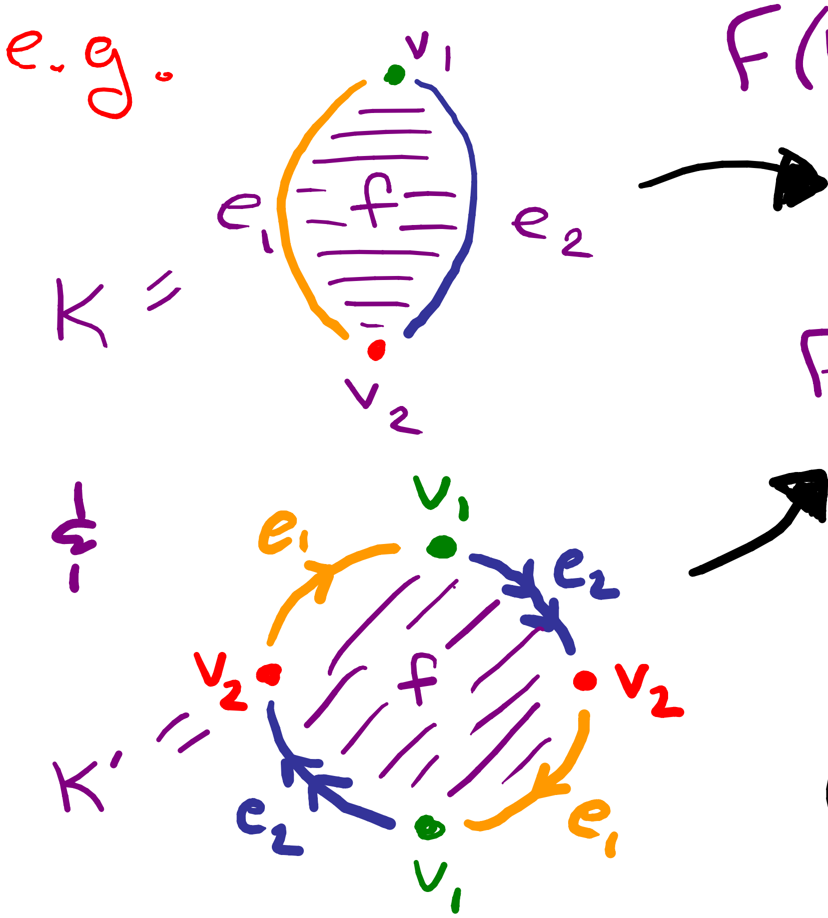
$$f_{(1,2,1)}(t_1,0,t_3)=\begin{pmatrix}1&t_1&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&t_3&\\&1&\\&&1\end{pmatrix}$$

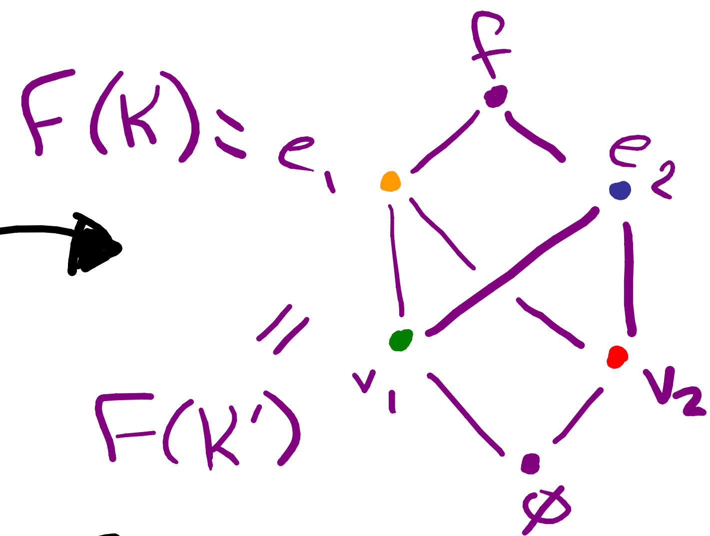
$$=\begin{pmatrix}1&t_1+t_3&\\&1&\\&&1\end{pmatrix}=x_1(t_1+t_3)$$

Thm (H.): Each "slice" of image is regular CW complex homeomorphic to closed ball.

II. Background:

e.g.





"closure poset"
or "face poset"

$(u \le v \iff u \le \bar{v})$

CW complexes: comprised of pieces called cells each homeomorphic to an open ball

· higher dimensional cells glued to unions of lower dimensional ones by attaching maps.

Regular CW Complexes

- A CW complex is regular if the attaching map for each cell is a homeomorphism (hence injective).
- e.g. all simplicial complexes & polytopes
- K regular $\Rightarrow K \cong \underbrace{\Delta(F(K) \setminus \{\hat{0}\})}$ $= sd(K)$

  order complex (i.e. nerve) of face poset of K

- Our focus: examples which arise from monomial & nonnegative polynomial maps as images & as fibers (from total positivity, algebraic statistics, electrical networks, etc.)

  e.g. $(t_1, t_2, t_3) \longmapsto (t_2, t_1 t_2, t_1 t_2 + t_3 t_2)$

Approximating Maps by Homeomorphisms

Combining CE-Approximation Theorem and Local Contractibility of Homeo(S^n, S^n) yields:

Let $g: B \to Z$ be continuous surjection from ball B to Hausdorff space Z such that:

(1) $g(\partial B) \cong \partial B = S^n$;

(2) $g(\partial B) \cap g(\operatorname{int}(B)) = \emptyset$,

(3) $g^{-1}(p)$ is contractible $\forall p \in g(\partial B)$

& (4) $g|_{\operatorname{int}(B)}$ is homeomorphism

Then $Z \cong B$, so Z is a closed ball.

(based on work of Siebenmann, Quinn, Armentrout, Kirby, Edwards, ...)

III. Toric Cubes

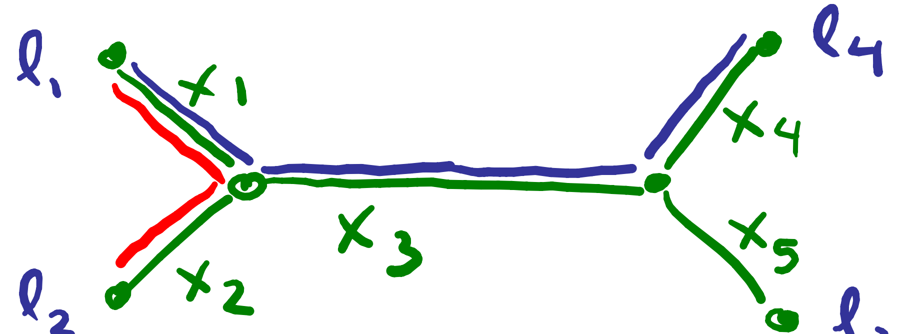
- based on:

A. Engström, P. H., B. Sturmfels, "Toric cubes", Rendiconti del Circolo Matematico di Palermo (2), 62, (2013), no. 1, 67-68.

Question Revisited:

Given a tree together with edge probabilities in $[0,1]$, consider the map yielding the probability for each pair of leaves that path between them is present.

e.g.



$$\underset{\displaystyle [0,1]^5}{\underset{\cap}{(x_1,\dots,x_5)}} \longmapsto \underset{\displaystyle [0,1]^{\binom{4}{2}} = [0,1]^6}{\underset{\cap}{(x_1x_2,\, x_1x_3x_5,\, x_1x_3x_4,\, x_2x_3x_5,\, x_2x_3x_4,\, x_4x_5)}}$$

- Will analyze images of such maps.

Edge Product Space of Phylogenetic Trees

- Each tree T with leaves $\{v_1, \dots, v_n\}$ gives rise to a monomial map

$$\underline{m}_T : [0,1]^{|E(T)|} \to [0,1]^{\binom{n}{2}}$$

- "edge-product space" is union of images:

$$\bigcup_{\substack{T \in \{\text{trees} \\ \text{with leaves } \{v_1, \dots, v_n\}\}}} \operatorname{im}(\underline{m}_T) \subseteq [0,1]^{\binom{n}{2}}$$

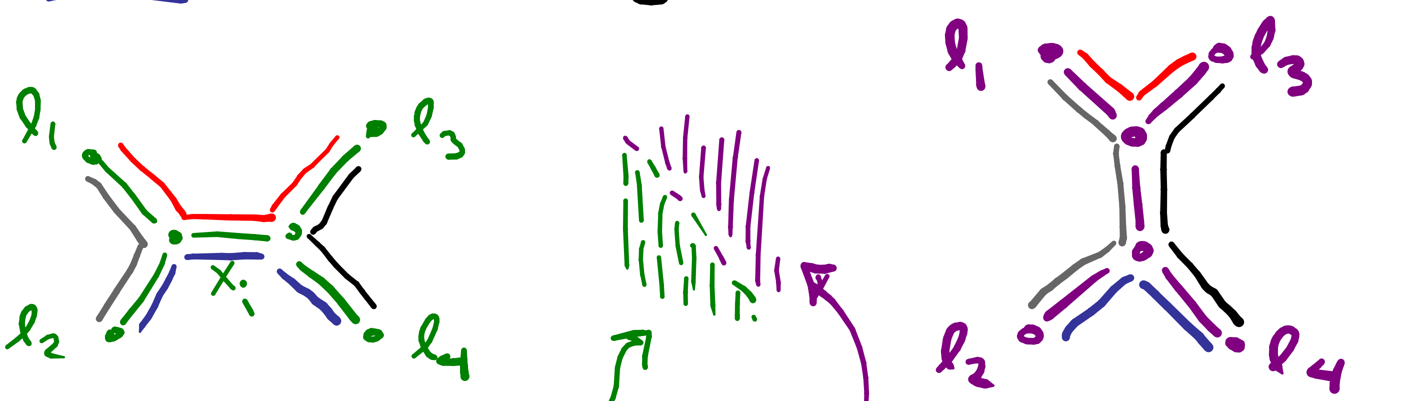
Theorem (Gill-Linusson-Moulton-Steele): This is regular CW complex.

Method of GLMS:

(1) Proved each closed interval in face poset ("Tuffley poset") is CL-shellable & thin, hence is the face poset of regular CW poset.

(2) Use induction on dimension + CL-shellability of Tuffley poset intervals to check sphericity hypothesis for approximating maps by homeomorphisms.

Open Question: Shellability of entire face poset?



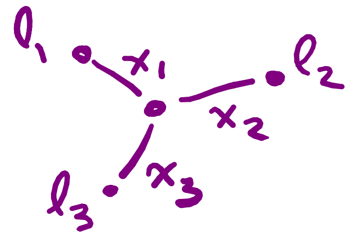
$$(P_{12})(P_{34}) \geq (P_{13})(P_{24}) = x_i^2\,(P_{12})(P_{34})$$

$$(P_{12})(P_{34}) \leq (P_{13})(P_{24})$$

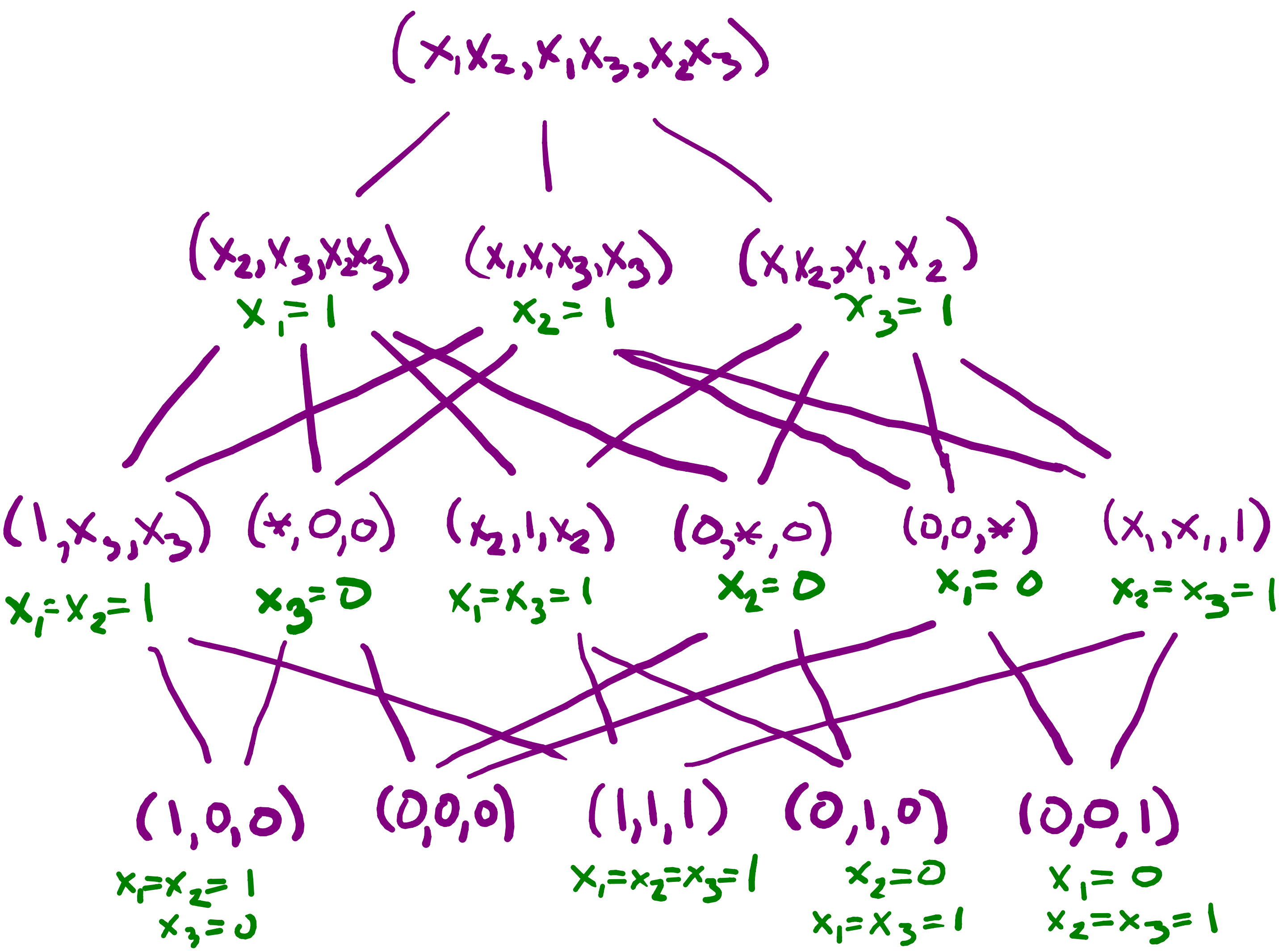
Example of Tuffley Poset for

Single Tree

$T =$



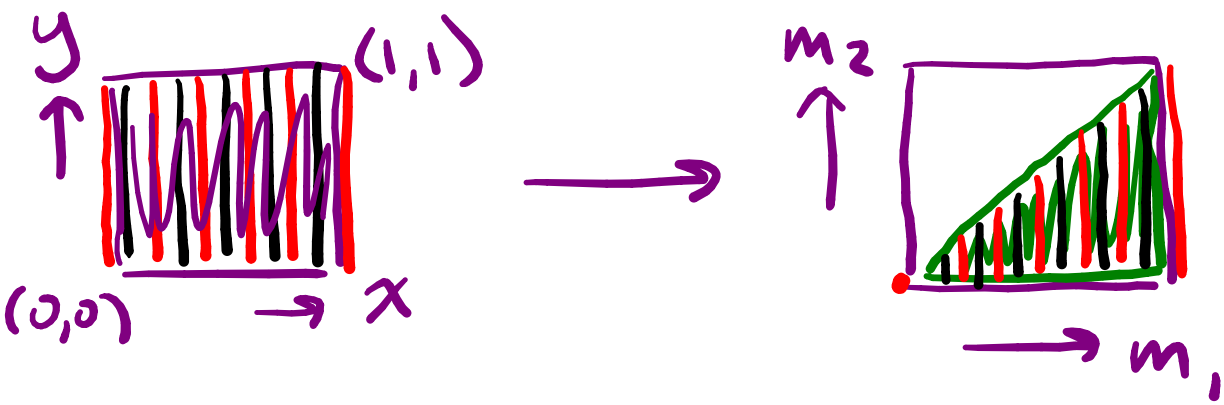
$$(x_1, x_2, x_3) \longmapsto (\underbrace{x_1x_2}_{p(\ell_1,\ell_2)}, \underbrace{x_1x_3}_{p(\ell_1,\ell_3)}, \underbrace{x_2x_3}_{p(\ell_2,\ell_3)})$$



Images (More Generally) of Monomial Maps on Cubes

e.g. $\underline{m} : [0,1]^2 \longrightarrow [0,1]^2$

$$(x,y) \longmapsto (x, xy) = (m_1, m_2)$$

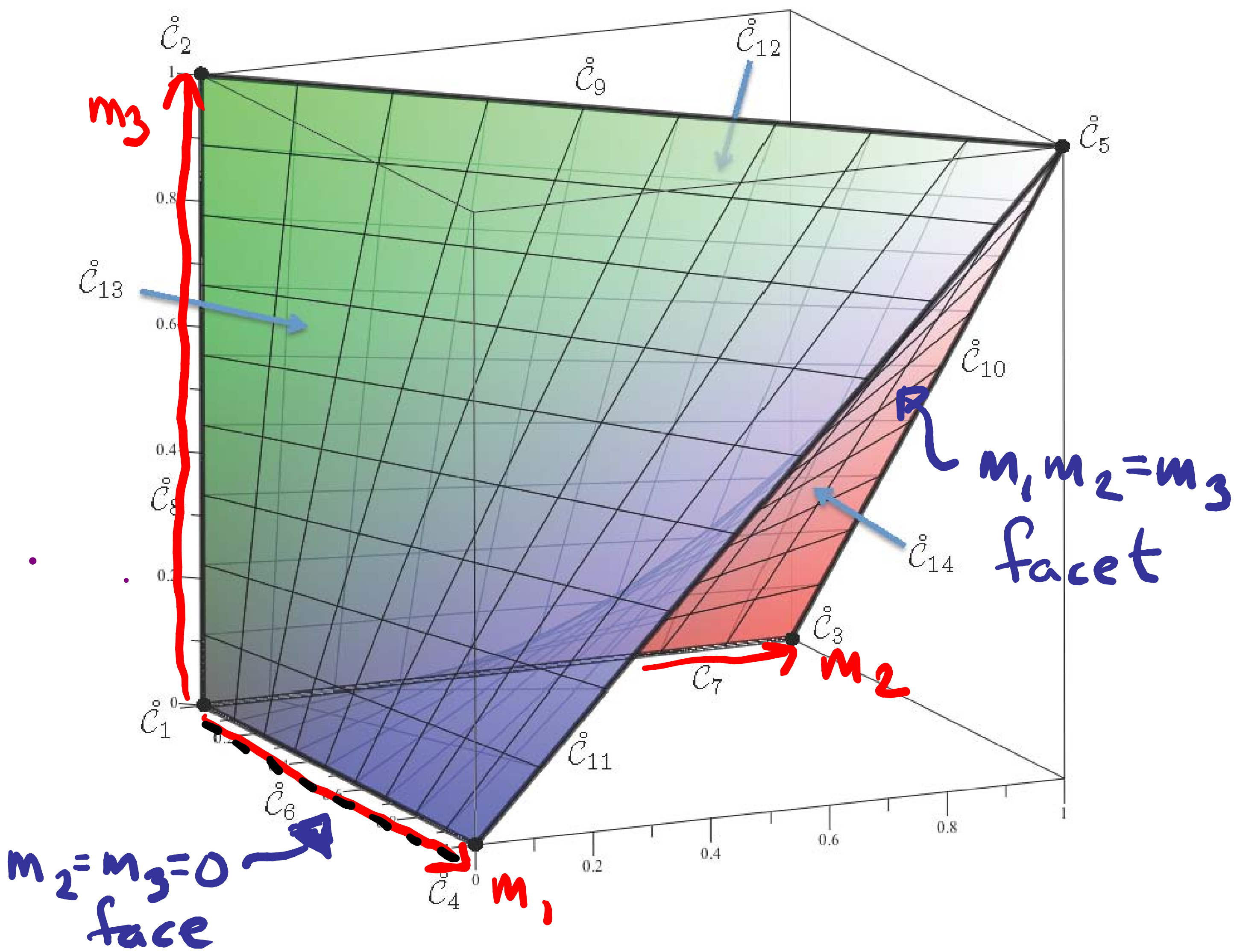


$$\mathrm{im}(\underline{m}) = \{ (m_1, m_2) \in [0,1]^2 \mid m_1 \geq m_2 \}$$

(variables as probabilities, so real-valued in $[0,1]$)

(m_1, m_2, m_3)

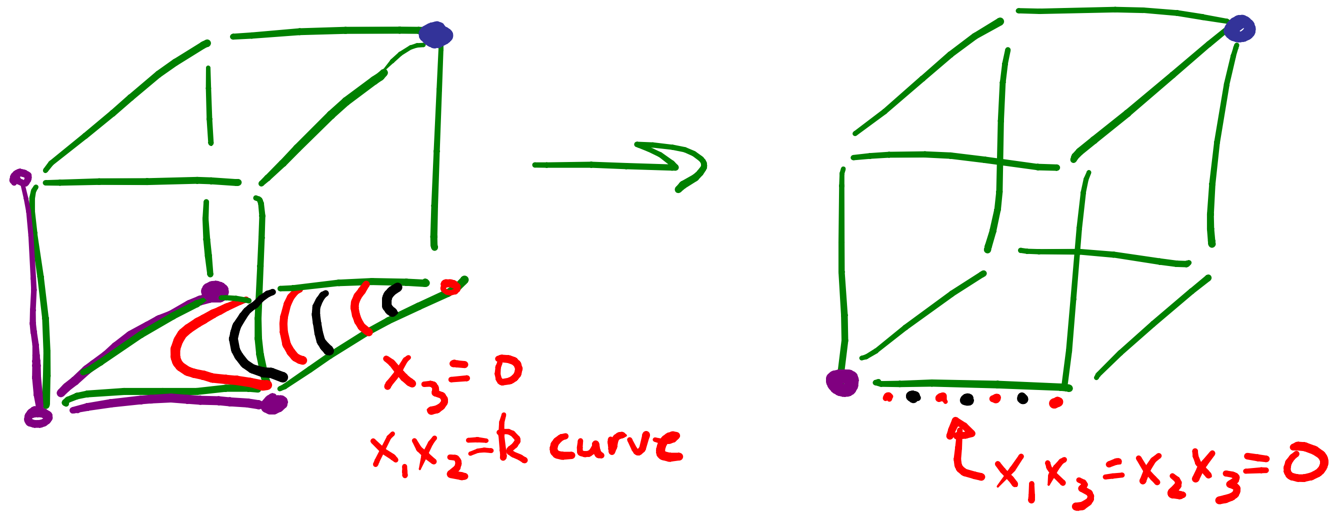
Image of $(x,y,z) \mapsto (xy, xz, yz)$



(bounding surfaces $\begin{array}{l} m_1 m_2 \le m_3 \\ m_1 m_3 \le m_2 \\ m_2 m_3 \le m_1 \end{array}$ in $[0,1]^3$)

"divisibility inequalities" due to $m_3 \mid m_1 m_2$, etc

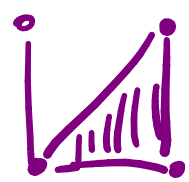
Example of Fibers



$$(x_1, x_2, x_3) \longmapsto (x_1x_2, x_1x_3, x_2x_3)$$

Combinatorial Objects Encoding Images of Monomial Maps

- A toric precube is a subset $\mathcal{C}$ of unit cube $[0,1]^n$ defined by finite set of binomial inequalities.
- A toric cube is a toric precube that equals the closure of its strictly positive part.



e.g. toric precube

$$\{(a,b,c,d) \mid ac \geq bd;\ bc \geq ad\}$$

which is not toric cube because

$(ac)(bc) \geq (bd)(bc) \geq (bd)(ad)$, hence

$ab(c^2-d^2) \geq 0$ so $c \geq d$ unless $ab=0$

Results about Toric Cubes

Theorem 1 (EHS): The toric cubes in $[0,1]^n$ are precisely the images of cubes $[0,1]^d$ under monomial maps.

Theorem 2 (EHS): Every toric cube is a CW complex whose cells are interiors of toric cubes. The boundary of each open cell is a subcomplex.

Theorem (Basu-Gabrielov-Vorobjov):

These CW decompositions are regular CW decompositions.

(via their theory of monotone maps)

Cautionary Example for Connection Between Monomial Maps & Binomial Inequalities (i.e. Theorem 1) Revisited

$$\left\{ (a,b,c,d) \in [0,1]^4 \;\middle|\; \begin{array}{l} ac \geq bd \\ bc \geq ad \end{array} \right\}$$

Then
$$\left.\begin{array}{c} (ac)(bc) \\ \vee\!\mid \\ (bd)(bc) \\ \vee\!\mid \\ (bd)(ad) \end{array}\right\} \Longrightarrow \underbrace{ab(c^2-d^2) \geq 0}_{\text{forces } c \geq d \text{ if } a,b>0}$$

$$\left\{ (a,b,c,d) \in [0,1]^4 \;\middle|\; \begin{array}{l} ac \geq bd \\ bc \geq ad \\ c \geq d \end{array} \right\}$$

$$\parallel$$

image of $\underline{m}$: $(x_1, x_2, x_3, x_4, x_5)$

$$\longmapsto (x_1x_2,\ x_1x_3,\ x_4,\ x_2x_3x_4x_5)$$

Algorithm for CW Decomposition of Toric Cubes

Step 1: Decompose based on which variables (and hence monomials) are strictly positive and which are 0.

$\underline{m}: (x_1, x_2, x_3) \longmapsto (x_1x_2, x_1x_3, x_2x_3)$

e.g. $(0, *, *) \longmapsto (0, 0, *)$

Step 2: Further decompose each such part by applying $-\ln\_$ map to positive coordinates to obtain cone.

Step 2:

e.g. $(x_1^1 x_2, x_1^1 x_3, x_2 x_3) \in (0,1]^3$

$$\Big\downarrow -\ln\_$$

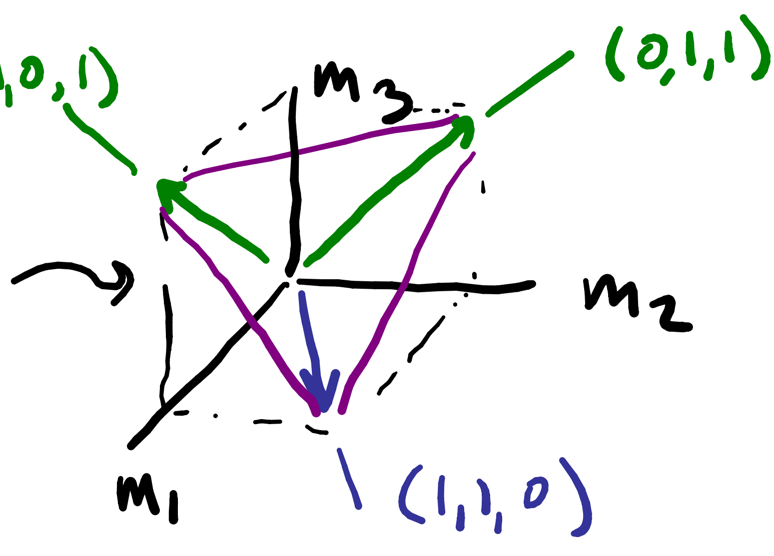
$$(\ell_1+\ell_2, \ell_1+\ell_3, \ell_2+\ell_3) \in \mathbb{R}^3_{\geq 0}$$

$$\|$$

$$\ell_1(1,1,0) + \ell_2(1,0,1) + \ell_3(0,1,1)$$

↑ exponents of x_1 in monomials

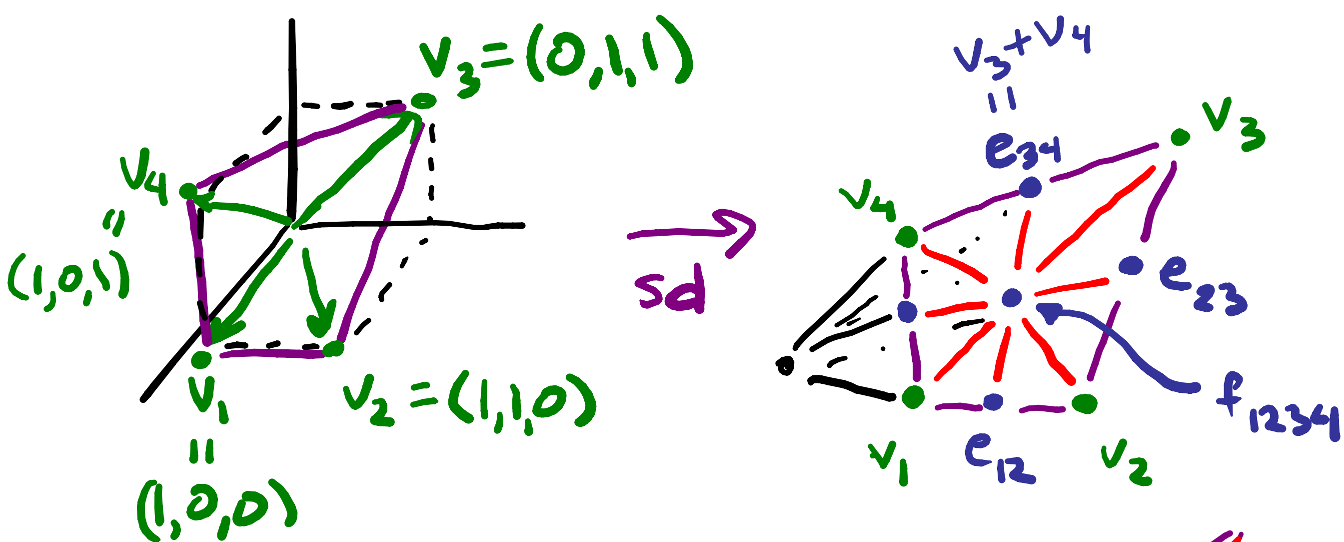
cone homeomorphic to positive part of $\mathrm{im}(\underline{m})$



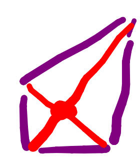
- If cone is simplicial, then $\underline{m}$ restricts to homeomorphism on int(cone) (& hence on int(cube))
- If not simplicial cone:

Step 3: Take barycentric subdivision of cone yielding collection of simplicial cones & new monomial map with same image

e.g. $\underline{m} : (x_1, x_2, x_3, x_4) \mapsto (x_1 x_2 x_4, x_2 x_3, x_3 x_4)$



Challenge: not injective on interior



e.g. $v_2 + v_4 = (2,1,1) = 2v_1 + v_3$

New Monomial Map:

(with faces as variables)

$$(x_1, x_2, x_3, x_4, x_{12}, x_{23}, x_{34}, x_{14}, x_{1234}) \longmapsto$$

$$(x_1 x_{12} x_{14} x_{1234} x_2 x_{12} x_{23} x_{1234} x_4 x_{34} x_{14} x_{1234},$$

$$x_2 x_{12} x_{23} x_{1234} x_3 x_{23} x_{34} x_{1234},$$

$$x_3 x_{23} x_{34} x_{1234} x_4 x_{34} x_{14} x_{1234})$$

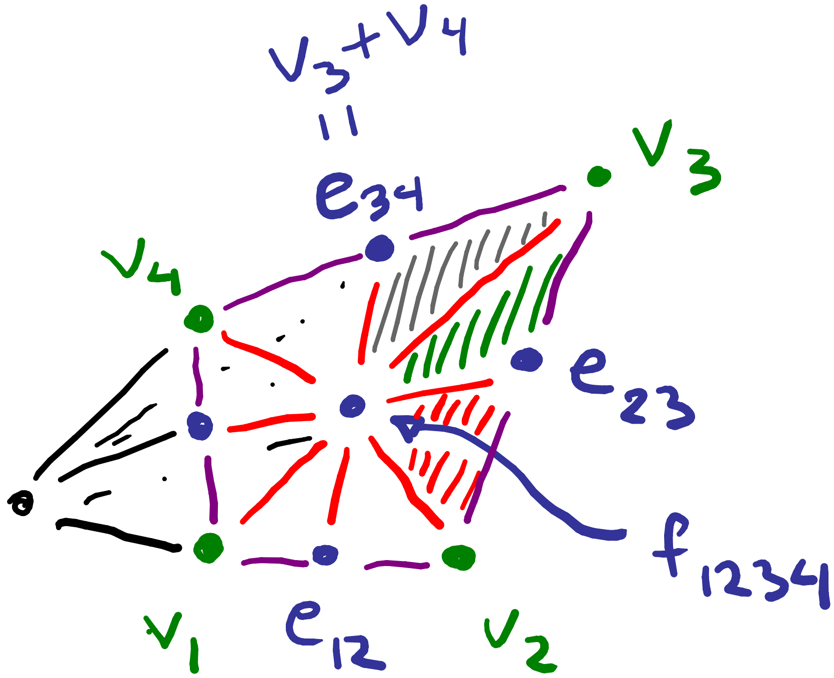
(1) same cone with more rays (from barycenters of faces)

(2) Same achievable $\{0,1,*\}$-patterns

(3) same bounding inequalities

Step 4: Choose ball of appropriate dimension within $[0,1]^{\#\text{faces}}$ as preimage so $\underline{m}|_{\text{int}(B)}$ is homeomorphism...

e.g.



$$\Big\updownarrow -\ln$$

$$\subseteq [0,1]^9$$

$$[0,1]_{v_3} \times [0,1]_{e_{34}} \times [0,1]_{f_{1234}} \times 1_{v_1} \times 1_{v_2} \times 1_{v_4} \times 1_{e_{12}} \times 1_{e_{23}} \times 1_{e_{14}}$$

$$\cup\ [0,1]_{v_3} \times 1_{e_{34}} \times [0,1]_{f_{1234}} \times 1_{v_1} \times 1_{v_2} \times 1_{v_4} \times 1_{e_{12}} \times [0,1]_{e_{23}} \times 1_{e_{14}}$$

$$\cup\ 1_{v_3} \times 1_{34} \times [0,1]_{f_{1234}} \times 1_{v_1} \times [0,1]_{v_2} \times 1_{v_4} \times 1_{e_{12}} \times [0,1]_{e_{23}} \times 1_{e_{14}}$$

$$\cup \dots$$

Picture of Preimage Ball: 

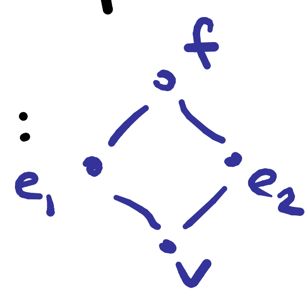
(polytope shellability guarantees a ball)

Open Question:

Is the face poset for image of monomial map CL-shellable?

Remarks:

- Is a "CW poset", since toric cube is regular CW complex.
- Is a "thin" poset:



- Special case of Tuffley poset intervals proven CL-shellable by Gill-Linusson-Moulton-Steele

IV: Other New Combinatorial-Topological Tools & Applications

- based on:

"Regular cell complexes in total positivity", to appear in Inventiones Mathematicae.

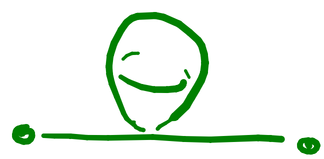
New Regularity Criterion:

Prop'n (H.) Let K be a finite CW complex w/ characteristic maps $\{f_\alpha\}$.

Suppose

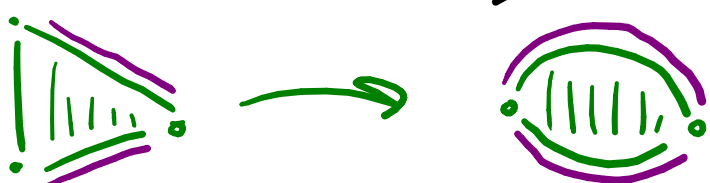
(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example:



(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example:



Then $F(K)$ is graded by cell dimension.

Remark: Next theorem "spreads around" injectivity requirement

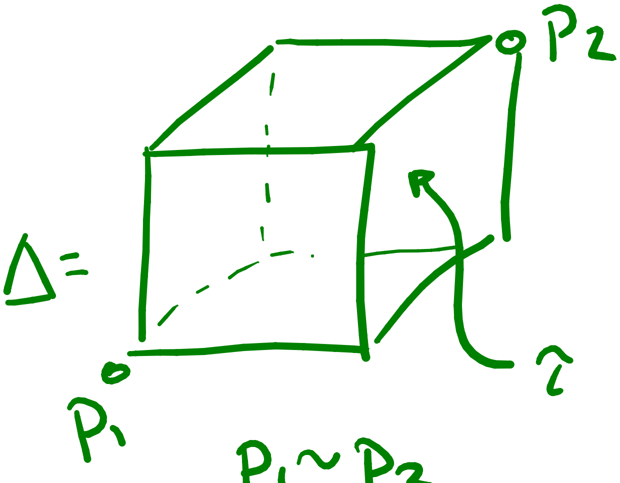
Thm (H.) Let K be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then K is regular w.r.t. $\{f_\alpha\}$ $\iff$

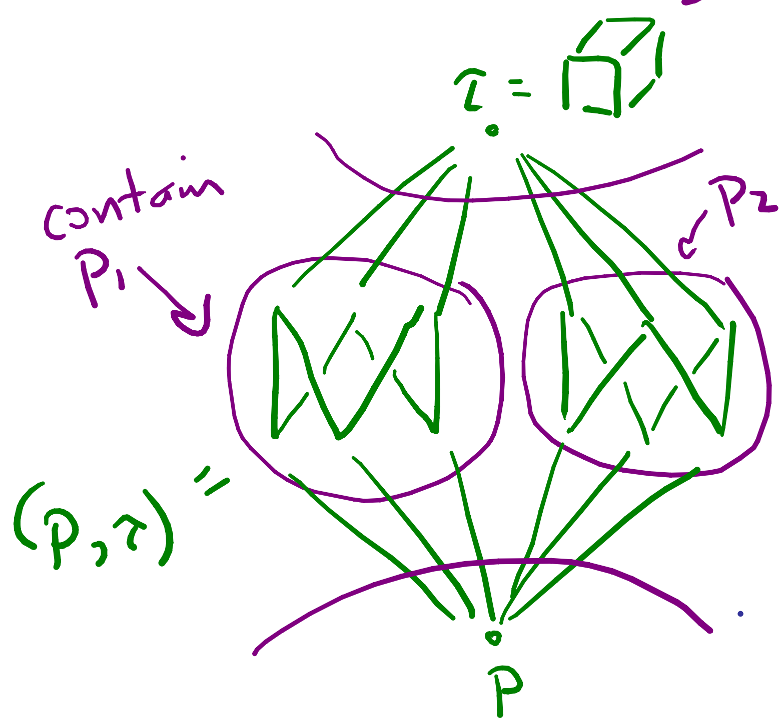
(1) K meets requirements of prop'n for $F(K)$ to be graded by cell dim.

(2) $F(K)$ is thin and each open interval (u,v) for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

Non-Example





(3) For each α, the restriction of f_α to preimages of codim. one cells in $\overline{e_\alpha}$ is injective.

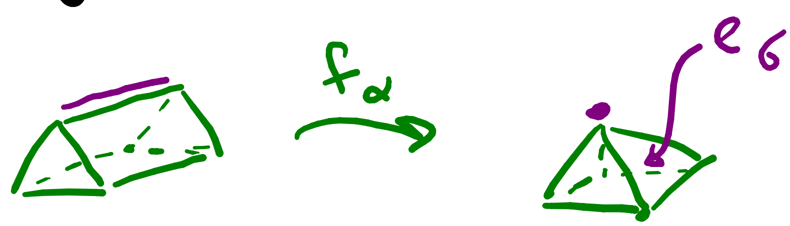
(topological condition)

Non-Example:

(4) $\forall\, e_\sigma \subseteq \overline{e_\alpha}$, f_σ factors as continuous inclusion $i: B^{\dim \sigma} \to B^{\dim \alpha}$ followed by f_α.

Non-Example:

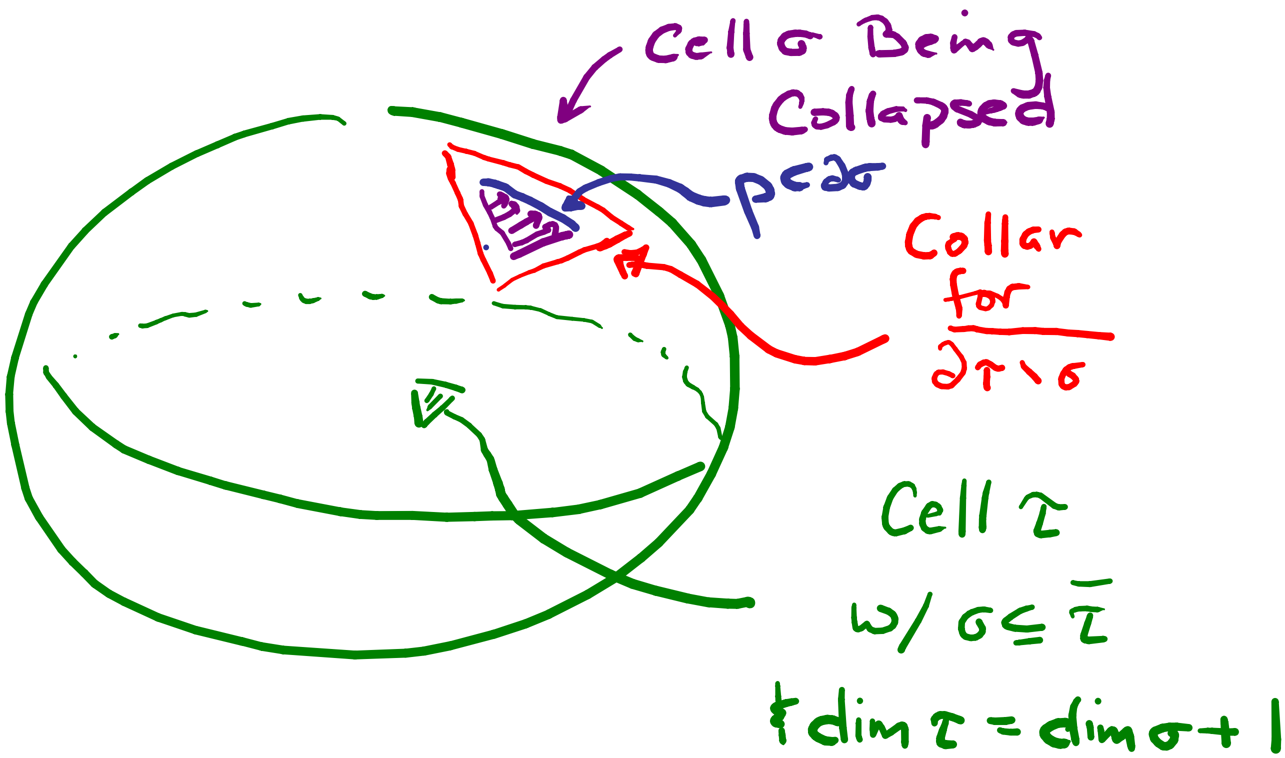
(due to David Speyer)



Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one

Homeomorphism Type Analogues of Elementary Collapses / Simple Homotopy Type (i.e. of Discrete Morse Functions)

- introduce maps f s.t. $f|_{\text{int}(B)}$ = homeomorphism and $f|_{\partial B}$ approximable by homeomorphisms.



Plan: Collapse $\overline{\sigma}$ onto $\overline{\rho} \subseteq \partial\sigma$ across curves, stretching collar for $\overline{\partial\tau \setminus \sigma}$ to cover $\overline{\sigma} \setminus \overline{\rho}$.

The Totally Nonnegative Part of a Space of Matrices (Revisited)

- $$x_i(t) = I_n + tE_{i,i+1} = \begin{array}{cc} & \text{column } i+1 \\ & \downarrow \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} & \leftarrow \text{row } i \end{array}$$

  $x_i(t) = \exp(te_i)$ (general type)

  $I_n +$ (type A)

- $$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$$

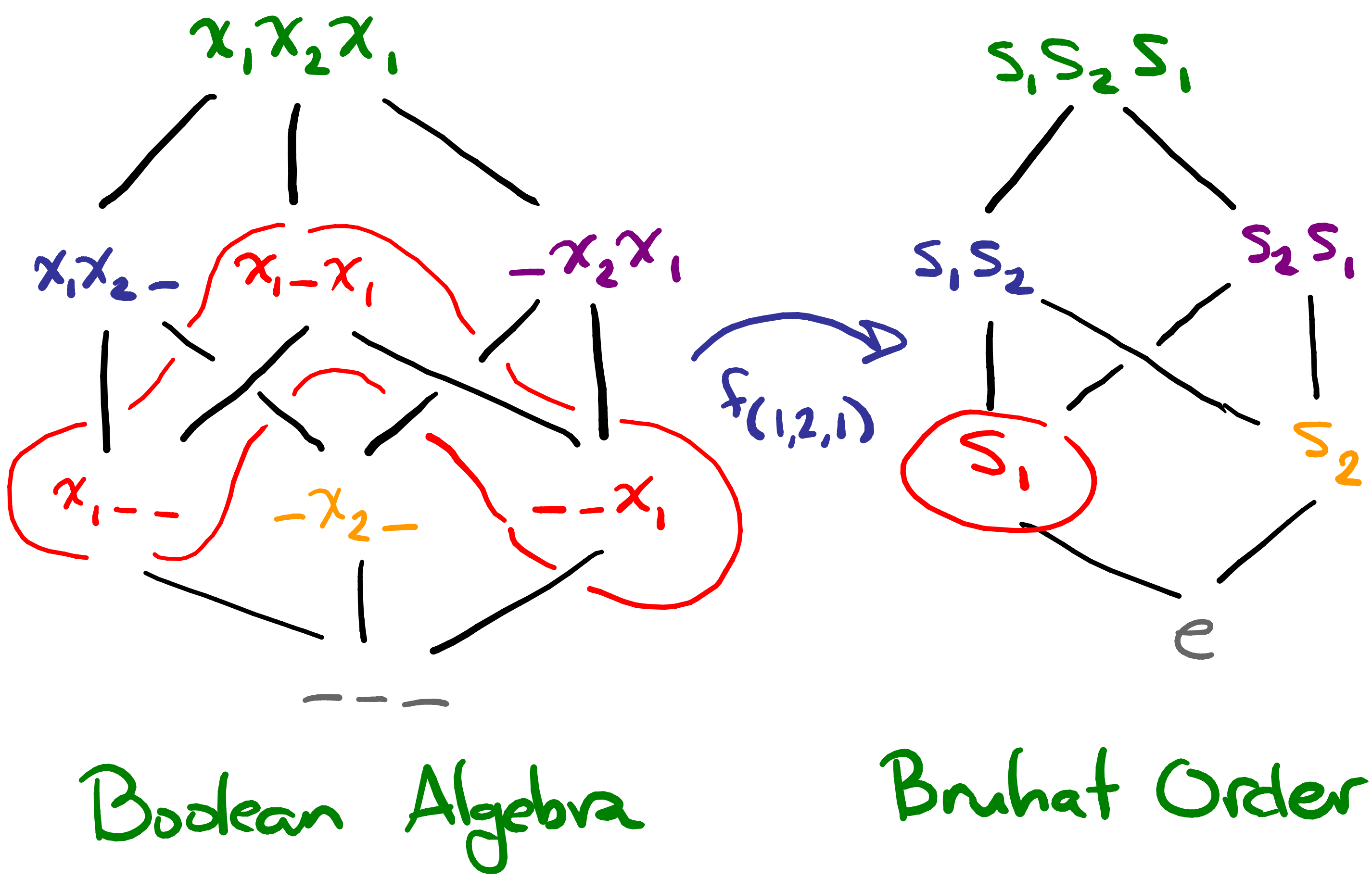
  $$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

e.g. $$f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

A Poset Map (on Face Posets) induced from $f_{(i_1,\dots,i_d)}$



- Apply braid moves & $x_i^2 \to x_i$ to get reduced expression; replace x_i's by s_i's
- Fibers $f_2^{-1}(u)$ are dual to face posets of subword complexes, which are shellable.

Conjecture (Fomin–Shapiro): The space of upper triangular, totally nonnegative matrices w/ 1's on diagonal stratified according to which minors are 0 and which are positive has lk(ID) regular CW complex homeomorphic to closed ball with Bruhat order as closure poset.

Theorem (H.): Fomin-Shapiro Conjecture indeed holds.

Proof: Collapses on preimage boundary, then regularity criterion.

Further Sources of Very Interesting Such Maps: Electrical networks (maps involving currents, voltages, Kirkhoff's laws...) & biological maps with probabilities as parameters

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