Fibers of Maps to Totally Nonnegative Spaces & the Fomin-Shapiro Conjecture

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Outline for Talk

1. Background & Motivations
2. Fomin-Shapiro Conjecture (2014 proof, H., Invent.)

   ◈ image$(f_{(i_1,\ldots,i_d)})$ is regular CW ball
3. Fibers of $f_{(i_1,\ldots,i_d)}$ (recent joint work with Jim Davis & Ezra Miller)
4. Spaces with (conjecturally & provably) analogous structure

(**Recall:** 

closed ball ↑ homeomorphic)

Topological Aspects of Total Positivity

- Lusztig (94), Fomin–M. Shapiro (00)... initiate study of: totally nonneg. real part of spaces of matrices, spaces of flags (i.e. GL_n/B)... (i.e. having all minors nonnegative)
- Conjecturably / provably homeomorphic to closed balls
- Proving this by studying fibers (as is done in H. & in DHM)
  - imposes restrictions on reln's amongst exp'd Chevalley gen's
  - reveals structure in Lusztig's canonical bases

Background on Coxeter Groups

- $s_i := \underbrace{(i, i+1)}_{\text{type A (i.e. } W = S_n)}$ a simple reflection in W
- $s_{i_1} \cdots s_{i_d}$ is reduced expression for $w \in W$ if $w = s_{i_1} \cdots s_{i_d}$ for d as small as possible
- length of w, denoted $\ell(w)$, is this smallest d.

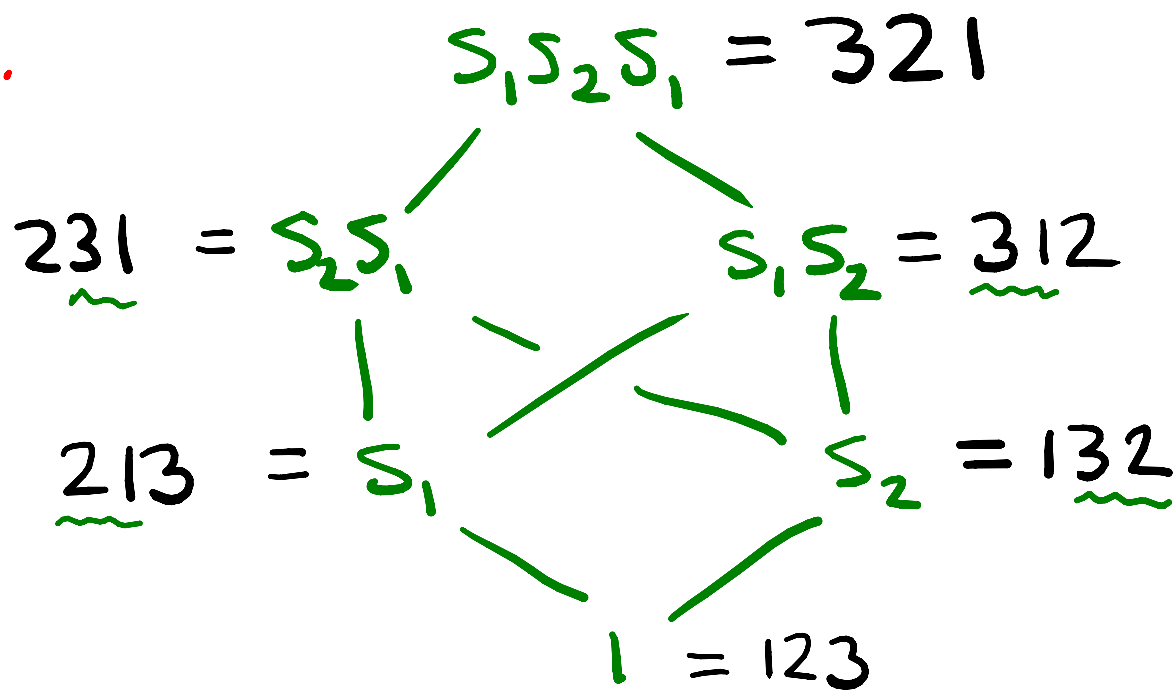
e.g. $s_1 s_2 s_1 = s_2 s_1 s_2$ has length 3

$$\underset{\parallel}{s_1 s_2 s_1}$$

$$321 \xrightarrow{s_1} 231 \xrightarrow{s_2} 213 \xrightarrow{s_1} 123$$

- $(i_1, \ldots, i_\ell)$ is **reduced word**
- **Bruhat order:** partial order

e.g.



on W with $u \leq v$ for $u, v \in W \iff$ any reduced word for v has subword that is reduced word for u.

Running Example: Totally Nonnegative Part of a Space of Matrices

column $i+1$

- $$x_i(t) = I_n + tE_{i,i+1} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & t & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \leftarrow \text{row } i$$

$= \exp(te_i)$ (general finite type, expon'd Chevalley generator)

(type A)

- $$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$$

reduced word

$$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

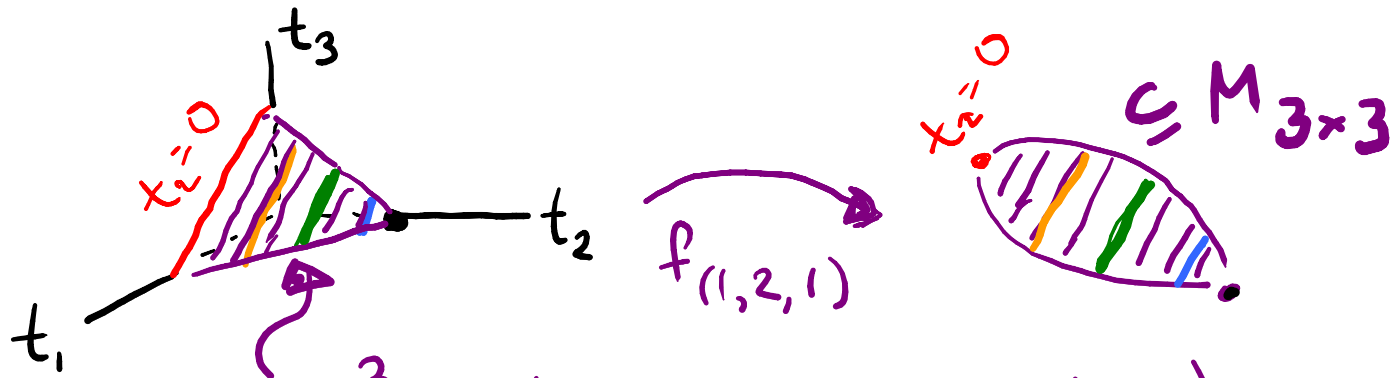
e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

w_0 case:

$$\left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \middle| \begin{array}{l} \text{tot.} \\ \text{nonneg} \end{array} \right\} = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1,\dots,i_d)}$



$\mathbb{R}^3_{\geq 0} \cap (\sum t_i = \underbrace{K}_{\text{positive constant}}$ hyperplane$)$

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow {\scriptstyle t_2 = 0}$$

$$f_{(1,2,1)}(t_1,0,t_3) = \underbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_1)}\underbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_3)}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

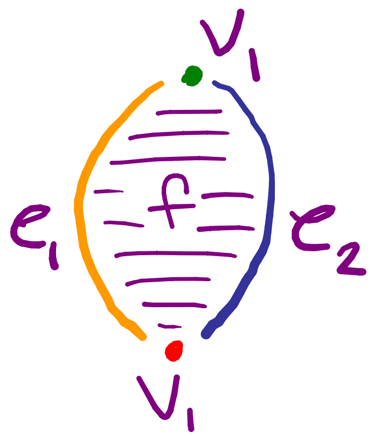
simplex faces w/ same image "$x_1^2 = x_1$"

e.g. $\{x_1(t) \mid t>0\} = \{x_1(t_1)x_1(t_2) \mid t_1,t_2>0\}$

CW Complexes & their Face Posets (Partially Ordered Sets)

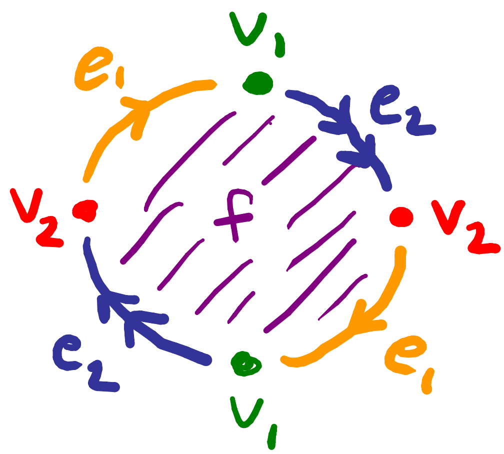
e.g.

$K =$



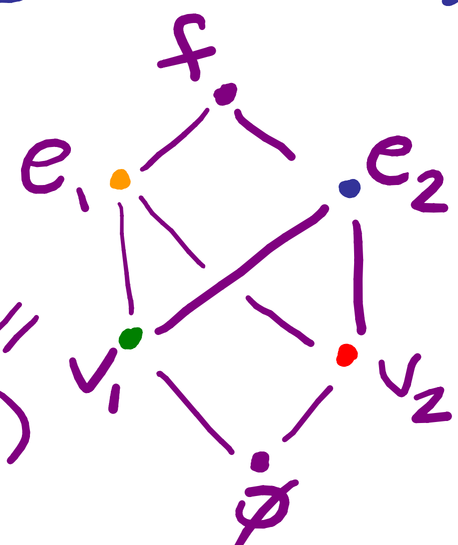
$\cong$ ball

$K' =$



$= \mathbb{RP}^2$

$F(K) =$



$= F(K')$

"face poset"

$(u \le v \iff u \le \bar{v})$

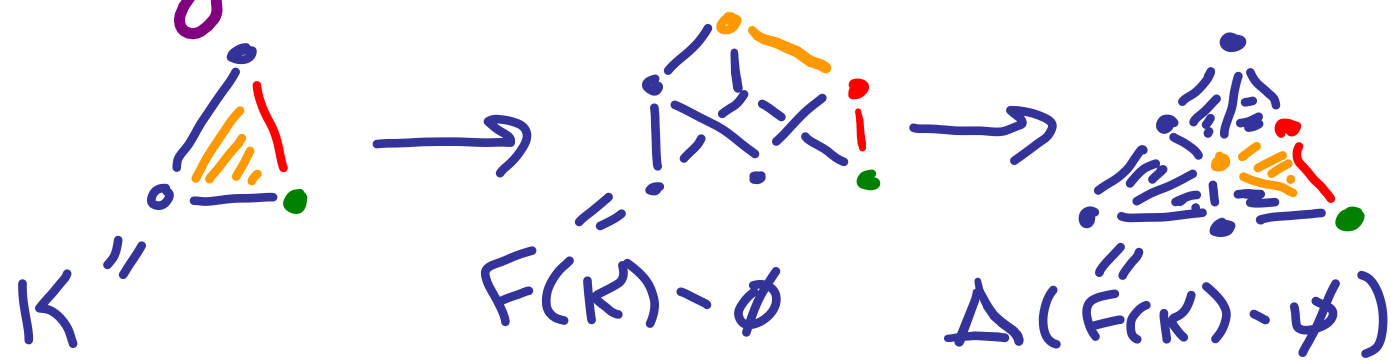
Recall: A CW complex: cells $e_\alpha \cong \mathbb{R}^{\dim(\alpha)}$,

characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \to \bigcup_{\beta \subseteq \bar{\alpha}} e_\beta$

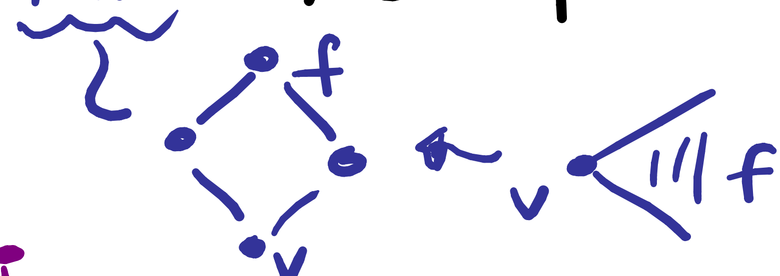
& attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

<u>Recall</u>: CW complex is regular if each f_α is homeomorphism.

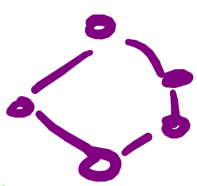
- $\Delta(P)$ = "nerve" or "order complex" of P
- K regular $\Rightarrow K \cong \Delta(F(K)\setminus\emptyset) = sd(K)$



- a CW poset is any face poset of regular CW complex
- "Shellable" + "Graded" + "thin" $\Rightarrow$ CW poset



e.g. not



e.g. Bruhat order (Björner-Wachs; Matthew Dyer)

Fomin-Shapiro Conjecture: The Bruhat stratification of $lk(id)$ in totally nonneg. real part of unipotent radical in Borel in algebraic group is regular CW complex homeomorphic to closed ball (w/ Bruhat order as face poset)

$$Y_w = \left[\overline{B^- w B^-} \cap \begin{pmatrix}\text{unipotent}\\ \text{subgp of } B\end{pmatrix}\right]_{\geq 0}$$

lower triangular opposite Borel B^-

permutation w

upper triang. w/ 1's on diagonal

totally nonneg. part

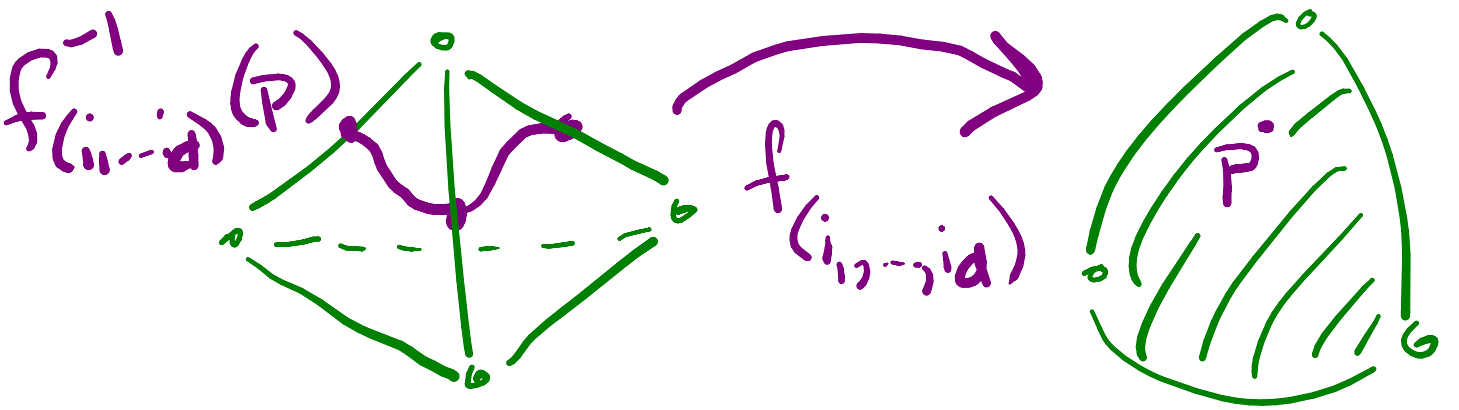
Theorem (H., 2014, Invent.):
Fomin-Shapiro Conjecture holds.

Special Case (Running example):
Space of totally nonneg. upper triang. matrices with 1's on diag. s.t. superdiagonal sums to $K > 0$

Concrete Realization: Products $x_{i_1}(t_1) \cdots x_{i_d}(t_d)$ of elementary matrices, by results of Whitney, Loewner & (generalizing beyond type A) Lusztig.

Conjecture (Davis-H-Miller):

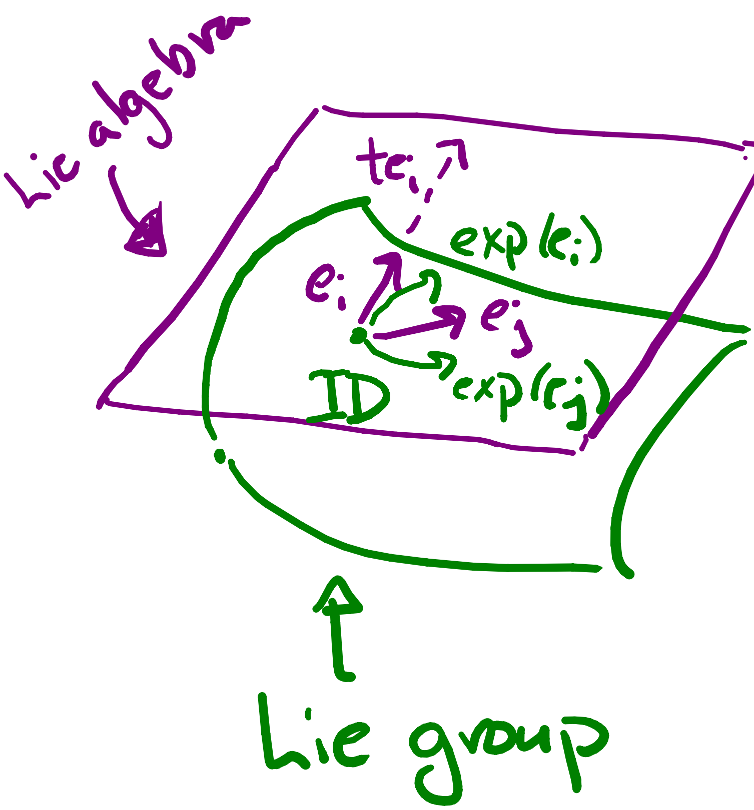
$f^{-1}_{(i_1,\dots,i_d)}(p)$ is regular CW complex homeomorphic to interior dual block complex of subword complex $\Delta((i_1,\dots,i_d),w)$ for $p\in Y^o_w$.



Thm (DHM): $f^{-1}_{(i_1,\dots,i_d)}(p)$ has cell decomposition & "correct" face poset.

Thm (DHM): Interior dual block complex of $\Delta((i_1,\dots,i_d),w)$ is contractible.

A Motivation to Study Fibers: Relations Among (Exponentiated) Chevalley Generators



$$te_i = \begin{pmatrix}0 & t & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\Big\downarrow \exp(\_)$$

$$\begin{pmatrix}1 & t & \\ & 1 & \\ & & 1\end{pmatrix}$$

$$\exp(t_1 e_i)\exp(t_2 e_j)$$

$$f_{(i,j)}(t_1,t_2) = x_i(t_1)\,x_j(t_2)$$

$$\exp(te_i) = \boxed{ID + te_i} + \underbrace{\frac{t^2 e_i^2}{2}}_{0=} + \underbrace{\frac{t^3 e_i^3}{6}}_{0=} + \cdots$$

Reln's $\longleftrightarrow$ Elts in same fiber of $f_{(i_1,\dots,i_\ell)}$

FS-Conj. Proof, 1st Idea: Describe Strata via 0-Hecke Algebra

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

$\Big\downarrow$ suppress parameters

$x_i x_i = x_i$

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2 t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1 t_2}{t_1+t_3}\right)$

for $t_1, t_2, t_3 > 0$

$\Big\downarrow$ (type A)

$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$

(& analogous relations outside type A)

Upshot: $\mathrm{im}(f_1) = \mathrm{im}(f_2) \iff \underbrace{x(f_1) = x(f_2)}_{\text{equal as 0-Hecke algebra elements}}$

Thm (Lusztig): If $(i_1, \dots, i_d)$ is reduced, then $f_{(i_1,\dots,i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

More Motivation for Nonnegative Real Part of Unipotent Radical

$f^{-1}_{(i_1,\ldots,i_d)}(p) \to f^{-1}_{(j_1,\ldots,j_d)}(p)$ coordinate change

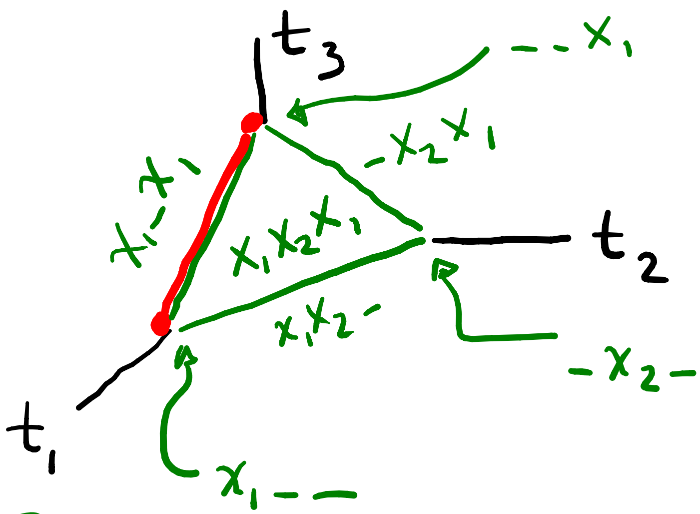
$$(t_1, t_2, t_3) \longmapsto \left(\frac{t_2 t_3}{t_1+t_3},\ t_1+t_3,\ \frac{t_1 t_2}{t_1+t_3}\right)$$

tropicalizes to coordinate change:

$$(a,b,c) \mapsto (b+c-\min(a,c),\ \min(a,c),\ a+b-\min(a,c))$$

for canonical bases w/ same braid move

Faces of (Preimage) Simplex as Subexpressions in 0-Hecke Algebra

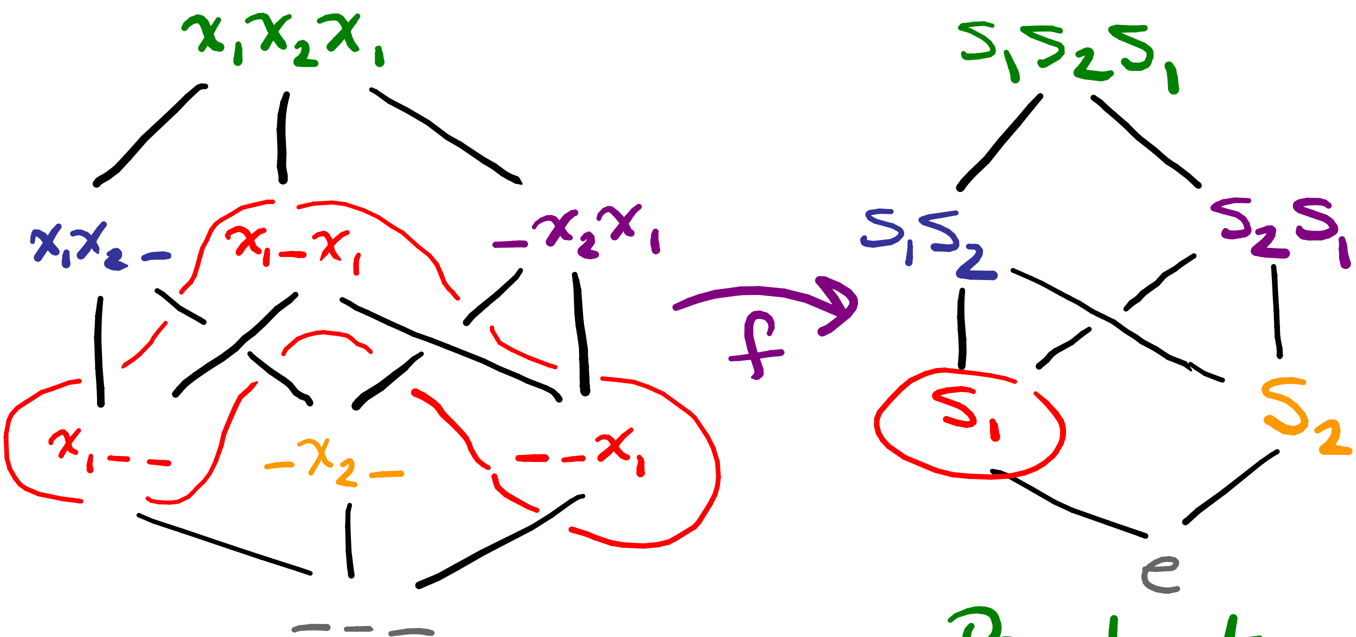


- let Y_w^o = open cell in $\mathrm{im}(f_{(i_1,\dots,i_d)})$ indexed by $w \in W$
- let $\delta(x_{i_1}\dots x_{i_d})$ denote (unsigned) 0-Hecke algebra product (a.k.a. "Demazure product")

e.g. $\delta(\underbrace{x_1x_2x_1}_{x_2x_1x_2}x_2x_1) = \delta(x_2x_1x_2x_1) = s_1s_2s_1$

since $x_2x_2 = x_2$

Map of Face Posets Induced by Map $f_{(i_1,\dots,i_d)}$ of Spaces



Boolean lattice B_n = Face poset of simplex

Bruhat order = $F(\mathrm{im}(f_{(i_1,\dots,i_d)}))$

$$f: x_{i_{j_1}} \cdots x_{i_{j_r}} \longmapsto \delta(x_{i_{j_1}} \cdots x_{i_{j_r}})$$

Proof of Fomin-Shapiro Conjecture

(H., 2014, Invent.)

Set-up: Surjective fn $f_{(i_1,\dots,i_d)}: \Delta_{d-1} \to Y$ s.t. $f_{(i_1,\dots,i_d)}\big|_{\mathrm{int}(\Delta_{d-1})}$ homeom. to $\mathrm{int}(Y)$.

Step 1: Perform "cell collapses" on $\partial(\Delta_{d-1})$ designed to preserve homeom. type & regularity yielding $(\Delta_{d-1})/\sim$ s.t.

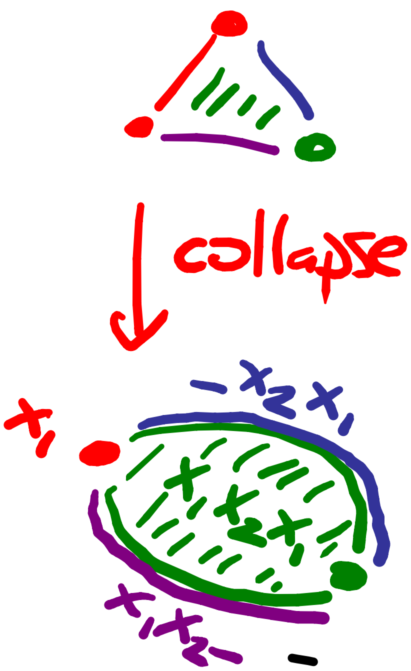
$$x_1 \sim x_2 \implies f_{(i_1 \dots i_d)}(x_1) = f_{(i_1 \dots i_d)}(x_2)$$

(eliminating faces given by nonreduced words)

Step 2: Prove $\bar{f}_{(i_1 \dots i_d)}: (\Delta_{d-1})/\sim \to Y$ is a homeomorphism via new regularity criterion for finite CW complexes.

Corollary of Proof: Contractibility of Fibers

Codimension One Injectivity of Attaching Maps via Exchange Axiom for Coxeter Groups



$s_1s_2s_1$

$-s_2s_1$ $\quad$ $\cancel{s_1 s_1}$ $\quad$ s_1s_2-

non-reduced

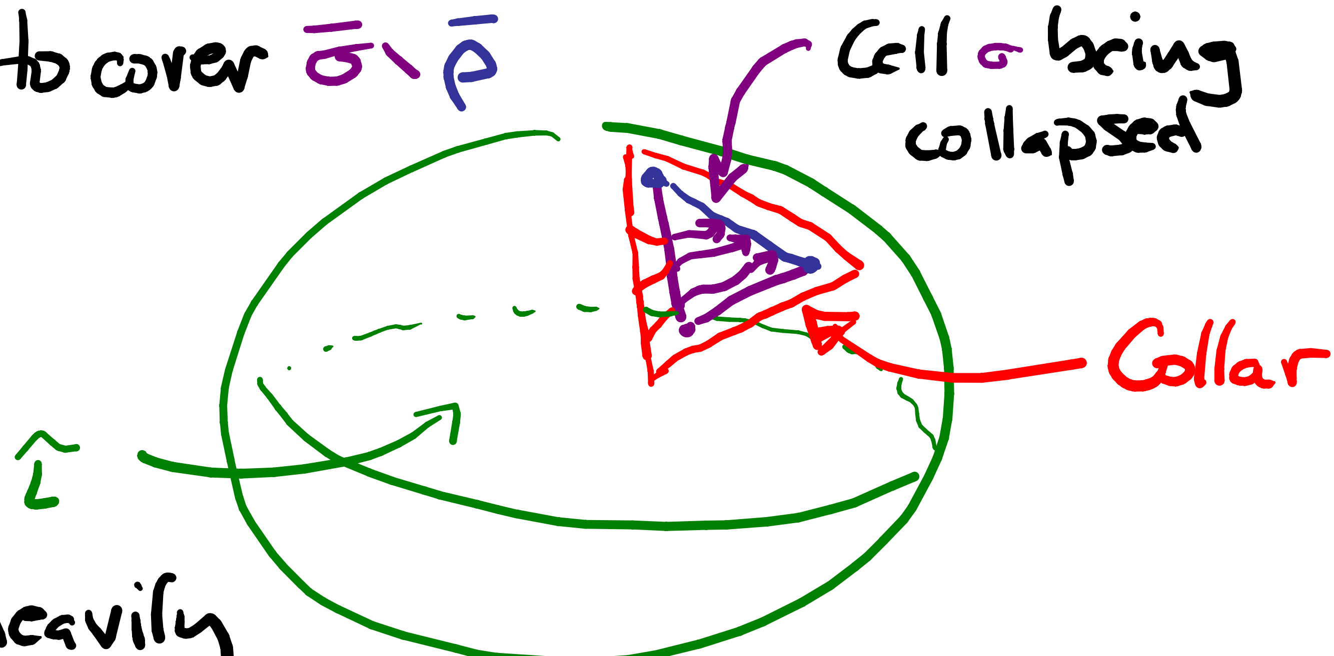
• reduced subexpressions of reduced expression by deleting one letter give distinct new w

• Need codim one! e.g. $s_1s_2s_1$

s_1-- $\qquad$ $--s_1$

Step 1: Cell Collapses Preserving Homeom. Type, Regularity & Topol. Manifold

- Cover each non-reduced σ with "nice" curves each in single fiber of $f_{(i_1,\ldots,i_d)}$
- Collapse $\overline{\sigma}$ onto $\overline{\rho} \subseteq \partial\sigma$ by stretching curve extensions in "collar" for $\partial z \setminus \sigma$ to cover $\overline{\sigma} \setminus \overline{\rho}$



* heavily uses combinatorics of reduced & nonreduced words in 0-Hecke algebra

Step 2 via: New Regularity Criterion

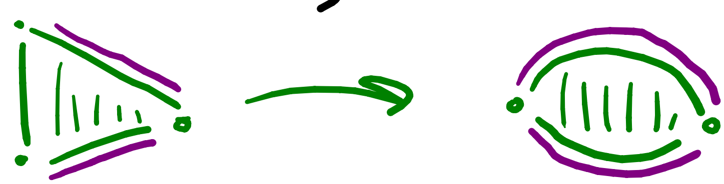
Preparatory Lemma (H.): Let K be a finite CW complex w/ characteristic maps $\{f_\alpha\}$. Suppose:

(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example:

(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example:



Then $F(K)$ is graded by cell dimension.

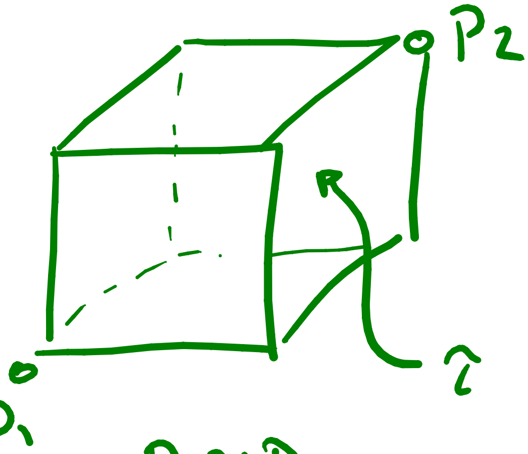
Thm (H.) Let K be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then K is regular w.r.t. $\{f_\alpha\}$ $\Longleftrightarrow$

(1) K meets requirements of prop'n for $F(K)$ to be graded by cell dim.

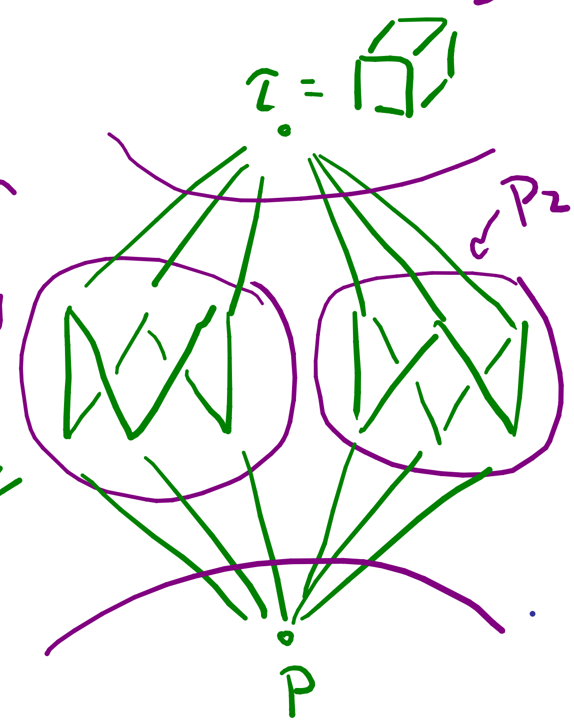
(2) $F(K)$ is thin and each open interval (u,v) for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

Non-Example

$\Delta =$ 

$p_1 \sim p_2$

$(p,\tau) \simeq$ 

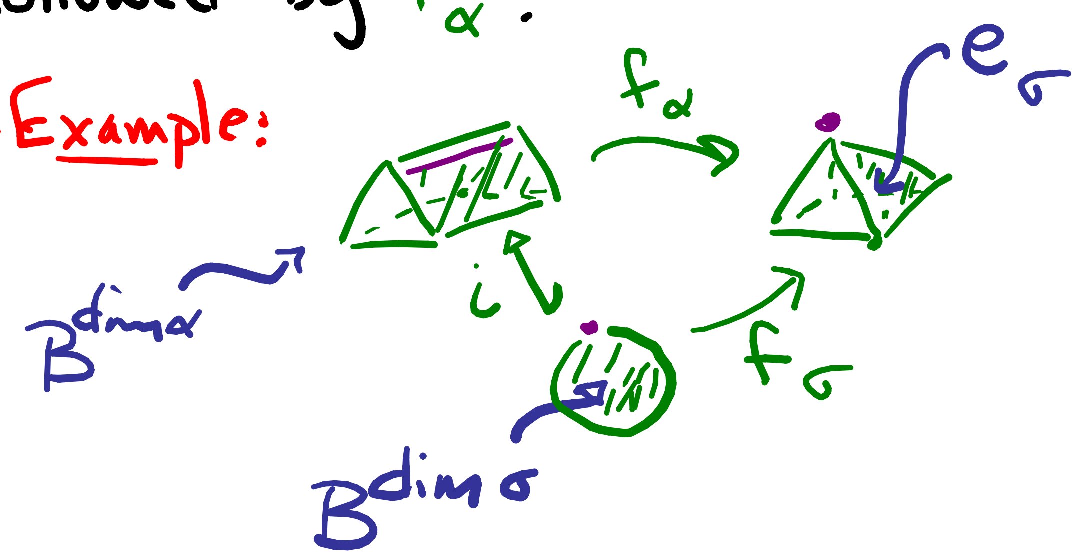
(3) For each α, the restriction of f_α to preimages of codim. one cells in $\overline{e_\alpha}$ is injective.

(topological condition)

Non-Example:

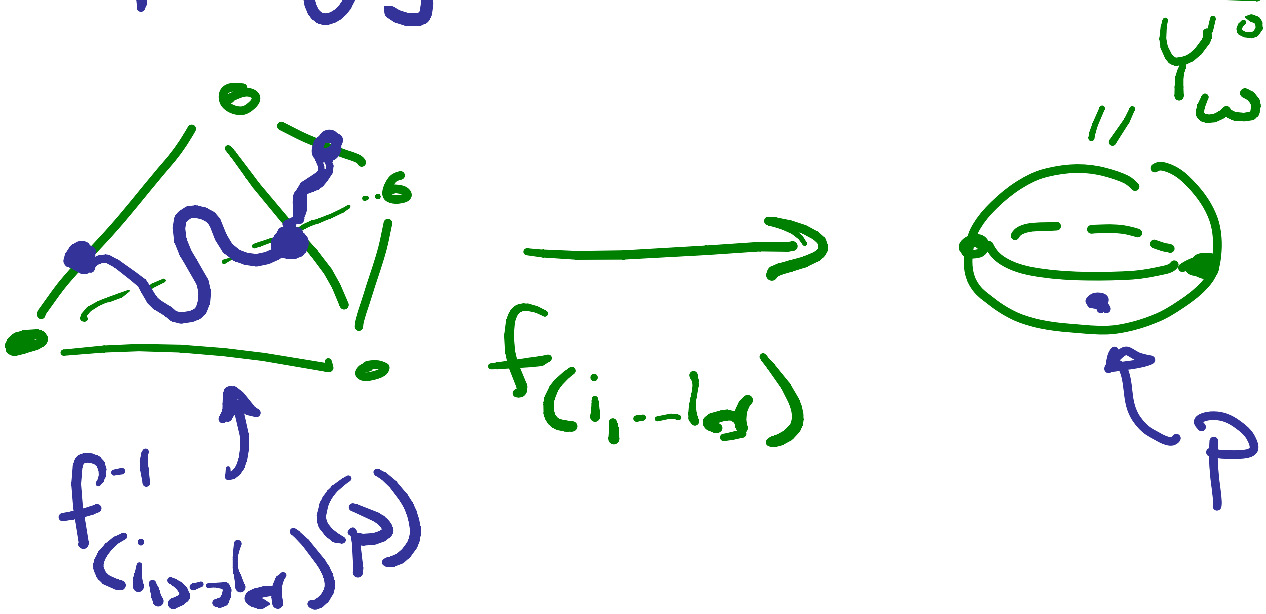
(4) $\forall\, e_\sigma \subseteq \overline{e_\alpha}$, f_σ factors as continuous inclusion $i: B^{\dim \sigma} \to B^{\dim \alpha}$ followed by f_α.

Non-Example:



Proof: Induction on difference in dim.

Topology of Fibers



Thm (Davis-H-Miller): Each fiber $f^{-1}_{(i_1,\dots,i_d)}(p)$ admits a cell decomposition induced by the natural cell decomposition of the simplex Δ_{d-1}.

Proof: Parametrization + continuity lemmas

Parametrization: Given $(i_1,\dots,i_d)$, consider rightmost subword that is reduced word for $w=\delta(i_1,\dots,i_d)$

e.g. $\delta(3,1,2,1,2,1,2,1)=s_3s_1s_2s_1$

Parametrize $f^{-1}_{(3,1,2,1,2,1,2,1)}(p)$ in $Y^{\circ}_{s_3s_1s_2s_1}$

$$x_3(t_1)x_1(t_2)x_2(t_3)x_1(t_4)x_2(t_5)x_1(t_6)x_2(t_7)x_1(t_8)$$

"free" parameters

"dependant" parameters

Thm (Davis-H-Miller):

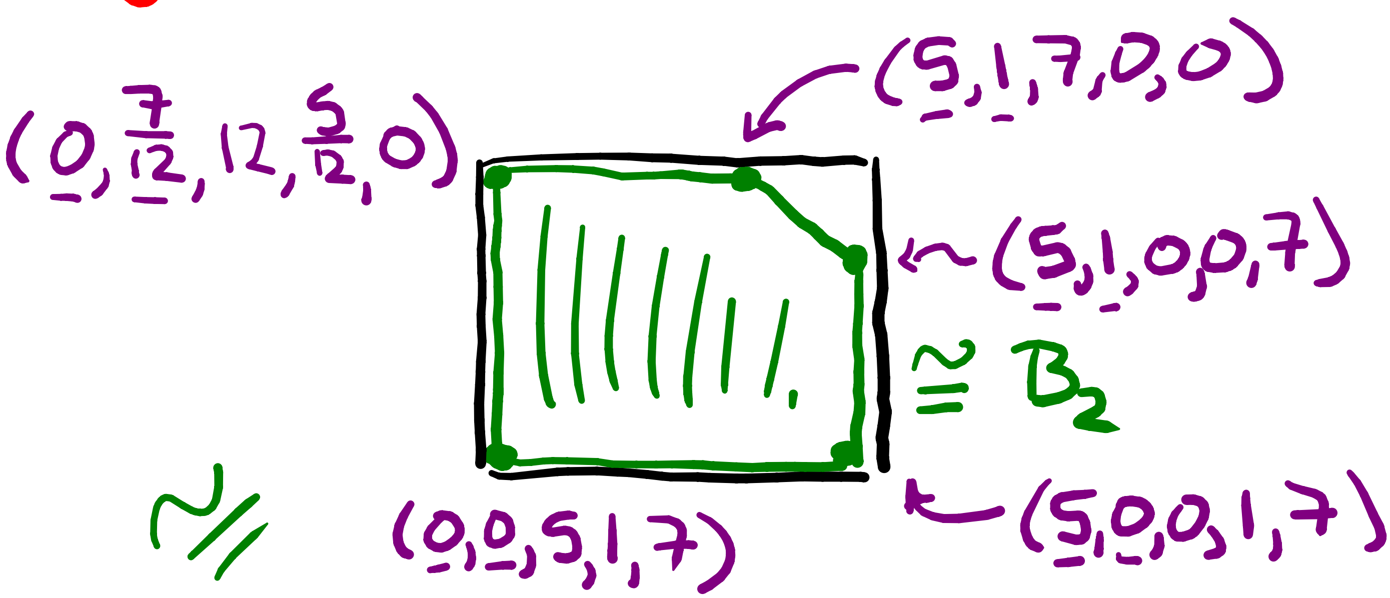
$$[0,1)^{\dim(\sigma)} \cong \bigcup_{v\le\bar{\tau}\le\bar{\sigma}} \tau$$

v source vertex of $\bar{\sigma}$

Examples of Fibers:

(with realizations as suggested by various results towards proof of DHM conjecture)

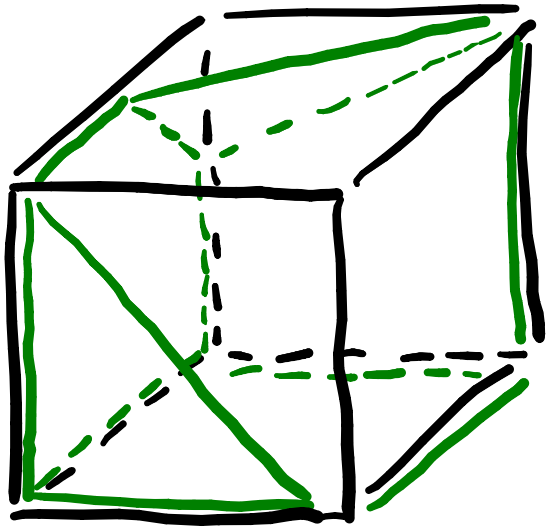
e.g.



$f^{-1}_{(1,2,1,2,1)}(M)$ for $M = x_1(5)\,x_2(1)\,x_1(7)$

$\cap$

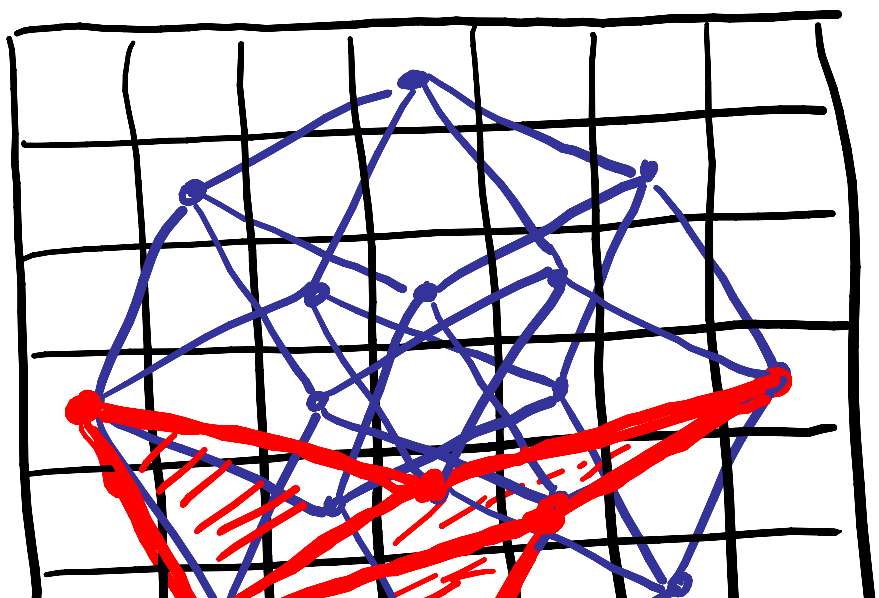
$U^{0}_{s_1 s_2 s_1}$



$\cong B_3$

$\cong$

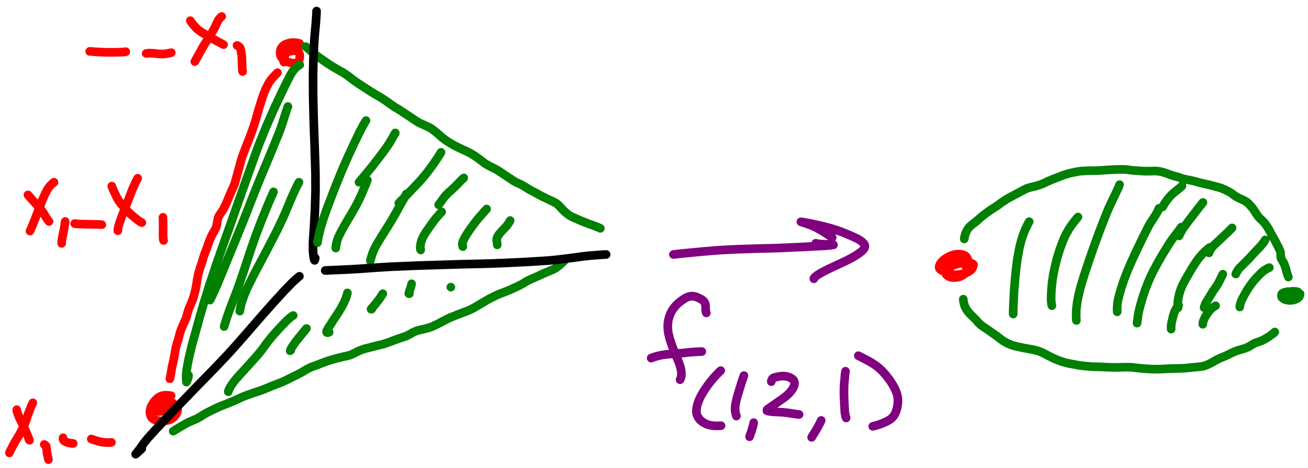
$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^{\circ}_{s_1 s_2 s_1}$



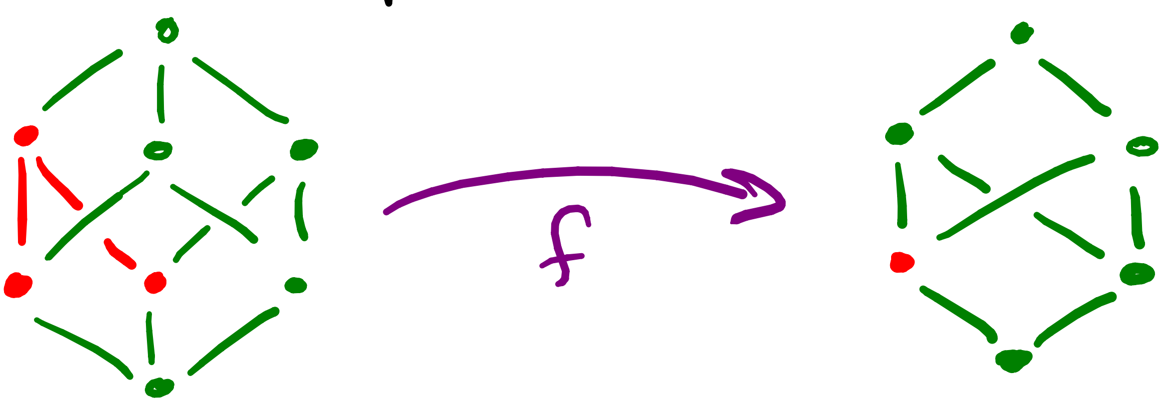
$\cong$

$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^{\circ}_{(1,2)}$

Combinatorics of Fibers



Induced map of face posets:



Thm (Armstrong-H., 2011): For each $u \in W$, $f^{-1}_{\geq}(u) = \{x \in B_n \mid f(x) \geq u\}$ is dual (i.e. upside-down) to face poset for subword complex $\Delta((i_1,\dots,i_k),u)$.

Thm (DHM, 2018): $f_{=}^{-1}(u)$ is face poset of interior dual block complex for subword complex $\Delta((i_1,\dots,i_d),u)$

Thm (DHM, 2018): Interior dual block complex of $\Delta((i_1,\dots,i_d),u)$ is collapsible, hence contractible.

Pf: Discrete Morse theory

Combining: DHM Conjecture would imply $f_{(i_1,\dots,i_d)}^{-1}(p) \cong$ interior dual block complex of $\Delta((i_1,\dots,i_d),u)$ for $p \in Y_u^o$, hence $f_{(i_1,\dots,i_d)}^{-1}(p)$ contractible.

Subword Complexes

(introduced by Allen Knutson & Ezra Miller)

$\Delta(\underset{\sim}{Q}, \underset{\sim}{w})$ = (abstract) simplicial complex of subwords Q' of Q s.t. $Q \setminus Q'$ contains a reduced word for w.

Q: reduced or nonreduced word

w: Coxeter group element

Aside:

1st arose as Stanley-Reisner complexes of initial ideals of coordinate rings of matrix Schubert varieties.

Thm (Knutson-Miller): $\Delta(Q,\omega)$ is homeomorphic to ball or sphere.

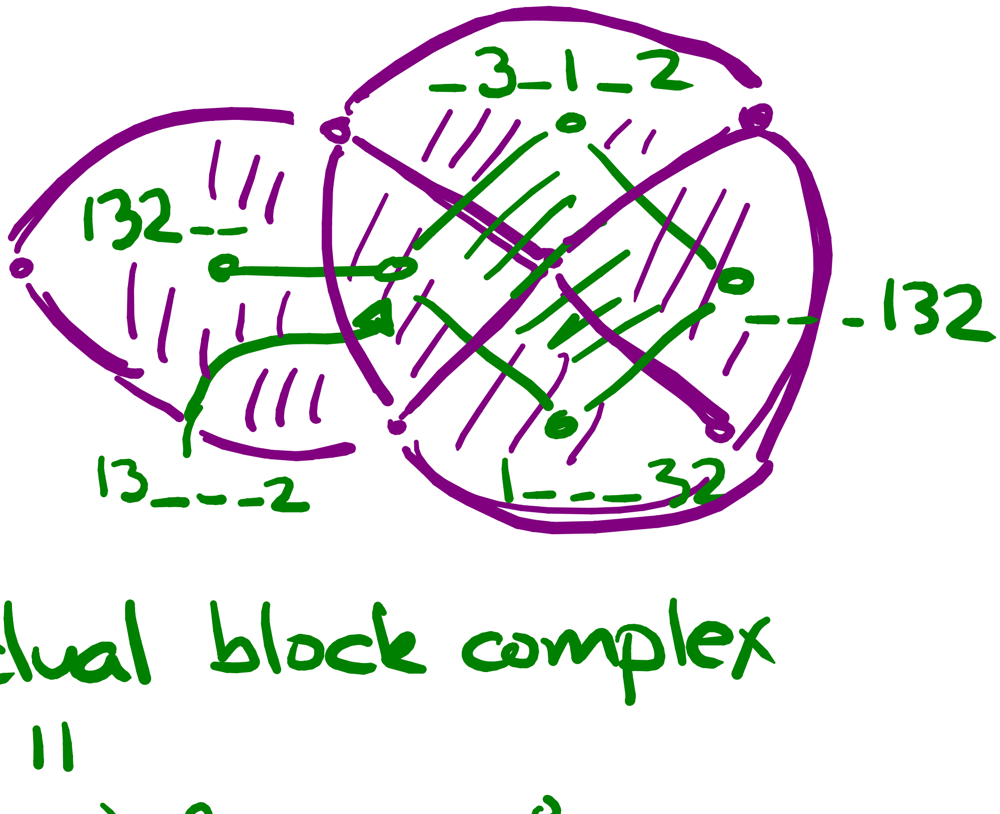
Example of Subword Complex & its Interior Dual Block Complex

$\Delta(Q,\omega) =$

$Q = 132132$

$\omega = s_1 s_3 s_2$

& its



Interior dual block complex

$$\| $$

$f^{-1}_{(1,3,2,1,3,2)}(M)$ for $M \in Y^{\circ}_{(1,3,2)}$

Totally Nonnegative Spaces w/ Seemingly Analogous Structure

1. Totally nonnegative real part of Grassmannian: $Gr_{\geq 0}(k,n) = (GL_n/P)_{\geq 0}$

Postnikov: polytope of "plabic graphs" w/ "measurement map" to $Gr_{\geq 0}(k,n)$ + theory of (reduced) plabic graphs

Postnikov-Speyer-Williams: $Gr_{\geq 0}(k,n)$ is CW complex (via attaching maps that are not homeomorphisms)

Galashin-Karp-Lam 2017 preprint: $Gr_{\geq}(k,n)$ is homeom. to closed ball (proof via curve shrinking as in [HK])

2. Totally nonneg. real part of Flag variety: $\mathcal{F}\ell_{\geq 0} = (GL/B)_{\geq 0}$

Rietsch: poset of closure rel'ns. Cells $R^{\circ}_{u,v}$ given by $u \leq v$ in Bruhat order

Marsh-Rietsch: parametrization for $R^{\circ}_{u,v}$

Williams: poset is CW poset

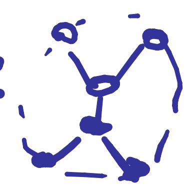
Rietsch-Williams: (1) CW complex w/ attaching maps via canonical bases.
(2) Contractibility of each cell closure

Galashin-Karp-Lam: Homeomorphism type for closure of "big cell".

3. Stratified Spaces E_n of Electrical Networks

(Curtis-Ingerman-Morrow, Kenyon-Wilson, ...)

- arises as image of:

Resp :  $\longmapsto$ response matrix

Lam:

- $E_n \hookrightarrow Gr_{\geq 0}(n-1,2n)$
- $F(E_n)$ is "Eulerian", i.e. $\mu(u,v)$

$$\tilde{\chi}(u,v) = \mu(u,v) = (-1)^{rk(v)-rk(u)}$$

H-Kenyon:

- $F(E_n)$ shelling & $F(E_n)$ is a CW poset (i.e. face poset of regular CW complex)

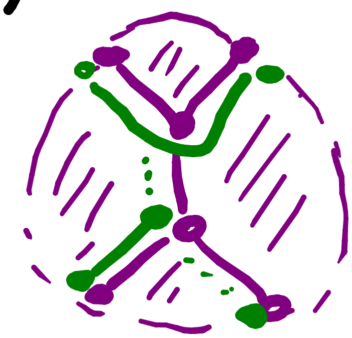
H-K Cor: shelling for each $[u,v]$ in face poset for edge-product space of phylogenetic trees, implying that is a CW poset.

Galashin-Karp-Lam:

- Homeomorphism type for closure of "big cell" for graph analogue of w_0, "well-connected graphs", via special structure of graphical w_0.
- recently announced in talks for other cells as well.

Some Further Questions

1. Subword complex analogues for fibers of other maps?
   e.g. H-Kenyon: $\Delta(G, \mathrm{elec}(H))$ for planar electrical networks (for H a minor of G)



2. Fiber structure for:
   electrical networks, $(GL_n)_{\geq 0}$, $(\mathcal{F}\ell_n)_{\geq 0}$, $(GL_n/P)_{\geq 0}$, ...?

3. Generalization of

Berenshtein-Fomin-Zelevinsky inverse map for positive part to: nonreduced word w/ unknown parameters as reduced subword

e.g.

$$x_1(t_1)x_2(t_3)x_1(5)x_2(7)x_1(t_3)=M$$

Qn: Sol'n existence $\Rightarrow$ uniqueness?

4. Proof of DHM Conjecture?

Thanks!