# Combinatorics & Topology of Totally Nonnegative Spaces

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(partly joint work with Jim Davis & Ezra Miller & with Rick Kenyon)

(slides available at: https://plhersh.math.ncsu.edu/ND18.pdf)

# Topological Aspects of Total Positivity

- Lusztig, Fomin, ... initiated study of: totally nonnegative, real part of spaces of matrices, spaces of flags (i.e. $GL_n/B$)...
(i.e. having all minors nonnegative)
- Conjecturably / provably homeomorphic to closed balls
- Proving this by studying fibers (method used in H. & in DHM)
  - imposes restrictions on rel'ns amongst exp'd Chevalley gen's
  - reveals structure in Lusztig's canonical bases

# Today:

1. **Review:** What is known about these spaces which all arise as images of "nice" maps
2. **Fomin-Shapiro Conj (2014 Proof)**
   $\mathrm{im}(f_{(i_1,\ldots,i_d)})$ is regular CW ball
3. **New Result Regarding Fibers of $f_{(i_1,\ldots,i_d)}$, including:**
   (a) cell decomposition
   (b) combinatorial model

(Recall: 

closed ball homeomorphic )

# Background on Coxeter Groups

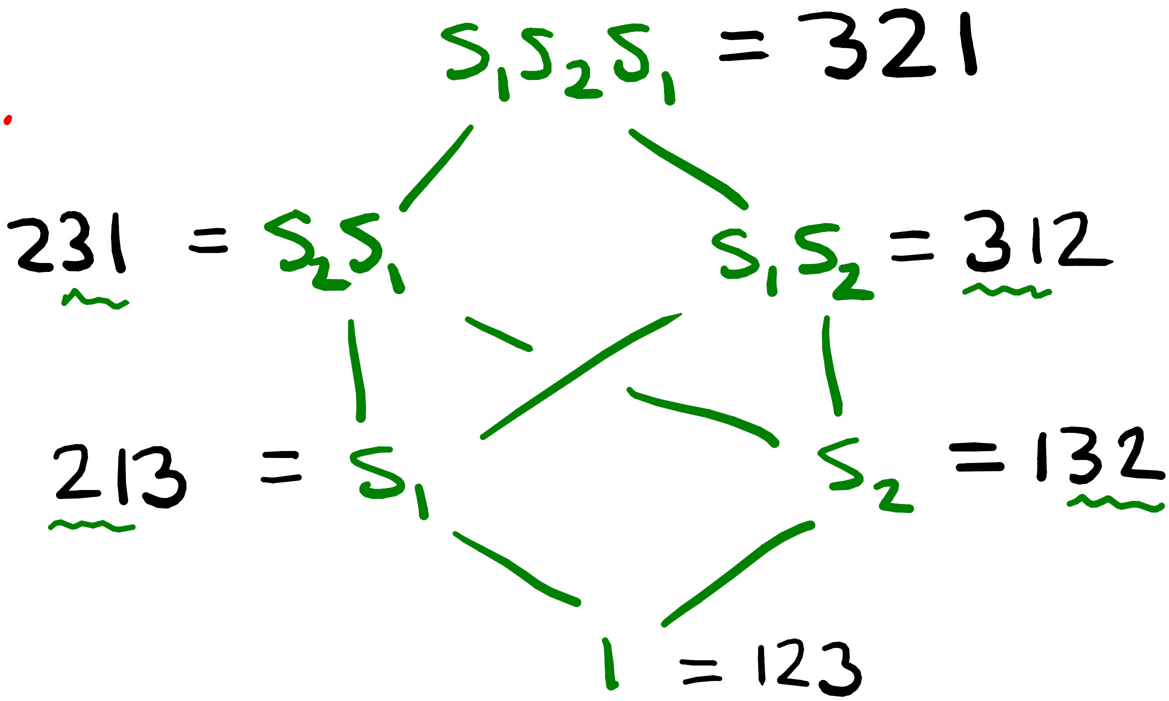
- $s_i := \underbrace{(i, i+1)}_{\text{type A (i.e. } W = S_n)}$ a simple reflection in $W$
- $s_{i_1} \cdots s_{i_d}$ is reduced expression for $w \in W$ if $w = s_{i_1} \cdots s_{i_d}$ for $d$ as small as possible
- length of $w$, denoted $\ell(w)$, is this smallest $d$.

e.g. $s_1 s_2 s_1 = s_2 s_1 s_2$ has length 3

$$\begin{array}{c} \| \\ 321 \end{array} \xrightarrow{s_1} 231 \xrightarrow{s_2} 213 \xrightarrow{s_1} 123$$

- $(i_1, \dots, i_\ell)$ is **reduced word**
- **Bruhat order:** partial order

e.g.




on $W$ (e.g. $S_n$) with $u \le v$ for $u, v \in W \iff$ any reduced word for $v$ has subword that is reduced word for $u$.

# Running Example: Totally Nonnegative Part of a Space of Matrices

$$\begin{array}{cc} & \text{column } i+1 \\ \bullet\ x_i(t) = I_n + tE_{i,i+1} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & t & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} & \text{row } i \end{array}$$

$x_i(t) = \exp(te_i)$ (general finite type, expon'd Chevalley generator)

$I_n + tE_{i,i+1}$ (type A)

$\bullet\ f_{\underbrace{(i_1,\dots,i_d)}_{\text{reduced word}}} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \subseteq \mathbb{R}^{n^2}$

$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1) \cdots x_{i_d}(t_d)$

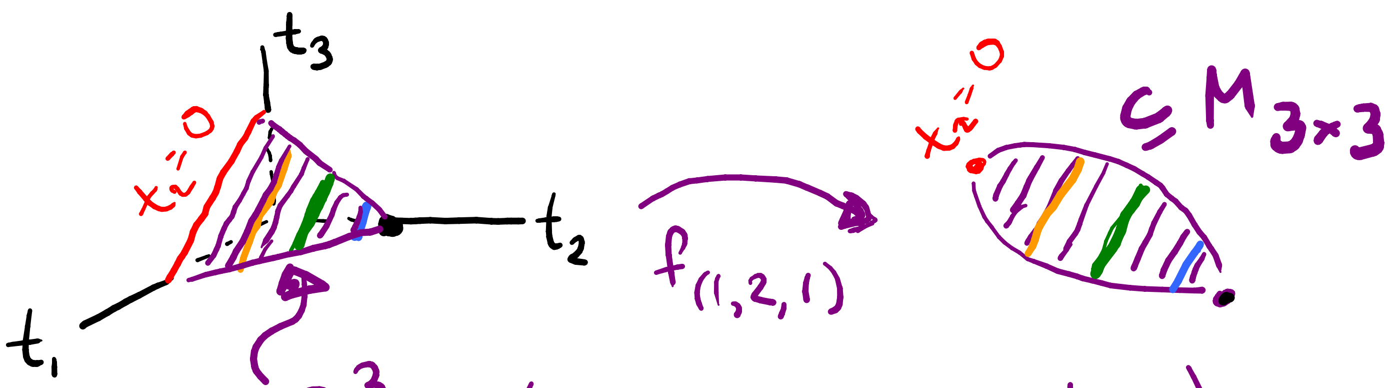
e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

e.g. $w_0$ case:

$$\left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \middle| \begin{array}{l}\text{tot.}\\ \text{nonneg}\end{array}\right\} = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

# "Picture" of Map $f_{(i_1,\dots,i_d)}$




$\mathbb{R}^3_{\geq 0} \cap (\sum t_i = \underbrace{K}_{\text{positive constant}} \text{ hyperplane})$

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow \; t_2 = 0$$

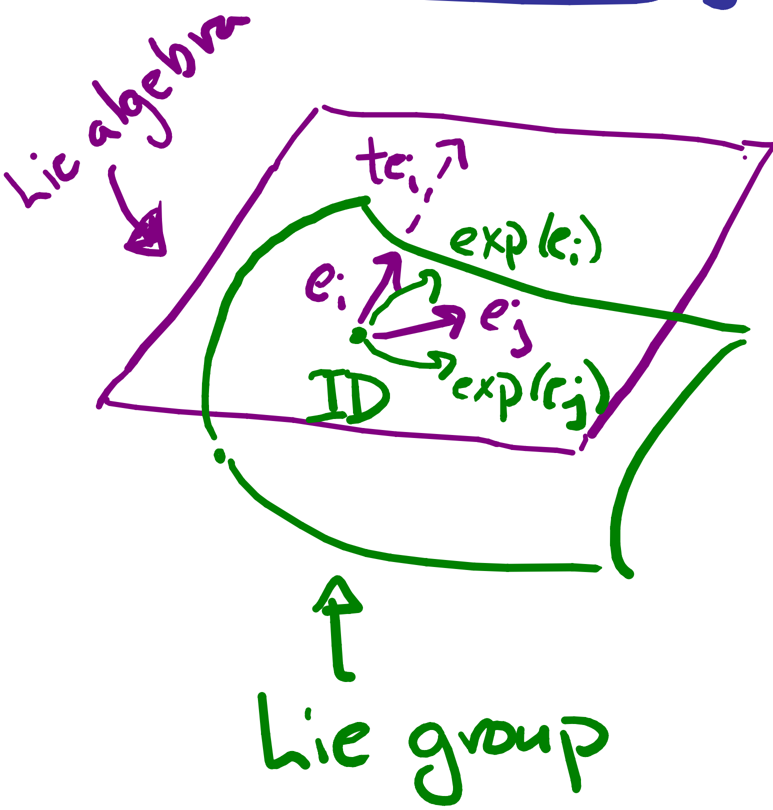
$$f_{(1,2,1)}(t_1,0,t_3) = \underbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_1)}\cdot\underbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_3)}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

simplex faces w/ same image "$x_1^2 = x_1$"

e.g. $\{x_1(t) \mid t > 0\} = \{x_1(t_1)x_1(t_2) \mid t_1, t_2 > 0\}$

# A Motivation: Understanding Relations Among (Exponentiated) Chevalley Generators




$$te_1 = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\exp(\_)} \begin{pmatrix} 1 & t & \\ & 1 & \\ & & 1 \end{pmatrix}$$

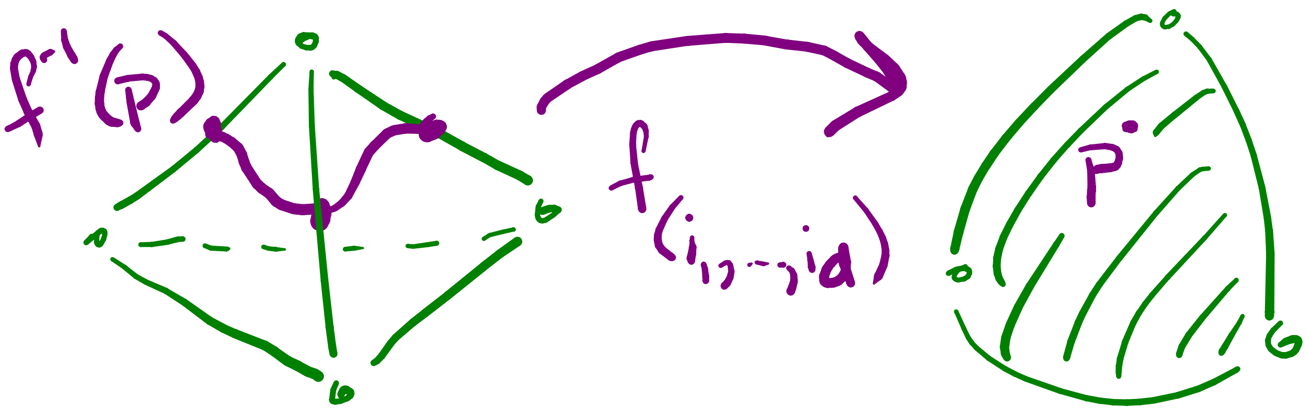
$$\exp(t_1e_i)\exp(t_2e_j)$$

$$f_{(i,j)}(t_1,t_2) = x_i(t_1)\,x_j(t_2)$$

$$\exp(te_i) = \boxed{ID + te_i} + \underbrace{\frac{t^2e_i^2}{2}}_{0=} + \underbrace{\frac{t^3e_i^3}{6}}_{0=} + \ldots$$

Reln's $\leftrightsquigarrow$ Elts in same fiber of $f_{(i_1 \ldots i_d)}$

**Conjecture (Davis-H-Miller):** $f^{-1}_{(i_1,\ldots,i_d)}(p)$ is regular CW complex homeomorphic to interior dual block complex of subword complex $\Delta((i_1,\ldots,i_d),w)$ for $p \in Y^o_w$.




**Thm (H. 2014):** $\mathrm{im}(f_{(i_1,\ldots,i_d)})$ is regular CW complex $\cong$ closed ball.

**Thm (DHM):** $f^{-1}_{(i_1,\ldots,i_d)}(p)$ has cell decomposition & correct face poset.

# Subword Complexes

(introduced by Allan Knutson & Ezra Miller)

$\Delta(\underset{\sim}{Q}, \underset{\sim}{w})$ = (abstract) simplicial complex whose faces are subwords $Q'$ of $Q$ whose complement $Q \setminus Q'$ contains a reduced word for $w$

$Q$: reduced or nonreduced word

$w$: Coxeter group element

**Aside:**

1st arose as Stanley-Reisner complexes of initial ideals of coordinate rings of matrix Schubert varieties.

**Thm (Knutson-Miller):** $\Delta(Q,\omega)$ is homeomorphic to ball or sphere.

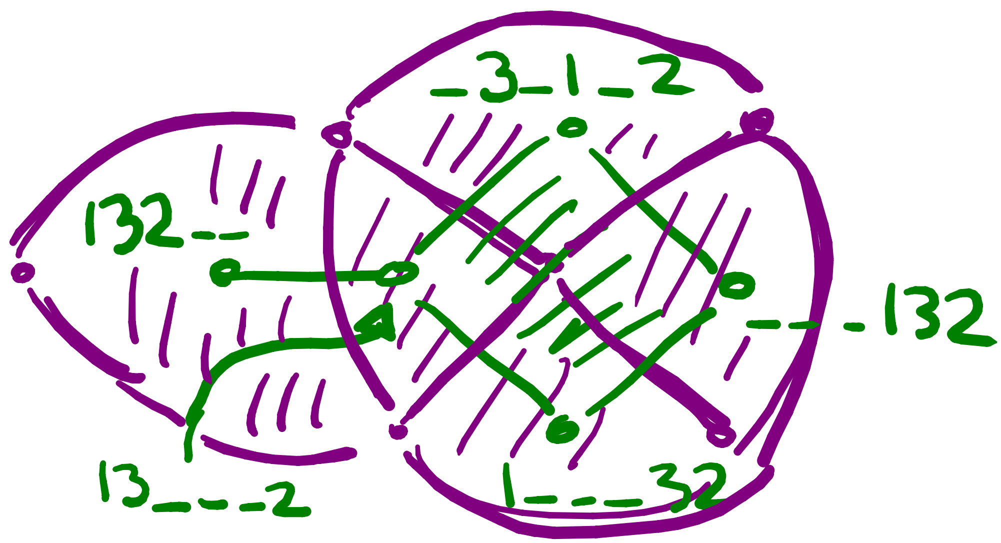
## Example of Subword Complex & its Interior Dual Block Complex

$\Delta(Q,\omega) =$

$Q = 132132$

$\omega = s_1 s_3 s_2$

& its




Interior dual block complex

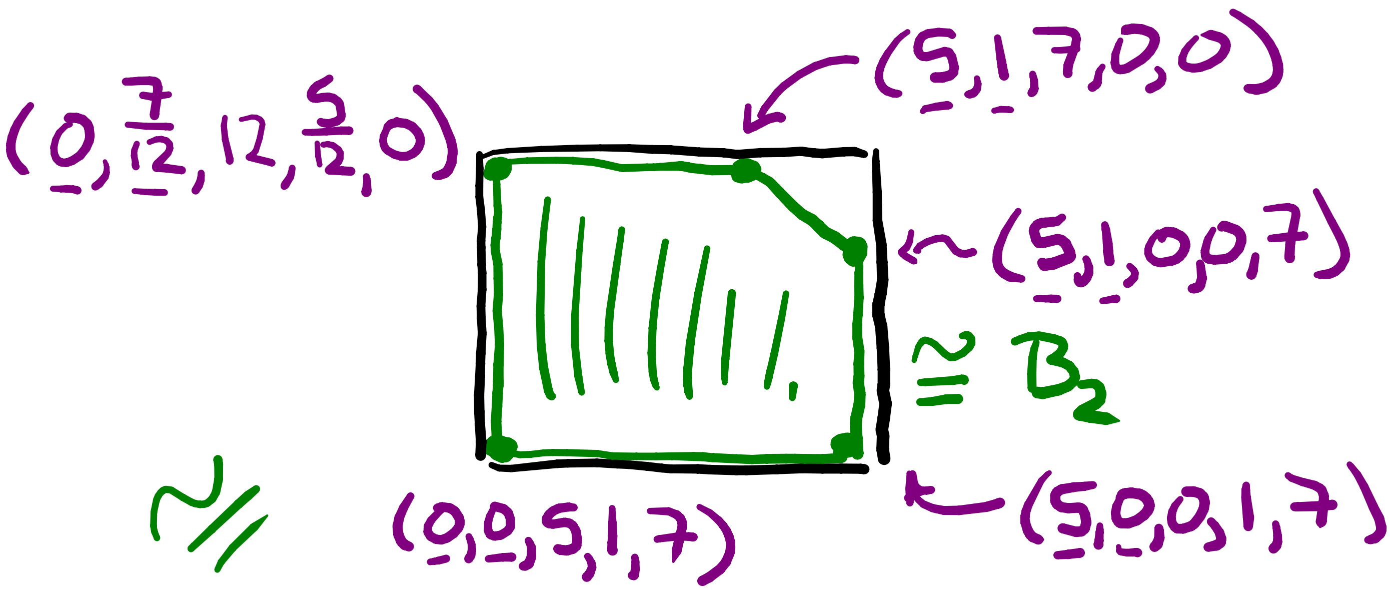
$$\Vert$$

$f^{-1}_{(1,3,2,1,3,2)}(M)$ for $M \in Y^{\circ}_{(1,3,2)}$

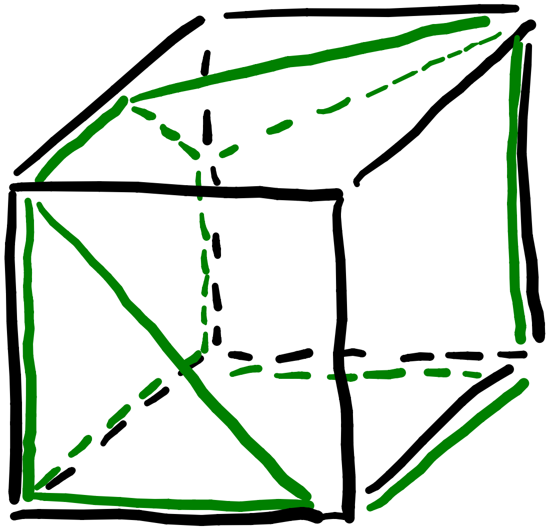
# Examples of Fibers:

(with realizations as suggested by various results towards proof of DHM conjecture)

e.g.



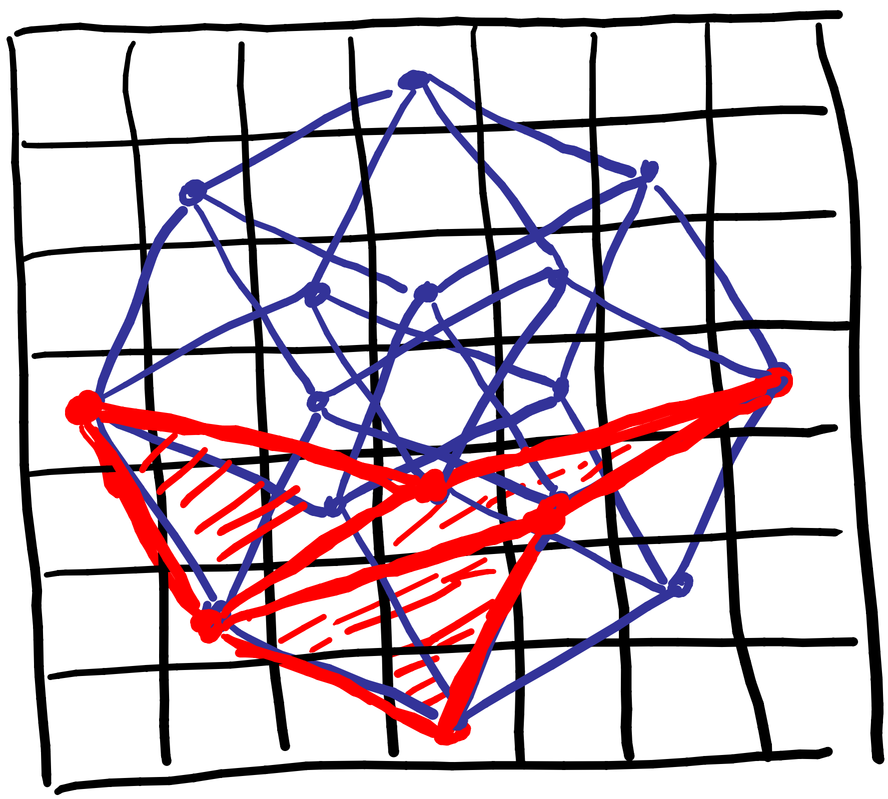

$f^{-1}_{(1,2,1,2,1)}(M)$ for $\underset{\substack{\cap \\ U^{0}_{s_1 s_2 s_1}}}{M} = x_1(5)\,x_2(1)\,x_1(7)$



$\cong B_3$

$\cong$

$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^\circ_{s_1 s_2 s_1}$

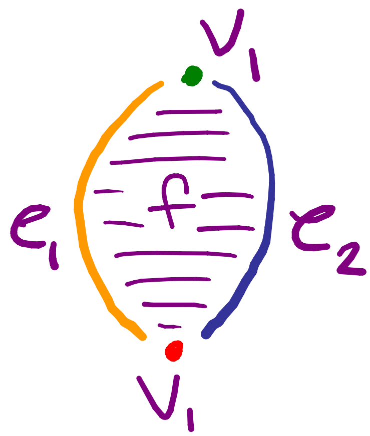


$\cong$

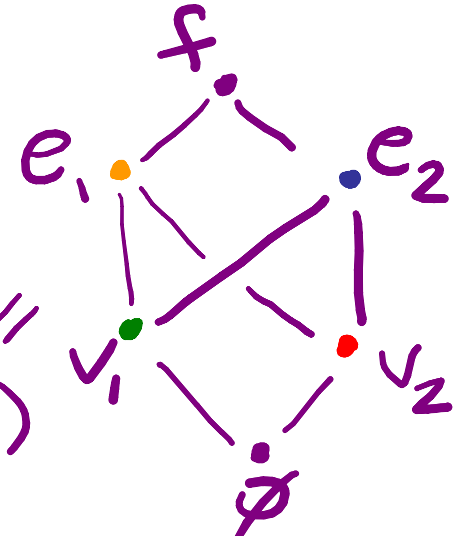
$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^\circ_{(1,2)}$

# CW Complexes & their Face Posets (Partially Ordered Sets)

e.g.

$K =$ 

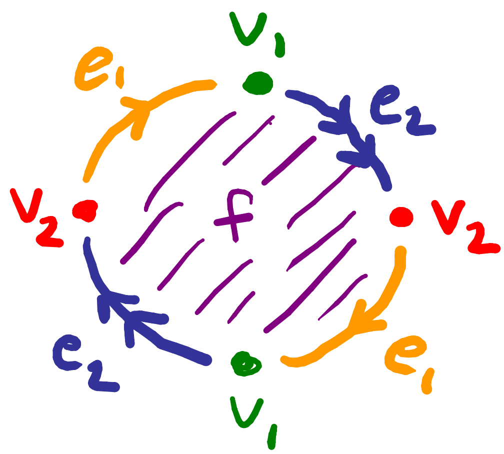

$\cong$ ball

$F(K) =$ 


$= F(K')$

"face poset"

$(u \le v \iff u \le \bar{v})$

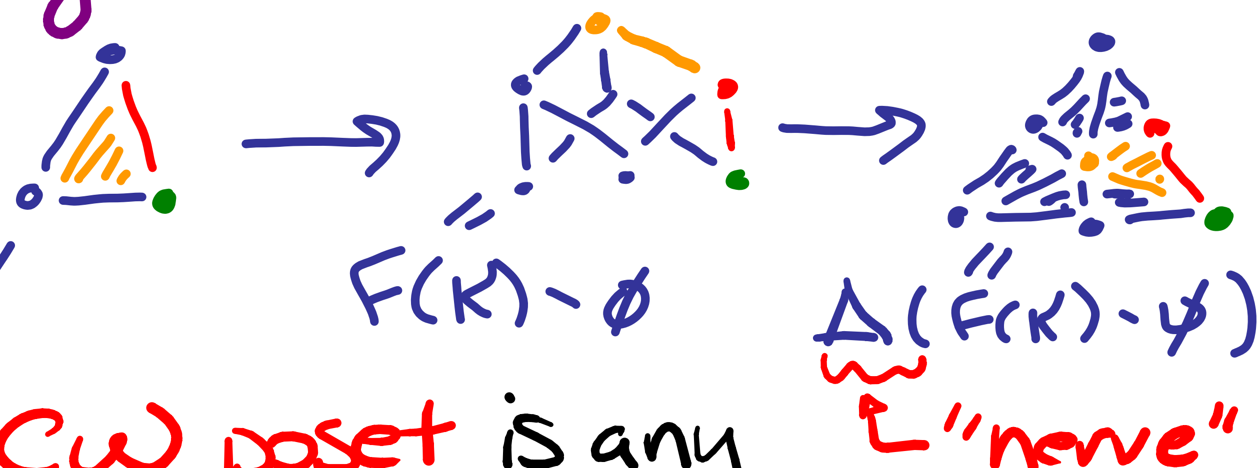
$K' =$ 


$= \mathbb{RP}^2$

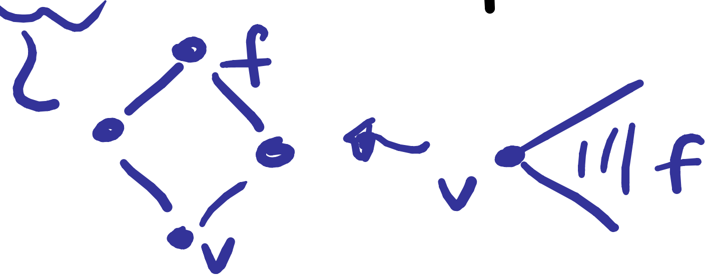
**Recall:** A **CW complex**: cells $e_\alpha \cong \mathbb{R}^{\dim(\alpha)}$, characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \to \bigcup_{\beta \subseteq \bar{\alpha}} e_\beta$ & attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

Recall: CW complex is regular if each $f_\alpha$ is homeomorphism.

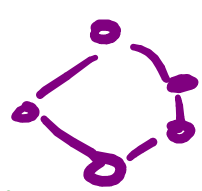
- $\Delta(P)$ = "nerve" or "order complex" of $P$
- $K$ regular $\Rightarrow K \cong \Delta(F(K)\setminus\emptyset) = sd(K)$




- a CW poset is any face poset of regular CW complex
- "Shellable" + "Graded" + "thin" $\Rightarrow$ CW poset




e.g. not



e.g. Bruhat order (Björner-Wachs; Matthew Dyer)

# Totally Nonnegative Spaces w/ Seemingly Analogous Structure

1. Totally nonnegative real part of Grassmannian: $Gr_{\geq 0}(k,n) = (GL_n/P)_{\geq 0}$

Postnikov: polytope of "plabic graphs" w/ "measurement map" to $Gr_{\geq 0}(k,n)$ + theory of (reduced) plabic graphs

Postnikov-Speyer-Williams: $Gr_{\geq 0}(k,n)$ is CW complex (via attaching maps that are not homeomorphisms)

Galashin-Karp-Lam 2017 preprint: $Gr_{\geq}(k,n)$ is homeom. to closed ball. (still open qn for cell closures)

2. Totally nonnegative real part of flag variety $\mathcal{F}\ell_{\geq 0} = (GL/B)_{\geq 0}$

Rietsch: poset of closure rel'ns. Cells $R^{\circ}_{u,v}$ given by $u \leq v$ in Bruhat order

Marsh-Rietsch: parametrization for $R^{\circ}_{u,v}$

Williams: poset is CW poset

Rietsch-Williams: (1) CW complex w/ attaching maps via canonical bases.
(2) Contractibility of each cell closure

Galashin-Karp-Lam: Homeomorphism type for closure of "big cell"

# 3. Stratified Spaces of Electrical Networks

(Curtis-Ingerman-Morrow, Kenyon-Wilson, ...)

- Image of map Resp :  $\mapsto$ response matrix

Lam: (1) $E_n \hookrightarrow Gr_{\geq 0}(n-1, 2n)$

(2) $F(E_n)$ is "Eulerian" ("i.e. correct Euler char")

H-Kenyon: $F(E_n)$ shelling & $F(E_n)$ is CW poset (uses Dyer Bruhat shelling)

H-K Cor: shelling for each $[u,v]$ in face poset for edge-product space of phylogenetic trees (so CW poset)

Galashin-Karp-Lam: Homeom. type for closure of "big cell" for graph analogue of $w_0$, "well-connected graph"

4. (Deconed) Totally nonneg. real part unipotent radical of Borel in semisimple simply connected algebraic group

(a) Vertex "link" (nbhd) in $(\mathcal{F}\ell_n)_{\geq 0}$ via Marsh-Rietsch parameters $\to 0$

(b) Image$(f_{(i_1,\ldots,i_d)})$ for $w_0 = w(i_1,\ldots,i_d)$

(c) Bruhat order as poset of closure rel'ns

(d) Each cell closure homeom. to closed ball.

- known as Fomin-Shapiro Conjecture
- proved by H., 2014, Invent.
- special case of $w_0$ in type A (shorter proof) GKL, 2017

# More Motivation for Nonnegative Real Part of Unipotent Radical

- Given quantized env. alg. $U = U^- \otimes_{\mathbb{Q}(v)} U^0 \otimes_{\mathbb{Q}(v)} U^+$ of Kac-Moody alg. (e.g. affine Lie alg.), then canonical basis is a basis $B$ for $U^-$ such that highest weight module with highest weight vector $v_\lambda$ has basis $\{v_\lambda b \mid b \in B, v_\lambda b \neq 0\}$ for each $\lambda$.

- $f^{-1}_{(i_1,\ldots,i_d)}(p) \to f^{-1}_{(j_1,\ldots,j_d)}(p)$ coordinate change

$$(t_1, t_2, t_3) \longmapsto \left(\frac{t_2 t_3}{t_1+t_3},\ t_1+t_3,\ \frac{t_1 t_2}{t_1+t_3}\right)$$

tropicalizes to coordinate change:

$$(a,b,c) \mapsto (b+c-\min(a,c), \min(a,c), a+b-\min(a,c))$$

for canonical bases w/ same braid move

# Role of 0-Hecke Algebra in Stratification for $\mathrm{im}(f_{(i_1,\ldots,i_d)})$

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

$\rightsquigarrow$ suppress parameters

$x_i x_i = x_i$

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2 t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1 t_2}{t_1+t_3}\right)$

for $t_1, t_2, t_3 > 0$

$\rightsquigarrow$ (type A)

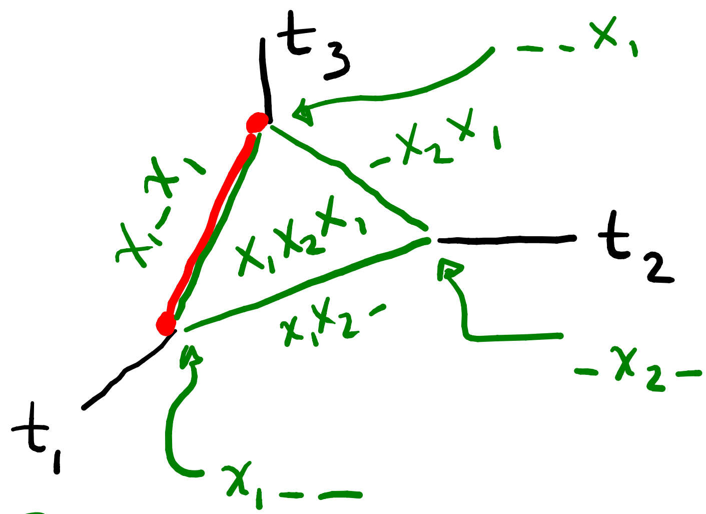
$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$

(& analogous relations outside type A)

**Upshot:** $\mathrm{im}(f_1) = \mathrm{im}(f_2) \iff \underbrace{x(f_1) = x(f_2)}_{\text{equal as 0-Hecke algebra elements}}$

**Thm (Lusztig):** If $(i_1,\ldots,i_d)$ is reduced, then $f_{(i_1,\ldots,i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

# Faces of (Preimage) Simplex as Subexpressions in 0-Hecke Algebra



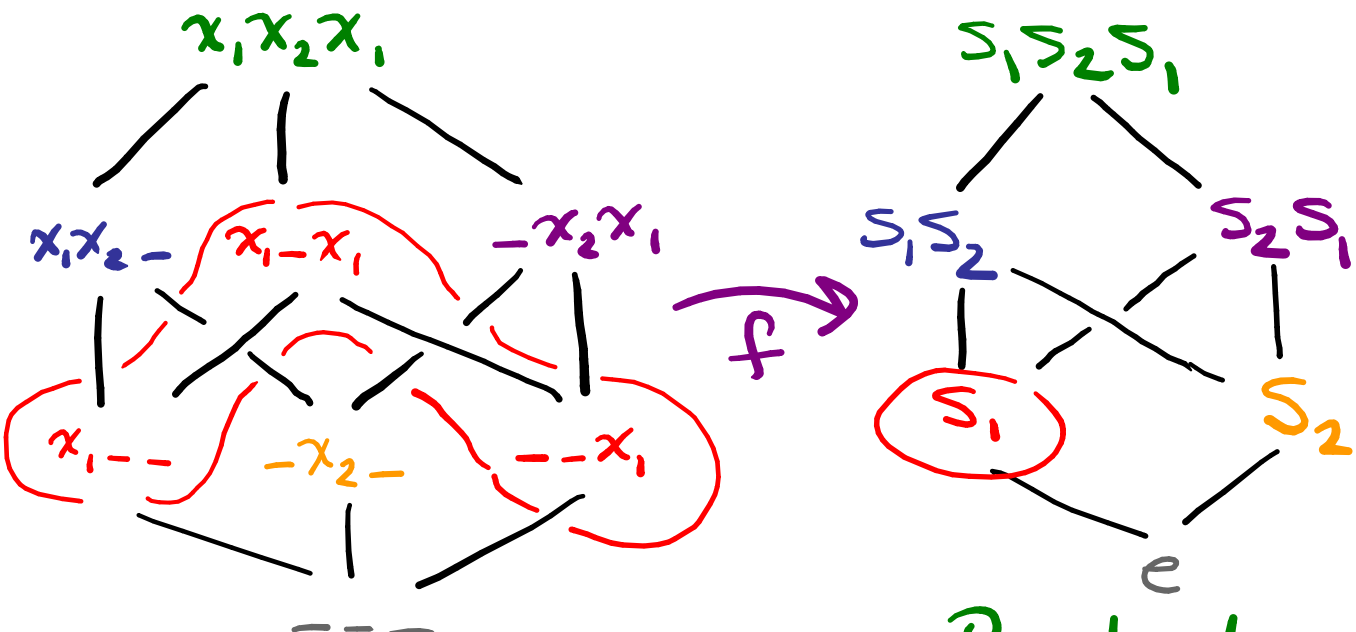

- let $Y_w^o$ = open cell in $\mathrm{im}(f_{(i_1 \ldots i_d)})$ indexed by $w \in W$

- let $\delta(x_{i_1} \ldots x_{i_d})$ denote (unsigned) 0-Hecke algebra product (a.k.a. "Demazure product")

e.g. $\delta(\underbrace{x_1x_2x_1}_{x_2x_1x_2}x_2x_1) = \delta(x_2x_1x_2x_1) = s_1s_2s_1$

since $x_2x_2 = x_2$

# Map of Face Posets Induced by Map $f_{(i_1,\ldots,i_d)}$ of Spaces

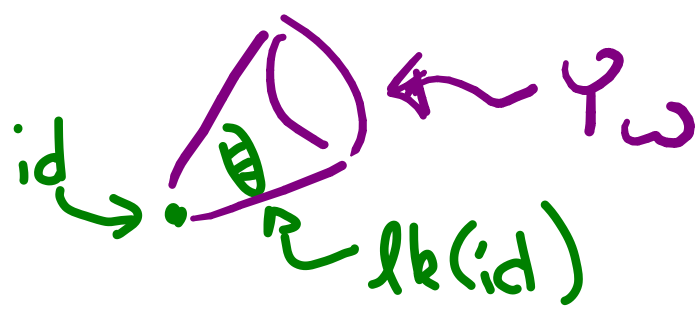



Boolean lattice $B_n$ = Face poset of simplex

Bruhat order = $F(\mathrm{im}(f_{(i_1,\ldots,i_d)}))$

$$f : x_{i_{j_1}} \cdots x_{i_{j_r}} \longmapsto \delta(x_{i_{j_1}} \cdots x_{i_{j_r}})$$

__Fomin-Shapiro Conjecture__: The Bruhat stratification of lk(id) in totally nonneg. real part of unipotent radical in Borel in algebraic group (e.g. im$(f_{(i_1,\ldots,i_d)})$) is regular CW complex homeom. to closed ball (with Bruhat order as face poset).




$$Y_w = \left[\overline{B^- w B^-} \cap \begin{pmatrix}\text{unipotent} \\ \text{subgp of } B\end{pmatrix}\right]_{\geq 0}$$

lower triangular opposite Borel $B^-$

permutation $w$

upper triang. w/ 1's on diagonal

totally nonneg. part

# Proof of Fomin-Shapiro Conjecture

(H., 2014, Invent.)

Set-up: Surjective fn $f_{(i_1,\dots,i_d)}: \Delta_{d-1} \to Y$ s.t. $f_{(i_1,\dots,i_d)}\big|_{\mathrm{int}(\Delta_{d-1})}$ homeom. to $\mathrm{int}(Y)$.

Step 1: Perform "cell collapses" on $\partial(\Delta_{d-1})$ designed to preserve homeom. type & regularity via cont. surjective fns $g_i: \Delta_{d-1} \to \Delta_{d-1}$ yielding $(\Delta_{d-1})/\sim$ s.t.

$$x_1 \sim x_2 \implies f_{(i_1,\dots,i_d)}(x_1) = f_{(i_1,\dots,i_d)}(x_2)$$

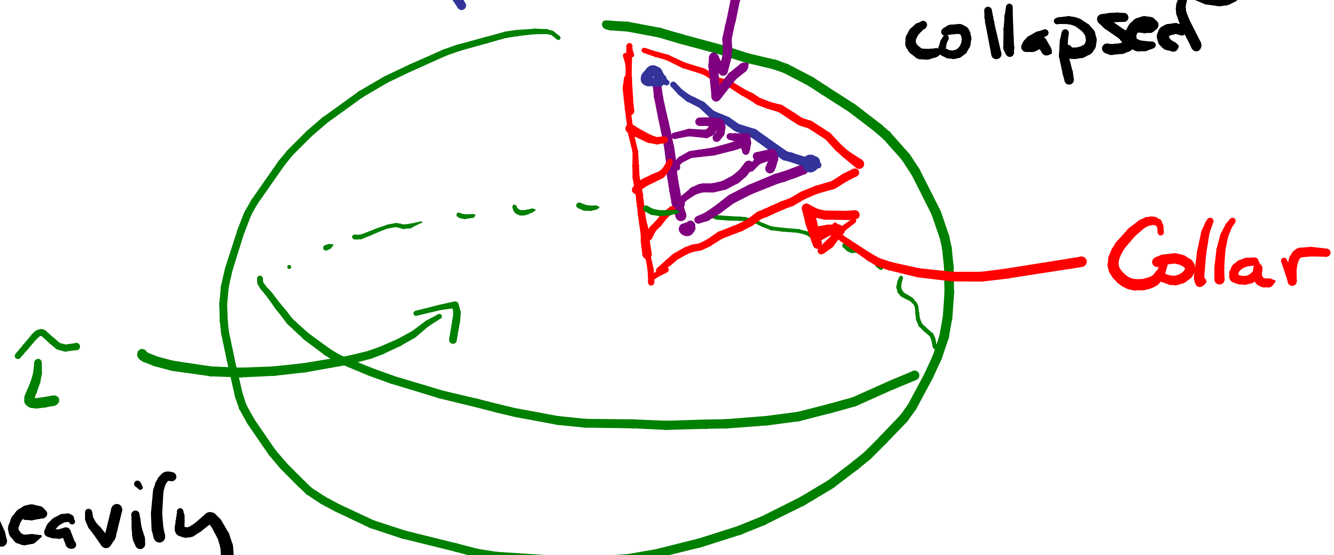
(eliminating faces given by nonreduced words)

Step 2: Prove $\overline{f}_{(i_1,\dots,i_d)}: (\Delta_{d-1})/\sim \to Y$ is a homeomorphism via new regularity criterion for finite CW complexes.

Corollary of Proof: Contractibility of fibers

## Step 1: Cell Collapses Preserving Homeom. Type, Regularity & Topol. Manifold

- Cover each non-reduced $\sigma$ with "nice" curves each in single fiber of $f_{(i_1,\dots,i_d)}$
- Collapse $\bar{\sigma}$ onto $\bar{\rho} \subseteq \partial\sigma$ by stretching curve extensions in "collar" for $\partial Z \setminus \sigma$ to cover $\bar{\sigma} \setminus \bar{\rho}$




⋆ heavily uses combinatorics of reduced & nonreduced words in 0-Hecke algebra

## Step 2 via: New Regularity Criterion

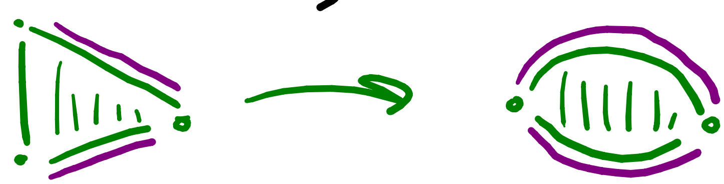
**Preparatory Lemma** (H.): Let $K$ be a finite CW complex w/ characteristic maps $\{f_\alpha\}$. Suppose:

(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example:

(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example:



Then $F(K)$ is graded by cell dimension.

Insightful feedback (Quinn): Next theorem "spreads around" injectivity requirement.

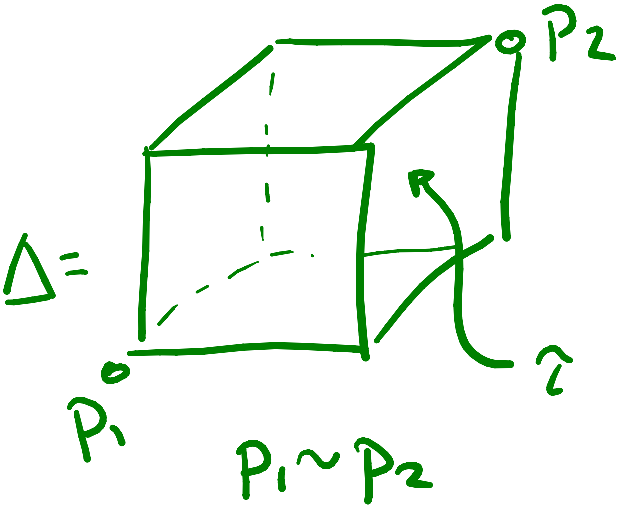
Thm (H.) Let $K$ be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then $K$ is regular w.r.t. $\{f_\alpha\}$ $\Leftrightarrow$

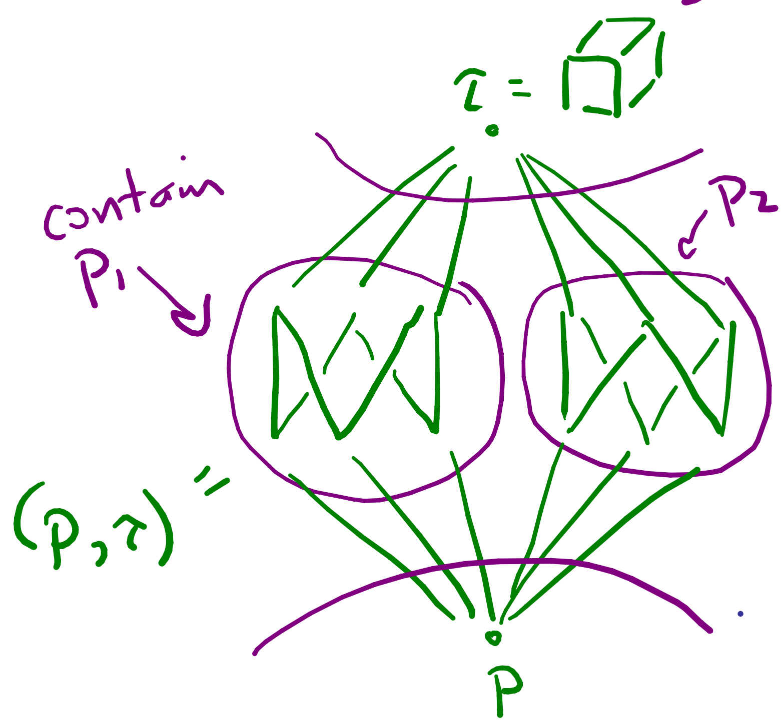
(1) $K$ meets requirements of prop'n for $F(K)$ to be graded by cell dim.

(2) $F(K)$ is thin and each open interval $(u,v)$ for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

Non-Example



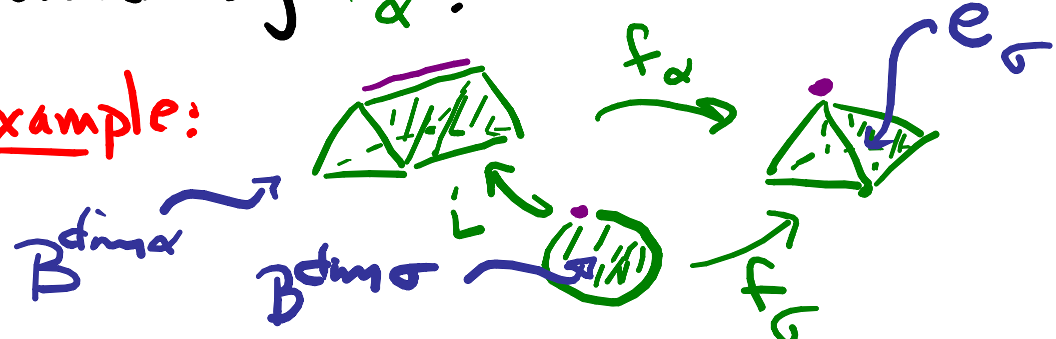
(3) For each $\alpha$, the restriction of $f_\alpha$ to preimages of codim. one cells in $\overline{e}_\alpha$ is injective.

(topological condition)

Non-Example:

(4) $\forall\, e_\sigma \subseteq \overline{e}_\alpha$, $f_\sigma$ factors as continuous inclusion $i : B^{\dim \sigma} \to B^{\dim \alpha}$ followed by $f_\alpha$.

Non-Example:




Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one.

Proof: Induction on difference in dim.

# Codimension One Injectivity via Exchange Axiom for Coxeter Groups

$$s_1 s_2 s_1 s_3$$

$-\,s_2 s_1 s_3$ $\qquad$ $\cancel{s_1 - s_1 s_3}$ $\qquad$ $s_1 s_2 - s_3$ $\qquad$ $s_1 s_2 s_1 -$

$\quad\|\qquad$ non-reduced $\qquad\|\qquad\qquad\|$

$3142 \neq 2341 \neq 3214$

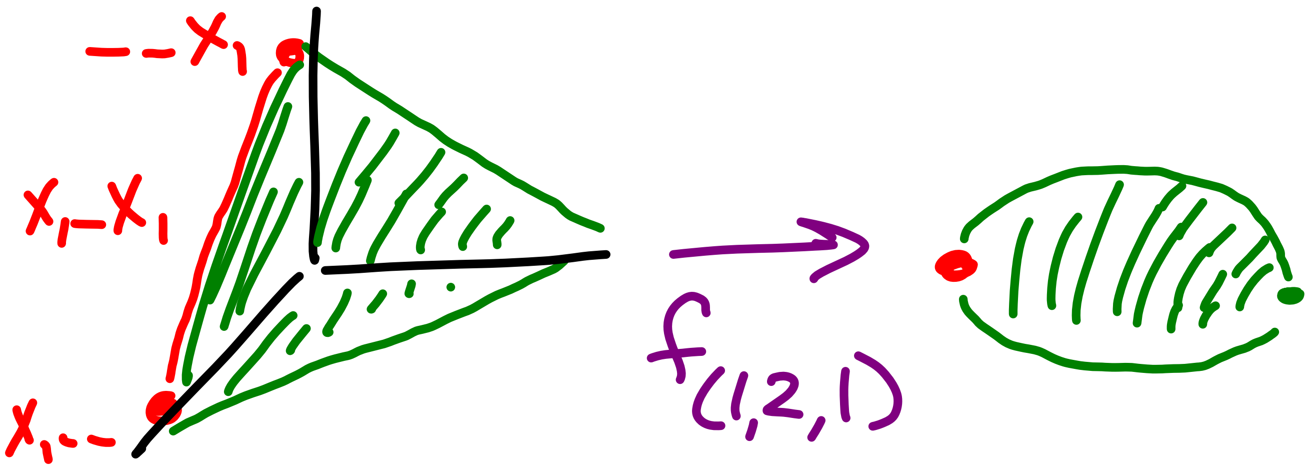
reduced subexpressions of a reduced expression obtained by deleting one letter must give distinct $u \in W$

This argument needs codim one!

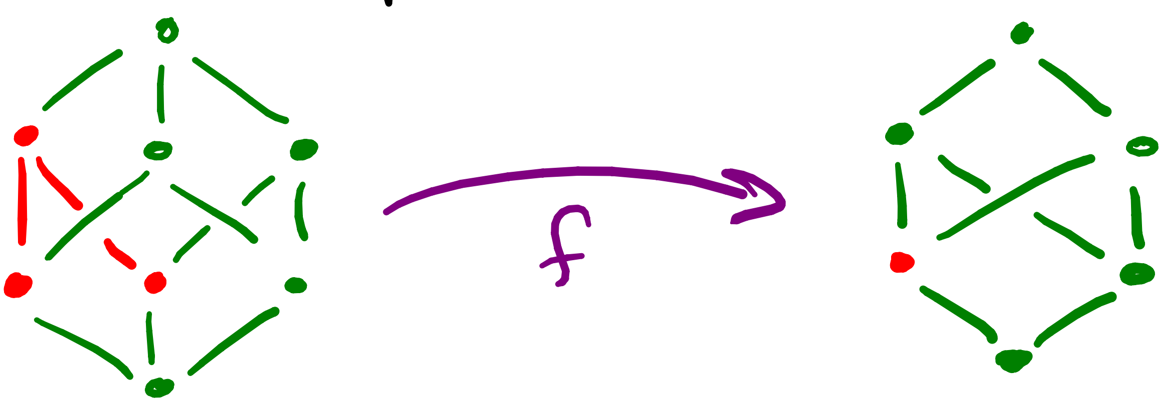
e.g. $s_1 s_2 s_1 s_3$

$s_1 - - -$ $\qquad$ $- - s_1 -$

# Combinatorics of Fibers




Induced map of face posets:




Thm (Armstrong-H., 2011): For each $u \in W$, $f^{-1}_{\geq}(u) = \{x \in B_n \mid f(x) \geq u\}$ is dual (i.e. upside-down) to face poset for subword complex $\Delta((i_1, \dots, i_k), u)$.

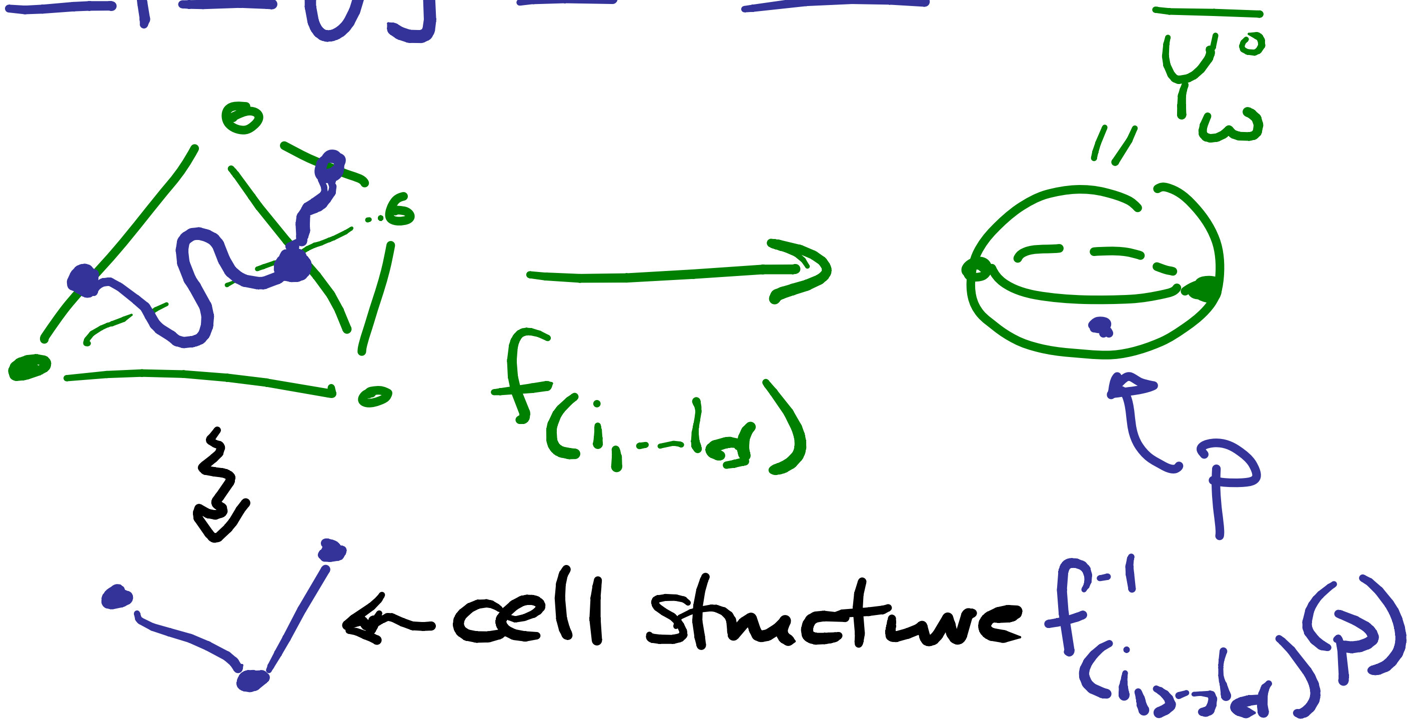
**Thm** (DHM, 2018): $f_{=}^{-1}(u)$ is face poset of interior dual block complex for subword complex $\Delta((i_1,\ldots,i_d),u)$

**Thm** (DHM, 2018): Interior dual block complex of $\Delta((i_1,\ldots,i_d),u)$ is contractible.

**Pf**: Discrete Morse theory

**Combining**: DHM Conjecture would imply $f^{-1}_{(i_1,\ldots,i_d)}(p) \cong$ interior dual block complex of $\Delta((i_1,\ldots,i_d),u)$ for $p \in Y_u^{o}$, hence $f^{-1}_{(i_1,\ldots,i_d)}(p)$ contractible.

# Topology of Fibers




**Thm** (Davis-H-Miller): Each fiber $f^{-1}_{(i_1,\dots,i_d)}(p)$ admits a cell decomposition induced by the natural cell decomposition of the simplex $\Delta_{d-1}$.

# Topology of Fibers: Key Lemmas

<u>Defn</u> (Davis-H-Miller): The letter $x_{i_1}$ is **redundant** in $x_{i_1} \cdots x_{i_d}$ if $\delta(i_1, \ldots, i_d) = \delta(i_2, \ldots, i_d)$

<u>Lemma</u> (Davis-H-Miller): $x_{i_1}$ is non-redundant in $x_{i_1} \cdots x_{i_d}$

$$\Longleftrightarrow f^{-1}_{(i_1 \ldots i_d)}(p) \text{ for } p \in Y^{o}_{\delta(i_1 \ldots i_d)}$$

$$\{(t_1, \ldots, t_d) \mid x_{i_1}(t_1) \cdots x_{i_d}(t_d) = p\}$$

has unique value $k_1$ for $t_1$.

$$\Longleftrightarrow f^{-1}_{(i_1 \ldots i_d)}(p) \cong f^{-1}_{(i_2 \ldots i_d)}(x_{i_1}(-k_1)p)$$

<u>Lemma</u> (Davis-H-Miller): Given $(t_1,\dots,t_d) \in f^{-1}_{(i_1,\dots,i_d)}(p)$ with $t_1 > 0$ and $x_{i_1}$ redundant, then $\exists (t_1',\dots,t_d') \in f^{-1}_{(i_1,\dots,i_d)}(p)$ for every $t_1' \in [0, t_1]$.

e.g. $M = \begin{pmatrix} 1 & \pi & e \\ & 1 & 14 \\ & & 1 \end{pmatrix}$ then $f^{-1}_{(1,2,1,2)}(M)$

$\mathbin{/\!/}$

$\{(t_1,t_2,t_3,t_4) \mid x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4) = M\}$

achieves every $t_1 \in [0, \frac{e}{14}]$.

Thm (DHM): DHM Conjecture $\Rightarrow$ (2nd proof of) Fomin-Shapiro Conjecture

- by Shellability of Bruhat Order + (Well Known) Topological Relationship

Fibers to Image: Let $g: B \to Z$ be continuous surjection from ball $B$ to Hausdorff space $Z$ whose restriction to $\text{int}(B)$ is an embedding. Suppose also:

(1) $g(\partial B) \cong \partial B = S^n$

(2) $g(\partial B) \cap g(\text{int}(B)) = \emptyset$

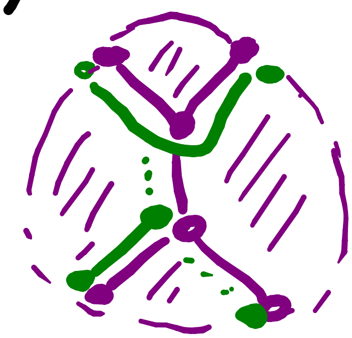
(3) $g^{-1}(p)$ is contractible $\forall p \in g(\partial B)$

Then $Z \cong B$.

# Some Further Questions

1. Subword complex analogues for fibers of other maps?

   e.g. H-Kenyon: $\Delta(G, \mathrm{elec}(H))$ for planar electrical networks (for $H$ a minor of $G$)



2. DHM Conjecture? Analogues?

3. Regular CW decomps for cell closures & fibers in cases of graphs, $Gr(k,n)_{\geq 0}$, $(\mathcal{F}\ell_n)_{\geq 0}$ & $(GL_n/P)_{\geq 0}$?

# Thanks!

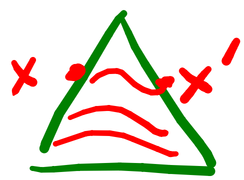
# Appendix: Further Remarks & Details

# (Mainly Combinatorial) Conditions Allowing Such Face Collapses Across Curves

$$
\begin{tikzcd}
 & \text{"}\Delta\text{"} \arrow[d, phantom, "\text{"}\cong"] & & & \\
K_0 \arrow[r, "g_1"'] \arrow[d, phantom, "\shortparallel"] & K_1 \arrow[r, "g_2"'] \arrow[d, phantom, "\cong"] & K_2 \arrow[r] \arrow[d, phantom, "\cong"] & \cdots \arrow[r] & K_i \\
\Delta & \Delta/\sim_1 & \Delta/\sim_2 & &
\end{tikzcd}
$$

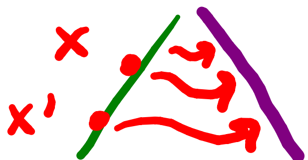
polytope!

- collapse face in $K_i$ across images of parallel line segments in $K_0$ satisfying:
  - Distinct endpoints condition (DE):



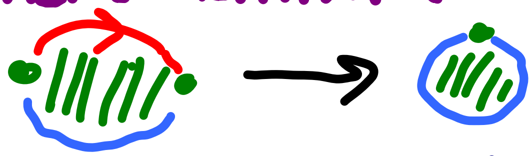

$x \neq x'$

  - Distinct initial points condition (DIP):




$x \neq x'$

  - Least upper bound condition (LUB)

to preclude 

(conditions checkable via 0-Hecke algebra)

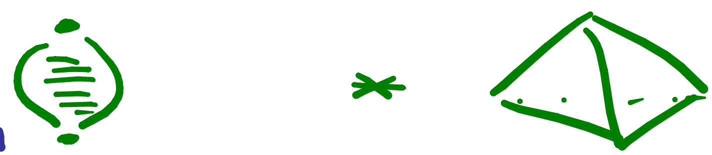
# Additional Key Challenges

1. 0-Hecke algebra lacks inverses & lacks cancellation law.

Key Idea: find ways to transfer properties from Coxeter gp

2. Need change-of-coords for braid moves as homeom's on closed cells

Key Idea: induction by embedding smaller instance



$$\big(\underbrace{i_1, i_2, i_3, \dots\dots, i_{d'-1}}_{\text{all possible collapses done}}, \underbrace{i_{d'}, \text{---}, i_d}_{\text{not in any collapses yet}}\big)$$

$\mathrm{im}(f_{(i_1 \text{---} i_{d'-1})})$

3. Need maps to extend to full complex

4. Tricky combinatorics to verify LUB at each collapsing step.

# Ingredients in this Relationship Between fibers & Image

- "CE-Approximation Thm" (Kirby-Siebenmann (all dim's); Quinn (dim 4); Armentrout (dim 3))
  - $g: \partial B \to \partial B$ as above may approximated by homeomorphisms

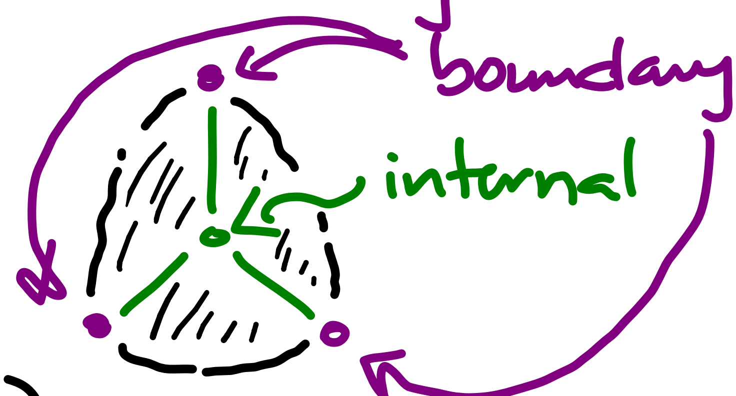
- Local Contractibility of $\mathrm{Homeo}^+(S^n, S^n)$
  - any two homeomorphisms "close enough" to each other connected by path of homeomorphisms

# One of Desired Analogues: Maps Arising from Electrical Networks

$$\underbrace{\Delta}_{\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}} \underbrace{\begin{pmatrix} v_N \\ v_I \end{pmatrix}}_{\text{voltages}} = \underbrace{\begin{pmatrix} c_N \\ 0 \end{pmatrix}}_{\text{currents}}$$

$I$ = internal nodes

$N$ = boundary nodes




$$(\underbrace{A - BC^{-1}B^T}) v_N = c_N$$

"response matrix" whose entries are nonneg. rat'l fns of conductances

(: input bdry voltages & output bdry currents)

**Goals:** Given a graph $G$, study:

(1) space of response matrices as image of

$$f : \left\{ \begin{matrix} \text{conductance} \\ \text{vectors} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{response} \\ \text{matrices} \end{matrix} \right\}$$

$$\overset{\shortparallel}{\left(\mathbb{R}_{\geq 0} \cup \{\infty\}\right)^{|E|}}$$

(2) fibers of $f$

**Note:**

contracting an edge $\longleftrightarrow$ sending conductance to $\infty$ (i.e. resistance to $0$)

deleting an edge $\longleftrightarrow$ sending conductance to $0$ (resistance to $\infty$)

# Face Posets of Spaces of Electrical Networks

**Thm (Lam):** The face poset $F(E_n)$, the "uncrossing poset", is Eulerian.

**Conj (Lam):** $F(E_n)$ is shellable.

**Thm (H-Kenyon):** $F(E_n)$ is dual EC-shellable, hence is a CW poset.

**Proof:** Extends Dyer's reflection order EL-labeling for Bruhat order, using isomorphism to subposet where labelings coincide.

Cor (H-Kenyon): Shelling for each interval in "Tuffley posets", i.e. face posets for edge product spaces of phylogenetic trees.

(previous shelling existence result by Gill-Linusson-Moulton-Steele)

- shelling ⟹ edge-product space CW-decomp of Moulton-Steele is regular CW complex

# Connection to Edge-Product Spaces of Phylogenetic Trees

- every interval in face poset of edge-product space of phylogenetic trees is isomorphic to interval in uncrossing posets (via result of Kenyon-Wilson)

# Face Poset