Shellability of Uncrossing Posets & the CW Poset Property

Patricia Hersh
North Carolina State University

(partly joint work with Rick Kenyon)

Outline for Talk

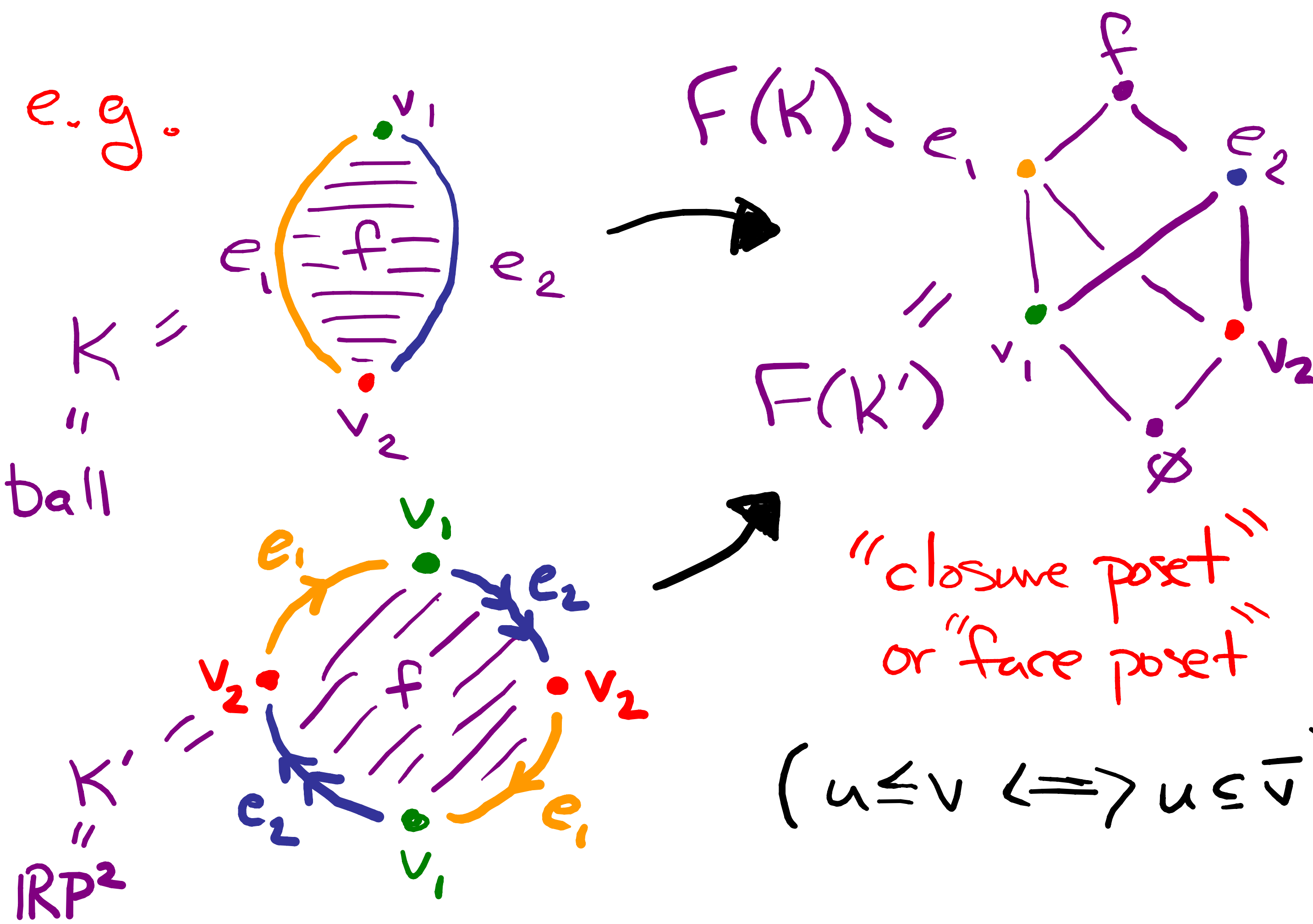
1. Background on CW Posets & (Lexicographic) Shellability
2. Two maps & face posets of their images
3. Close Look at Dyer's Reflection Order Labeling for Bruhat order
4. Shellability of uncrossing posets (joint work w/ Kenyon)

Motivations & Context

- Want to study images & fibers of interesting maps $f: K \to L$ of stratified spaces which induce poset maps $\tilde{f}: F(K) \to F(L)$ on face posets
- It helps to prove $F(L)$ is "CW poset" which follows from "thinness" & "shellability"
- Today: Discuss such maps and poset shellability

  for
  - $F(L) =$ Bruhat order
  - $F(L) =$ uncrossing order (w/ Rick Kenyon)

CW Complexes & their Face Posets

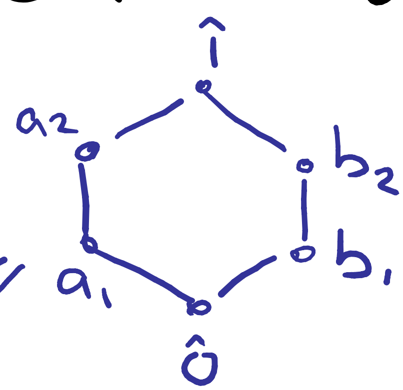


$(u \leq v \iff u \leq \bar{v})$

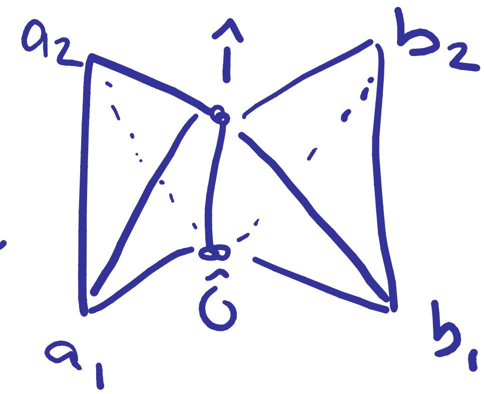
Recall: A **CW complex** is comprised of **open cells** each homeomorphic to an open ball. A **regular CW complex** further has cell closures homeomorphic to closed balls. e.g. simplicial complexes

Defn: The order complex (or nerve) of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-"chains" $v_0 < \ldots < v_i$ in P

e.g. $P =$



$\Delta(P) =$



Key Property (Hall; popularized by Rota):

$$\mu_P(x,y) = \tilde{\chi}(\Delta_P(x,y)) = \begin{array}{l} -1 + \#\text{vertices} \\ \quad - \#\text{edges} \\ \quad + \#\text{2-faces} \ldots \end{array}$$

$$= -1 + \beta_0 - \beta_1 + \beta_2 - \ldots$$

- $(u,v) = \{ z \in P \mid u < z < v \}$
- $u \lessdot v$ means $u < v$ & $\nexists z$ s.t. $u < z < v$
- saturated chains u to v := $u \lessdot \ldots \lessdot v$

Background on Face Posets & CW Posets

- A graded poset with $\hat{0}$ & $\hat{1}$ is **Eulerian** if $\mu(u,v) = (-1)^{rk(v)-rk(u)}$ for all $u \leq v$.

- A graded poset P is a **CW poset** if
  (1) $\hat{0} \in P$
  (2) P has at least one other element
  (3) $\Delta(\hat{0}, u) \cong S^{rk(u)-2}$ for $u \neq \hat{0}$

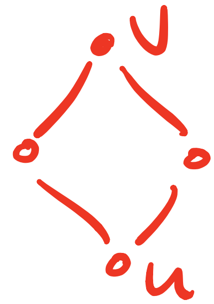
Thm (Björner): P is CW poset $\iff$ there exists regular CW complex with P as poset of closure rel'ns

Cor: CW Poset $\Rightarrow$ Eulerian

Some CW Posets

- all graded, thin, shellable posets (Danaraj-Klee)

"thin" $\Longleftrightarrow$



rk(v) − rk(u) = 2 then $|[u,v]| = 4$

- Bruhat order (Björner-Wachs; Dyer)

- Face posets of stratified spaces of electrical networks
  - conjectured by Thomas Lam
  - proved by H.-Kenyon (a main topic for today's talk)

Shellability

- simplicial complex is **pure** of dim. d if all maximal faces ("facets") are d-dimensional
- simplicial complex is **shellable** if there is total order $F_1, F_2, \ldots, F_k$, a **shelling**, on facets s.t. $\overline{F_j} \cap \left(\bigcup_{i<j} \overline{F_i} \right)$ is pure, codimension one subcomplex of $\overline{F_j}$ for each $j>1$ (hence is $\partial \overline{F_j}$ or has a cone point).

OR

- Each facet attachment preserves homotopy type or closes off a new sphere

Lexicographic Shellability

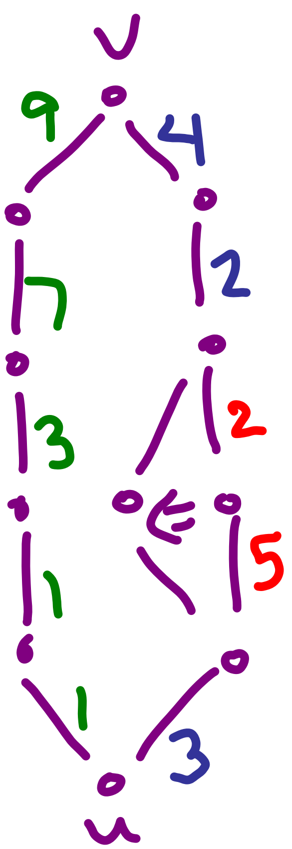
(Björner & Björner–Wachs)

A poset P is **EL-shellable** if it admits labeling λ (called an **EL-labeling**) of its cover relations $x \lessdot y$ w/ integers s.t. $u < v$ implies:

(1) there is unique saturated chain $u \lessdot u_1 \lessdot \ldots \lessdot u_k \lessdot v$ s.t. $\lambda(u,u_1) \le \lambda(u_1,u_2) \le \ldots \le \lambda(u_k,v)$

and

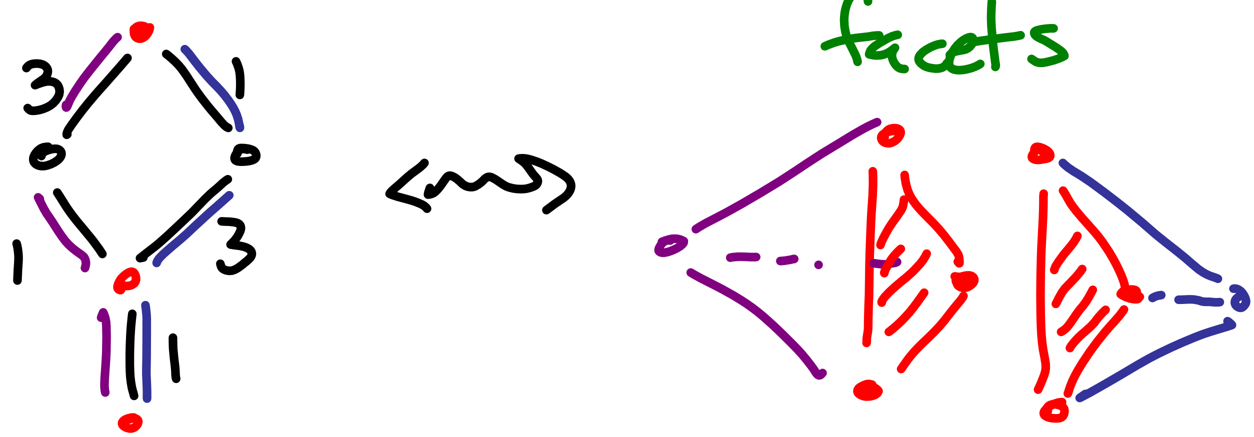
(2) $(\lambda(u,u_1), \lambda(u_1,u_2), \ldots, \lambda(u_k,v))$ is lexicographically smaller than the label sequences on all other saturated chains from u to v.



Thm (Björner): EL-labeling $\Rightarrow$ Shelling

Idea: Lexicographic order on maximal chains (breaking ties arbitrarily) induces shelling order on corresponding facets of $\Delta(P)$.

- "descents in labeling" $\leftrightsquigarrow$ codim. one overlap of facets



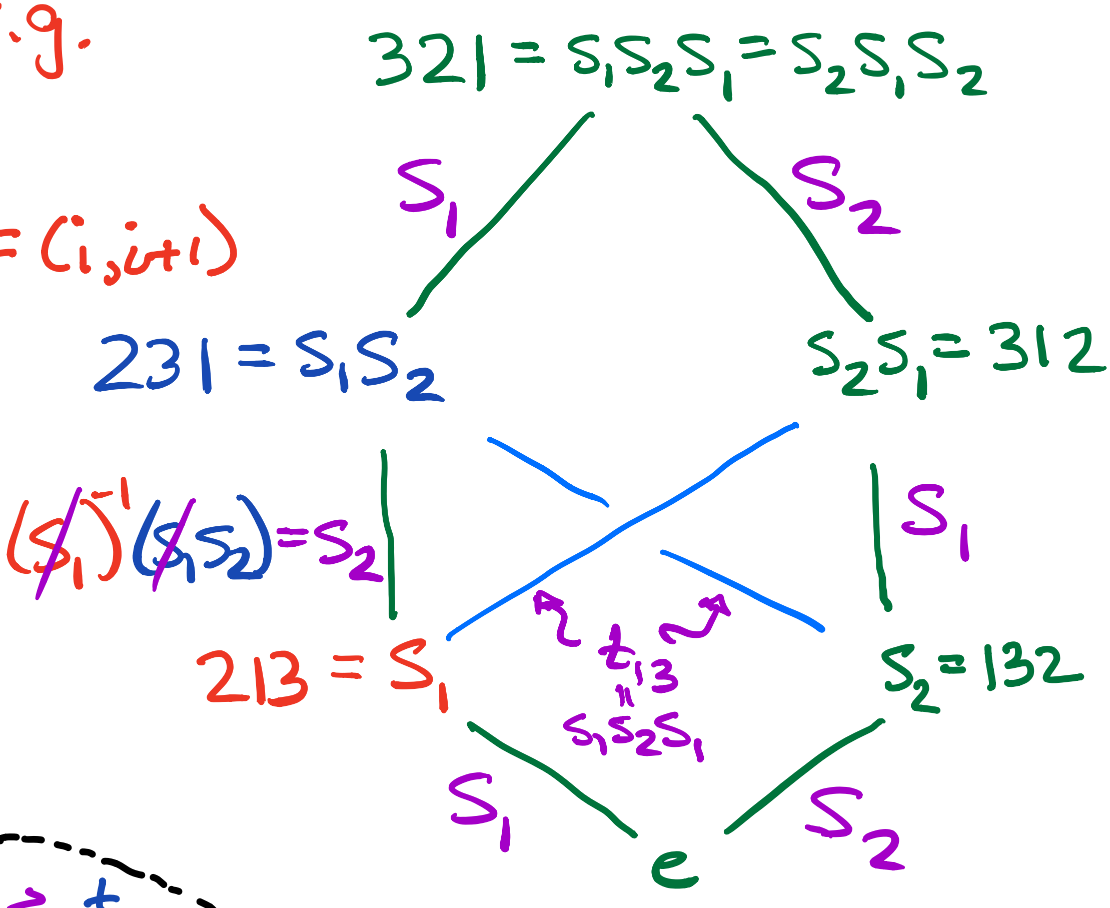
- "descending" chains $\leftrightsquigarrow$ facets attaching along entire bdry $\leftrightsquigarrow$ spheres

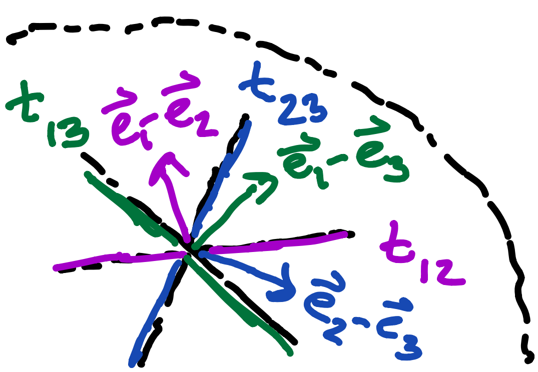
- $\mu_P(u,v) = \pm$ # descending chains u to v (for P graded)

Bruhat Order of a (Finite) Reflection Group / Coxeter Group & Dyer's EL-Labeling

e.g.

$s_i := (i, i+1)$





EL-labeling: $u \lessdot v = ut$

$\lambda(u,v) := u^{-1} v = t$

Dyer's EL-labeling (cont)

- Use any "reflection order" to totally order edge labels
- Dyer proved these exist & induce EL-labelings

Defn: A total order on positive roots (& assoc'd reflections) is **reflection order** if $\alpha < c_1\alpha + c_2\beta < \beta$ or $\beta < c_1\alpha + c_2\beta < \alpha$ for each such triple of positive roots w/ $c_1, c_2 > 0$

e.g $(1,2) < (1,3) < (2,3)$ or $(2,3) < (1,3) < (1,2)$ in type A

A Useful Characterization of Bruhat Order Cover Relations (& Label Sequences)

Thm (Dyer, preprint 2011; rediscovered H. 2017)

Given $u \in W$ & reflection $t_\gamma \in W$,

$u \lessdot u \cdot t_\gamma \iff$ (1) $\gamma \notin R(u)$

(2) $\nexists\, \alpha, \beta \notin R(u)$ s.t. $\gamma = c_1\alpha + c_2\beta$ for $c_1, c_2 > 0$

Recall: $\gamma \in R(u) \iff \ell(u \cdot t_\gamma) < \ell(u)$

e.g. $e \not\lessdot (1,3)$ but $(1,2) \lessdot (1,2)\cdot(1,3)$

$(1,2) \lessdot (1,2)\cdot(2,3)$

$(1,3) \leftrightarrow e_1 - e_3$

$\shortparallel$

$(e_1 - e_2) + (e_2 - e_3)$

Bruhat Order as Face Poset of Map Image & as CW Poset

- $$x_i(t) = I_n + tE_{i,i+1} = \begin{array}{cc} & \text{column } i+1 \\ & \downarrow \\ \text{row } i \rightarrow & \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \end{array}$$

  $x_i(t) = \exp(te_i)$ (general finite type, expon'd Chevalley generator)

  $I_n + tE_{i,i+1}$ (type A)

- $$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$$

  $$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

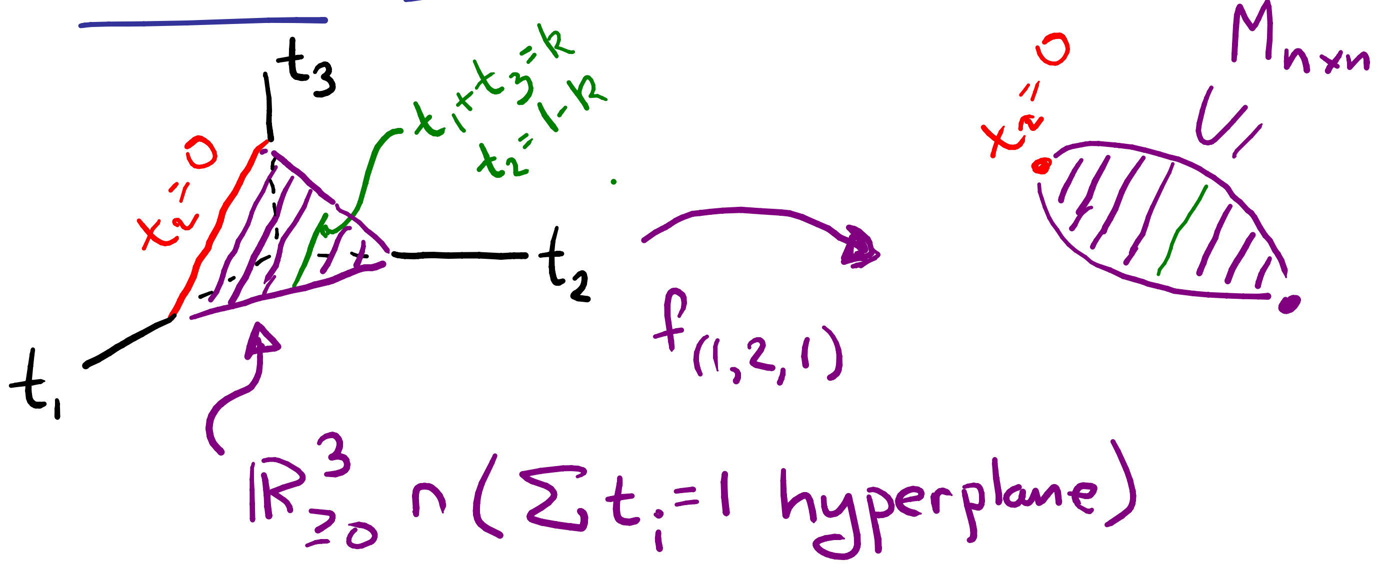
e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Rk: $(i_1,\dots,i_d)$ word for $w_0 \Rightarrow \mathrm{im}(f_{(i_1,\dots,i_d)})$ = totally nonneg. part unipotent radical of Borel

"Picture" of Map



$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow \; t_2 = 0$$

$$x_1(t_1) \circ x_1(t_3)$$

$$f_{(1,2,1)}(t_1,0,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

Non-injectivity: results from "nil-moves" $x_i(u)x_i(v) = x_i(u+v)$ & "long braid moves"

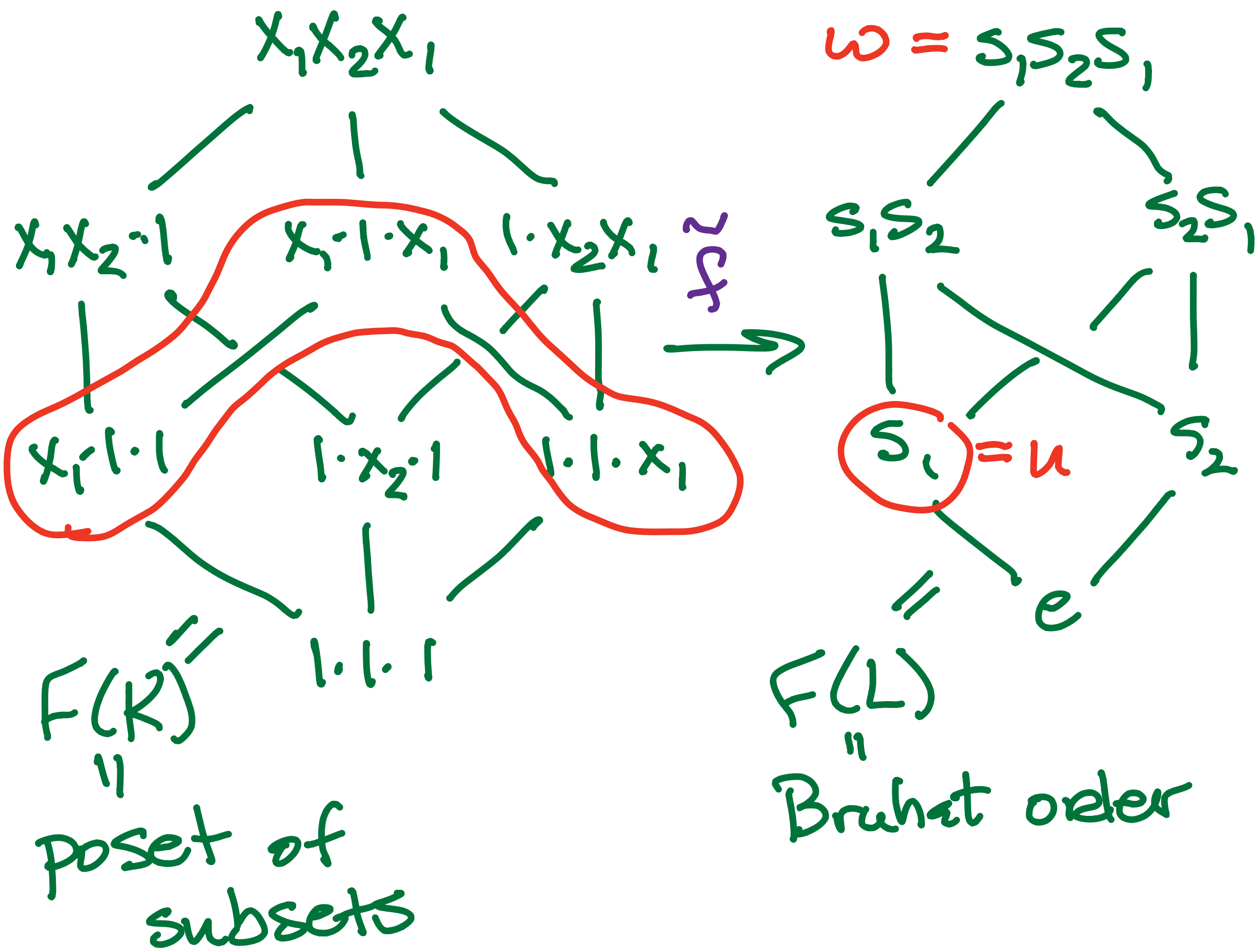
Thm (Fomin-Shapiro): Face poset for image of $f_{(i_1,\dots,i_d)}$ is Bruhat interval $[\hat{0},w]$ for $(i_1,\dots,i_d)$ reduced word for w.

Thm (H., 2014): Image of $f_{(i_1,\dots,i_d)}$ is regular CW complex homeomorphic to closed ball. ("Fomin-Shapiro Conj.")

Lusztig: Connections to "dual canonical bases"

Galashin-Karp-Lam (July 2017): clever, shorter proof of homeom. type for closure of "big cell" for $w=w_0$ in type A; other spaces including $Gr_{\geq}(n,k)$ & closed big cell for electrical networks in "well-connected graphs"

Induced Map of Face Posets



Obs (Armstrong-H.): Each subposet $\tilde{f}^{-1}_{\geq u}$ of $F(K)$ is dual to face poset of "subword complex" $\Delta(Q, u)$ of Knutson-Miller.

reduced word for w

Maps Arising from Electrical Networks (see R. Kenyon, "The Laplacian on Planar Graphs & Graphs on Surfaces")

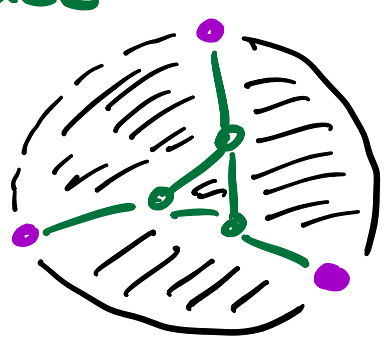
$$\underbrace{\Delta}_{\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}} \begin{pmatrix} v_N \\ v_I \end{pmatrix} = \begin{pmatrix} c_N \\ 0 \end{pmatrix}$$

vector of voltages

vector of currents

I = internal nodes

N = boundary nodes



$$(\underbrace{A - BC^{-1}B^T}_{\text{"response matrix of network"}}) v_N = c_N$$

(entries are rat'l fns of conductances)

A Goal: Given a graph G, study the space of response matrices as image of

$$f : \left\{ \begin{matrix} \text{conductance} \\ \text{vectors} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{response} \\ \text{matrices} \end{matrix} \right\}$$

$$\overset{\shortparallel}{\left(\mathbb{R}_{\geq 0} \cup \{\infty\}\right)^{|E|}}$$

Note:

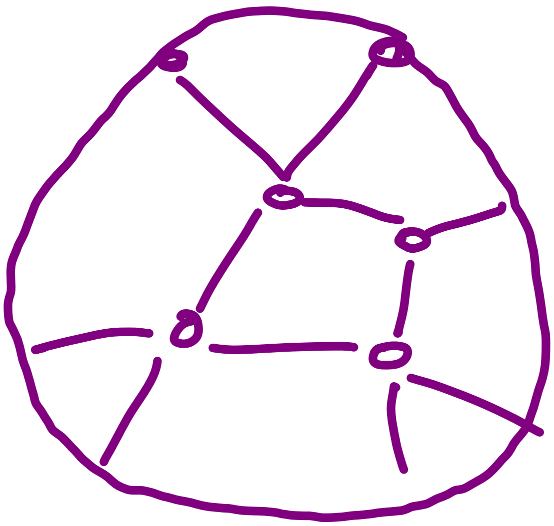
contracting an edge $\leftrightsquigarrow$ sending conductance to ∞ (i.e. resistance to 0)

deleting an edge $\leftrightsquigarrow$ sending conductance to 0 (resistance to ∞)

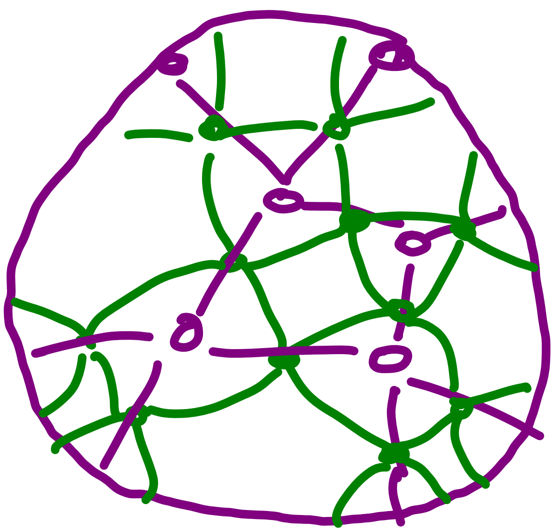
Secondary Goal: Study fibers of f

Correspondence: From Graphs to Uncrossing Wire Diagrams "Medial Graphs"

$\Gamma =$

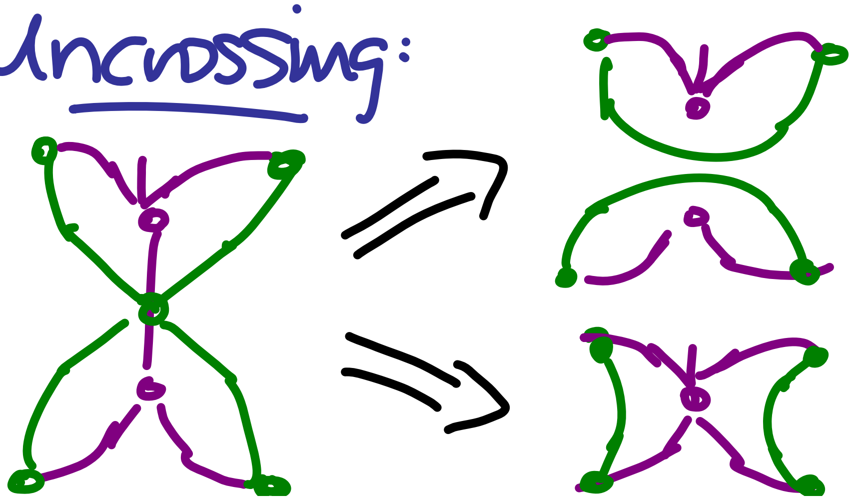


$G(\Gamma) =$



= wire diagram

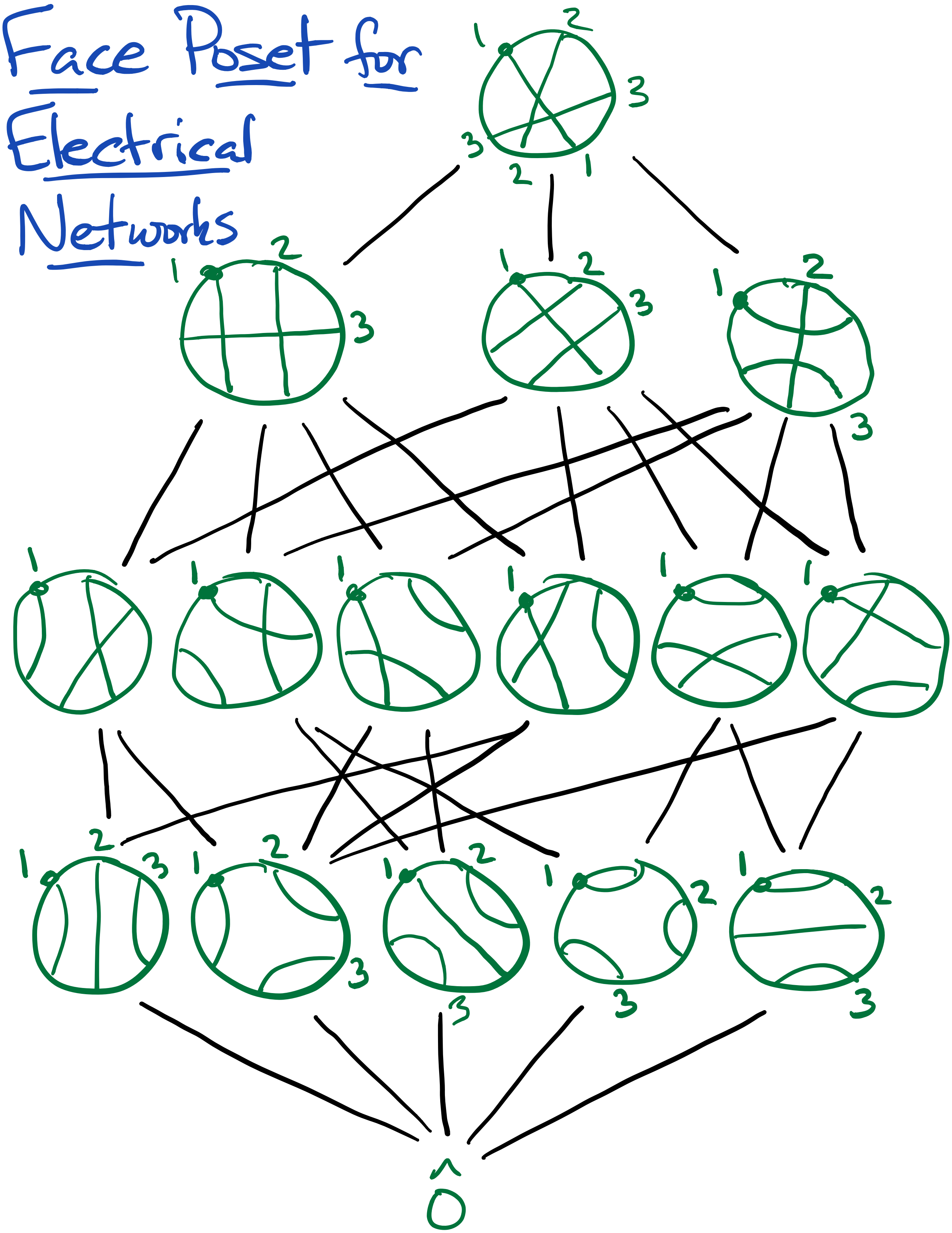
Uncrossing:



deletion

contraction

Face Poset for Electrical Networks



A Conjecture of Thomas Lam

Thm (Lam): The uncrossing poset is Eulerian.

Conjecture (Lam): The uncrossing poset is lexicographically shellable.

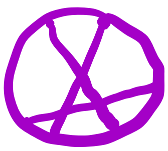
Thm (H.-Kenyon): Uncrossing posets are dual EC-shellable.

Cor: They are CW posets.

The Uncrossing Poset (Face Poset for Electrical Networks)

- $\hat{1}$:= wire diagram w/ all $\binom{n}{2}$ crossings of n wires

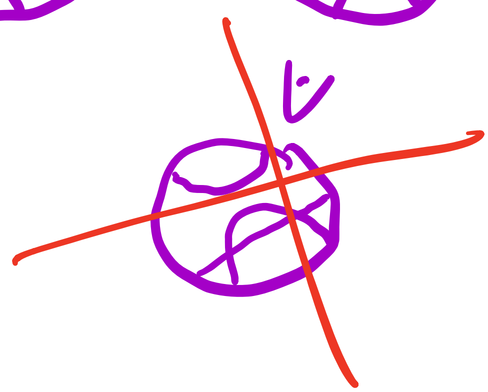
  $\rotatebox{90}{=}$

- $u \lessdot v$ if u obtained from v uncrossing pair of wires without introducing double crossing

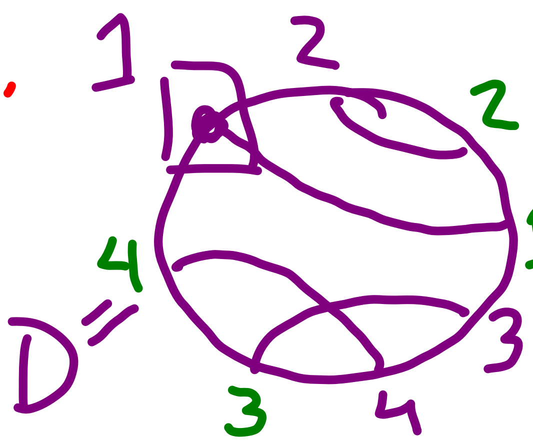
- $\hat{0}$ adjoined below Catalan many atoms

Edge Labeling

Step 1: Define **word** of wire diagram D, denoted $w(D)$, as sequence of $2n$ wire endpts encountered clockwise starting with "root" 1.

e.g.

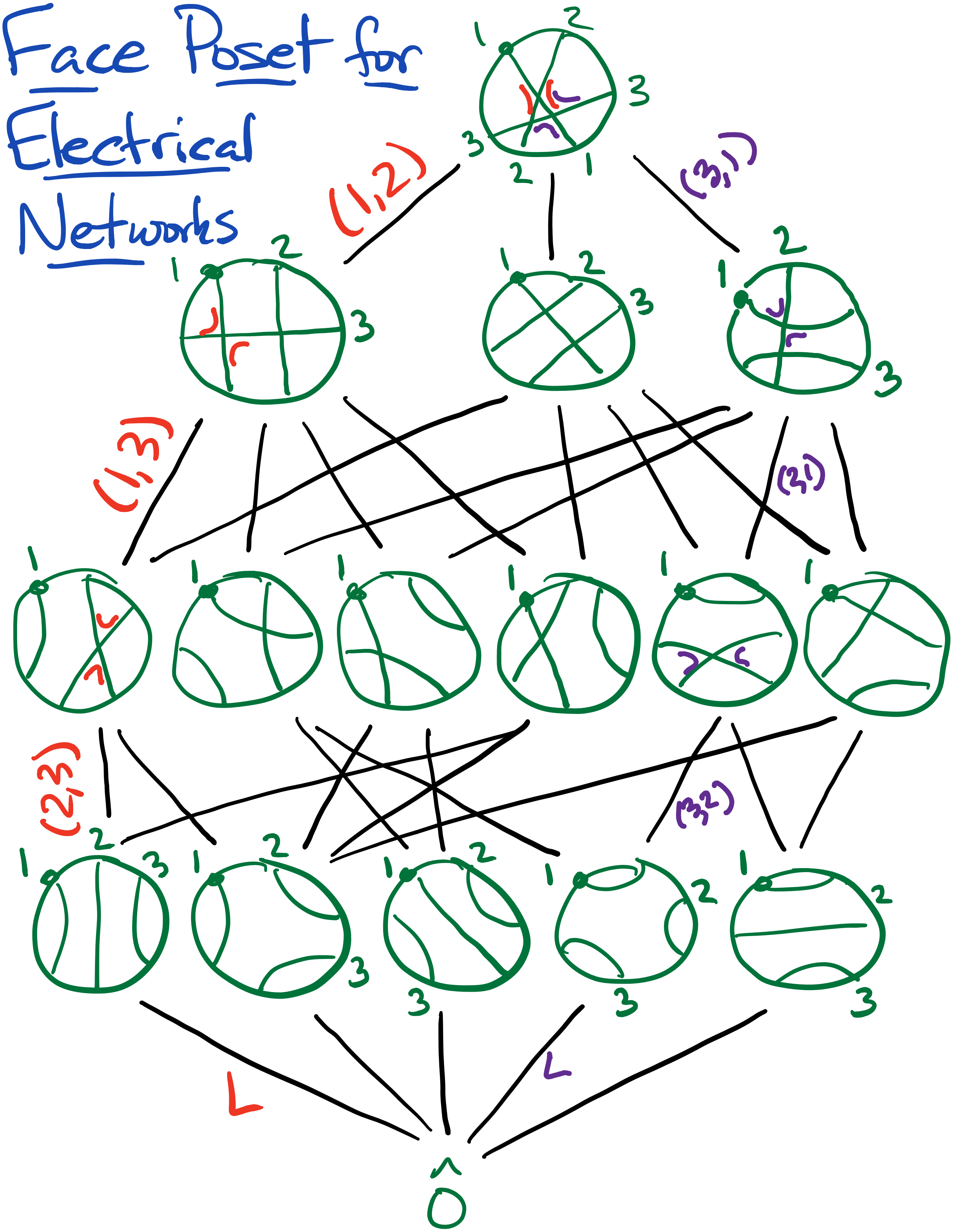


$\rightsquigarrow$ 12213434 $= w(D)$

Step 2: Label $D \lessdot D'$ for $i<j$ as

- (i,j) if $ijij$ in $w(D')$ becomes $ijji$ in $w(D)$
- (j,i) if $ijij$ becomes $iijj$

Face Poset for Electrical Networks



Label Ordering

- labels $\{(i,j) \mid 1 \le i < j \le n\} \cup \{L\}$
  $\cup \{(j,i) \mid 1 \le i < j \le n\}$

- $(i,j) < L < (r,s)$ for all $i<j$
  and all $r>s$

- $(1,2) < (1,3) < (1,4) < \cdots < (1,n) < (2,3)$
  $< (2,4) < \cdots < (2,n) < (3,4) < \cdots < (3,n)$
  $< \cdots < (n-1,n)$

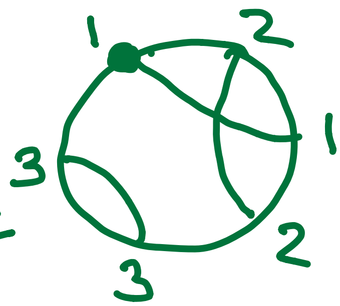
- $(n,n-1) < (n,n-2) < (n-1,n-2) < (n,n-3)$
  $< (n-1,n-3) < (n-2,n-3) < \cdots$
  $< (2,1)$

Rk: finite type A reflection order $\{(i,j)\}$ then L, then reversal for $\{(j,i)\}$

"Start Sets" and Connection to Type A Bruhat Order

The start set of D, denoted $S(D)$, is the subset of $\{1,2,\ldots,2n\}$ of positions in $w(D)$ where 1st copies of letters occur.

e.g.

$D =$ 

$w(D) = \underline{1}\,\underline{2}\,1\,2\,\underline{3}\,3$

$S(D) = \{1,2,5\}$

Prop'n: If $D' < D$ and $S(D') = S(D)$, then $[D',D] \cong [\underbrace{\pi(D'),\pi(D)}_{\text{subwords of } w(D') \,\&\, w(D)}]_{\text{Bruhat}}$

- uncrossing order labeling coincides here w/ Dyer's Bruhat order labeling

Prop'n: $D' \leq D \Rightarrow S(D') \leq_{lex} S(D)$

(very helpful for proving $[D', D]$ has unique topologically ascending chain)

Noncrossing Sets

Given a wire diagram D, its noncrossing set is defined as $N(D) = N_1(D) \cup N_2(D)$ for

- $N_1(D) = \{\underbrace{(i,j)}_{i<j} \mid w(D) \text{ includes } ijji\}$
- $N_2(D) = \{\underbrace{(j,i)}_{j>i} \mid w(D) \text{ includes } iijj\}$

New Description of Cover Relations in Uncrossing Order

Given wire diagram D, there is $D' \lessdot D$ uncrossing wires k & m with $\lambda(D', D) = (k,m) \in N(D)$

$\iff$ (1) $(m,k) \notin N(D)$

(2) $k < m \Rightarrow$ for $k < \ell < m$
$|\{(k,\ell),(\ell,m)\} \cap N(D)| = 1$

(3) $m < k \Rightarrow$ for $\ell < m$ or $\ell > k$
$|\{(k,\ell),(\ell,m)\} \cap N(D)| = 1$

Type A Specialization of Bruhat Order Cover Relation Description

$\pi = \pi(1)\pi(2)\ldots\pi(n)$ (one line notation)

has

$\pi < \cdot\ \pi \cdot (i,k)$ ← swap letters i and k

$\Updownarrow$

- i appears to left of k
- for each j s.t. $i<j<k$, either j appears to left of i or j appears to right of k

e.g. $\pi = 3\underline{2}\underline{5}1\underline{4}$ ← inversion 5,4

$\pi \cdot (2,4) = 3\underline{4}\underline{5}1\underline{2}$ ← inversion 5,2

Dyer's Proof Reflection Labelings are EL-labelings

- interpreted number of ascending chains from u to v, namely $u \lessdot u_1 \lessdot \cdots \lessdot u_k \lessdot v$ s.t. $\lambda(u,u_1) \le \lambda(u_1,u_2) \le \ldots \le \lambda(u_k,v)$, as leading coef. of $\tilde{R}_{u,v}(q)$

- observed EL-shellability then followed from result of Kazhdan & Lusztig that $\tilde{R}_{u,v}(q)$ is monic.

Thm (H.): Elementary proof in type A (without properties of KL-polys)

$\tilde{R}$-poly's: Unique polys $\{\tilde{R}_{u,v}[q]\}$, $\tilde{R}_{u,v}[q] \in \mathbb{N}[q]$

s.t.

$$\underbrace{R_{u,v}(q)}_{} = q^{\frac{\ell(u,v)}{2}} \tilde{R}_{u,v}(q^{1/2} - q^{-1/2})$$

used to define Kazhdan-Lusztig poly's

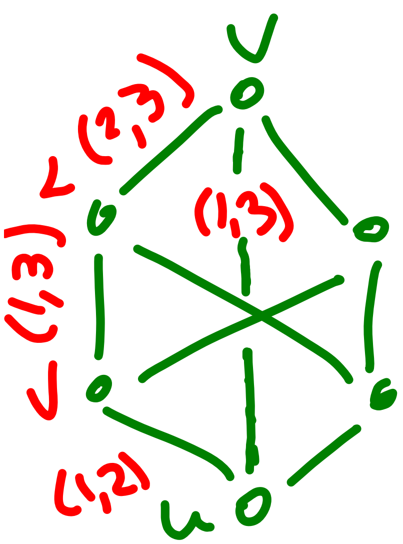
Thm (see e.g. Bjorner-Brenti 5.3.4)

$$\tilde{R}_{u,v}(q) = \sum_{\Delta \in B(u,v)} q^{\ell(\Delta)}$$

$\ell(\Delta)$: # edges in upward path Δ in Bruhat graph $B(u,v)$

s.t. $D(\Delta, <) = \emptyset$

$<$: any fixed reflection order



$\tilde{R}_{u,v}(q) = q^3 + q$

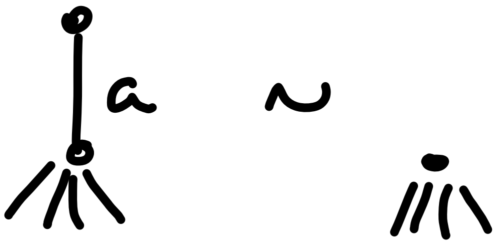
Appendix: Some Further Details

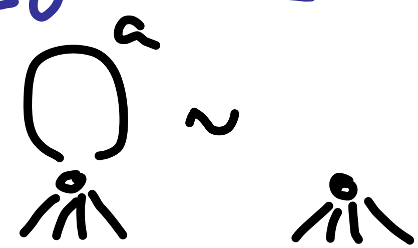
slides available at:

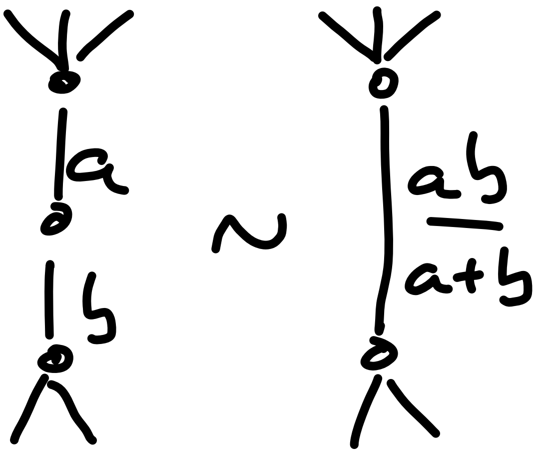
- http://www4.ncsu.edu/~plhersh/

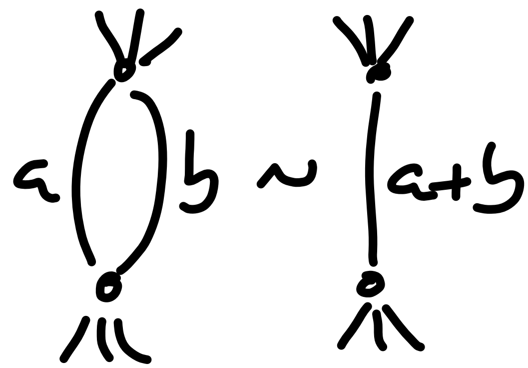
Thank you!

Turning to Fibers via "Electrical Equivalence"

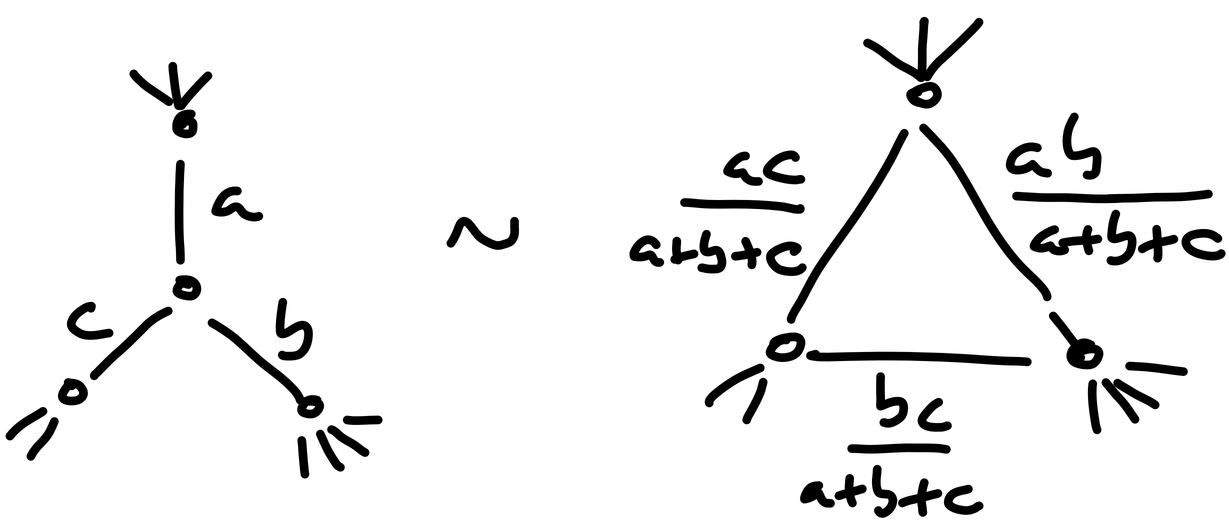
(1) 

(2) 

(3) 

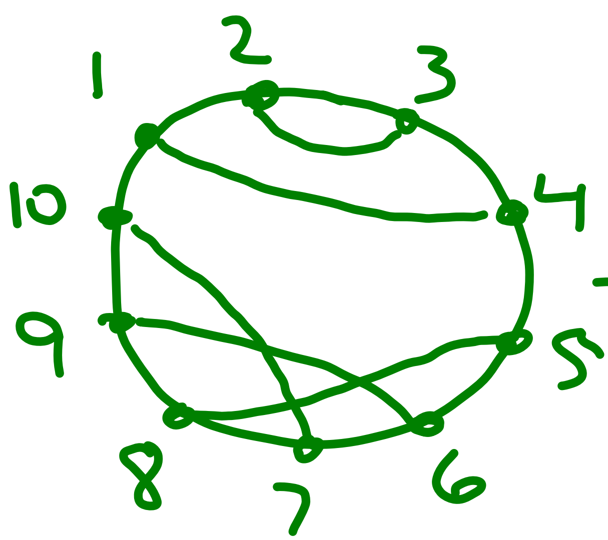
(4) 

(5) "Y-Δ moves"



Connection to Bruhat Order in Affine Type A

$\tilde{S}_{2n} := \{ f : \mathbb{Z} \to \mathbb{Z} \mid f$ bijection s.t.
$f(i+2n) = f(i) + 2n \;\&$
$\sum_{i=1}^{2n} f(i) = 2n^2 + \sum_{i=1}^{2n} i \}$



medial graph

$\longrightarrow (23)(14)(58)(69)(7,10)$

$\parallel$

$\tau \in S_{2n}$
fixed pt
free involution

$\downarrow$

$g_\tau \in \tilde{S}_{2n}$

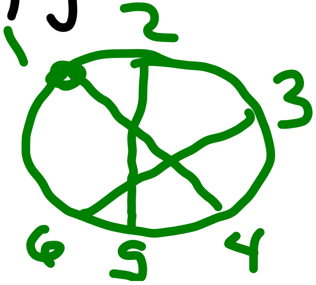
$g_\tau(2) = 3 \qquad g_\tau(1) = 4$

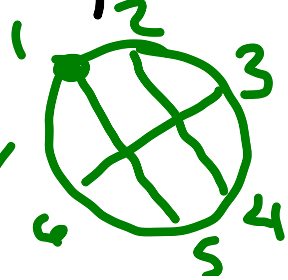
$g_\tau(3) = 2+5 \quad g_\tau(4) = 1+5 \quad \ldots$

Lam's Dual Embedding of Uncrossing Order into $\tilde{S}_{2n}$ Bruhat Order

- D_e = fully crossed diagram $\rightsquigarrow$ $1_D \in \tilde{S}_{2n}$
  $= g_2$ s.t. $i \to i+n$

- $D \lessdot D'$ where D obtained from D' by i,j wire endpoint swap $\rightsquigarrow$ $g_D = (i,j)\,g_{D'}\,(i,j)$

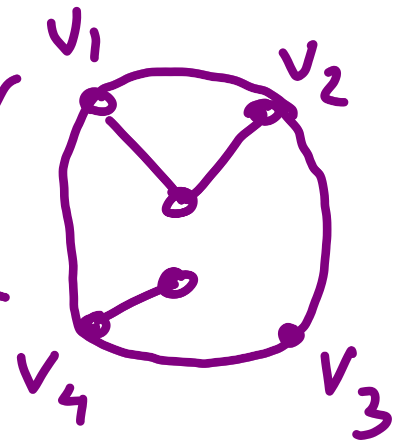
e.g. $D' =$ 

$D =$ 

$g_D = (1,2)\,g_{D'}\,(1,2)$
$= (4,5)\,g_{D'}\,(4,5)$

Minors of Response Matrix via "Groves"

$\pi(\underset{\text{graph}}{G}) :=$ set partition of bdry nodes into connected components

e.g. $\pi\Bigg($ 

$\Bigg) = 12|3|4$

Thm (Special Case of Next Result):

$$L_{ij}(G) = \frac{\displaystyle\sum_{\substack{G' \leq G \\ \pi(G') = ij|\text{singletons}}} wt(G')}{\displaystyle\sum_{\substack{G' \leq G \\ \pi(G') = \text{all singletons}}} wt(G')}$$

$wt(G') =$ product of edge weights (i.e. conductances)

Thm (Kenyon-Wilson; Curtis-Ingerman-Morrow)

For $|S|=|R|$ with $S\cap R=\emptyset$,

$$\det\left(L_{R\cup T}^{S\cup T}\right)=(-1)^{|T|}\cdot\sum_{p\in S_{|R|}}\operatorname{sgn}(p)\cdot K_p$$

for $$K_p=\frac{\displaystyle\sum_{\substack{G'<G\\ \pi(G')=r_1,\pi(r_1)|r_2,\pi(r_2)|\cdots\\ \cdots r_k,\pi(r_k)|\text{singletons}}} wt(G')}{\displaystyle\sum_{\substack{G'<G\\ \pi(G')=\text{all singletons}}} wt(G')}$$

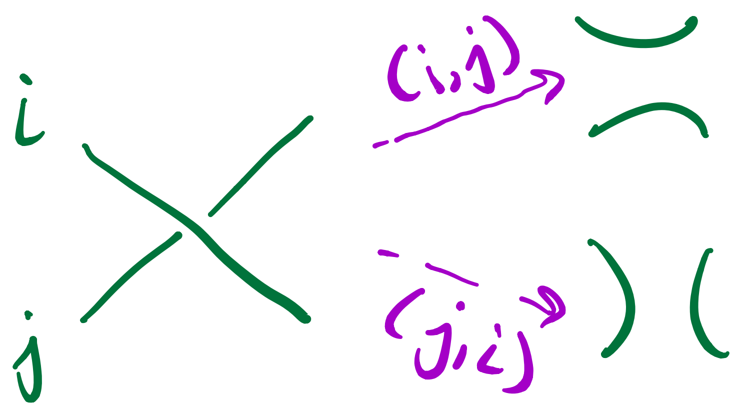
Tempting Idea Which Doesn't (Quite) Work

- label uncrossing of wires i and j for $1 \le i < j \le n$ as (i,j) or (j,i) ← exchange i with $j-n$

exchange i & j

$$(i,j) \in \tilde{S}_n \ni (j,i)$$

$$\tilde{S}_n = \left\{ f : \mathbb{Z} \to \mathbb{Z} \;\middle|\; f(i+n) = f(i)+n,\ \sum_{i=1}^{n} f(i) = \binom{n+1}{2} \right\}$$



e.g. $\boxed{n=3}$

$(1,3) \longleftrightarrow (e_1 - e_2) + (e_2 - e_3)$

$1 \mapsto 3 \mapsto 1 \quad 2 \mapsto 2 \quad -2 \mapsto 0 \ldots$

$\underset{s_0}{\overset{\shortparallel}{(3,1)}} \longleftrightarrow \delta - (e_1 - e_3)$

$1 \mapsto 0 = 3-3 \mapsto 1 \quad 2 \mapsto 2 \ldots$

Subword Complexes (introduced by Knutson & Miller)

$Q :=$ (not necessarily reduced) expression

$w :=$ Coxeter group element

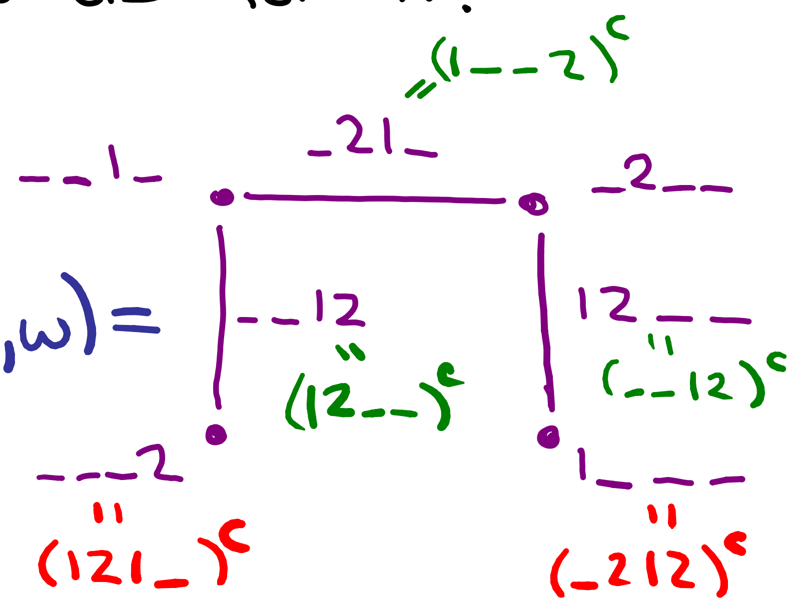
Facets of $\Delta(Q,w)$ are the subwords of Q whose complements are reduced words for w.

e.g.

$Q = (1,2,1,2)$

$w = s_1 s_2$

$\Delta(Q,w) =$



Thm (Knutson-Miller): $\Delta(Q,w)$ is "vertex decomposable" (hence shellable) ball or sphere.

(Used to study matrix Schubert varieties via "Gröbner degeneration")

Kazhdan-Lusztig Polynomials

KL-poly's: Unique $\{P_{u,v}(q) \in \mathbb{Z}[q]\}$

s.t. (1) $P_{u,v}(q) = 0$ for $u \not\leq v$

(2) $P_{u,u}(q) = 1$

(3) $\deg(P_{u,v}(q)) \leq \frac{1}{2}(\ell(u,v) - 1)$ for $u < v$

(4) $q^{\ell(u,v)} P_{u,v}\left(\frac{1}{q}\right) = \sum_{a \in [u,v]} R_{u,a}(q) P_{a,v}(q)$ for $u \leq v$

KL-poly $P_{u,v}(q)$
$\|$
"local intersection homology Euler characteristic of $\bar{O}_v$ at generic pt in O_u"

R-poly's: Unique $\{R_{u,v}(q)\}$ s.t.

(1) $R_{u,v}(q) = 0$ for $u \not\leq v$

(2) $R_{u,u}(q) = 1$

(3) for $s \in D_R(v)$, then

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } s \in D_R(u) \\ q R_{us,vs}(q) + (q-1) R_{u,vs}(q) & \text{otherwise} \end{cases}$$

Recall: A real matrix is **totally nonnegative** if all minors are nonnegative.

e.g. $\left\{ \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| t_1, t_2, t_3 \geq 0 \right\}$

Since $t_1+t_3 \geq 0$ $\quad t_2(t_1+t_3) - t_1t_2 \geq 0$

$t_2 \geq 0$

$t_1t_2 \geq 0$

also: $\left\{ \begin{pmatrix} 1 & t_2' & t_2't_3' \\ 0 & 1 & t_1'+t_3' \\ 0 & 0 & 1 \end{pmatrix} \middle| t_1', t_2', t_3' \geq 0 \right\}$

unipotent radical e.g. $\left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$

First: A Motivation for Nonneg. Real Part of Unipotent Radical

e.g. $(t_1, t_2, t_3) \mapsto \left(\frac{t_2 t_3}{t_1+t_3}, t_1+t_3, \frac{t_1 t_2}{t_1+t_3}\right)$
("simply laced" case)

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{t_1'} \quad \overset{\shortparallel}{t_2'} \quad \overset{\shortparallel}{t_3'}$

- Tropicalizes to change-of-basis map for Lusztig's "canonical bases":

$(a,b,c) \mapsto (b+c-\min(a,c), \min(a,c), a+b-\min(a,c))$

(applying braid move to reduced expression for w_0 w.r.t. which canonical basis is defined)

- Given quantized env. alg. $U = U^- \otimes_{\mathbb{Q}(v)} U^0 \otimes_{\mathbb{Q}(v)} U^+$

then **canonical basis** is a basis B for U^- such that highest weight module with highest weight vector v_λ has basis $\{v_\lambda b \mid v_\lambda b \neq 0\}$ for each λ.

Some Other Related Work

- Lauren Williams:
  - shelling face posets of nonneg flag varieties (2007)
- Galashin-Karp-Lam
  - homeomorphism type for closure of type A big cell for w_0 & for big cell for well-connected electrical networks (top element of full uncrossing poset) (preprint, July 2017)