Posets Arising as 1-Skeleta of Simple Polytopes, the Nonrevisiting Path Conjecture & Poset Topology

Patricia Hersh

North Carolina State University

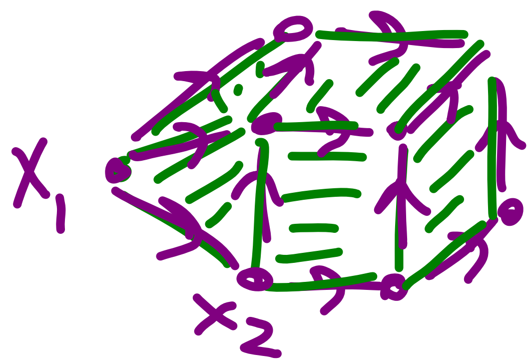
- with thanks to Karola Mészáros for fruitful discussions early in project
- slides will be posted at:

https://plhersh.math.ncsu.edu/talks.html

Linear Programming

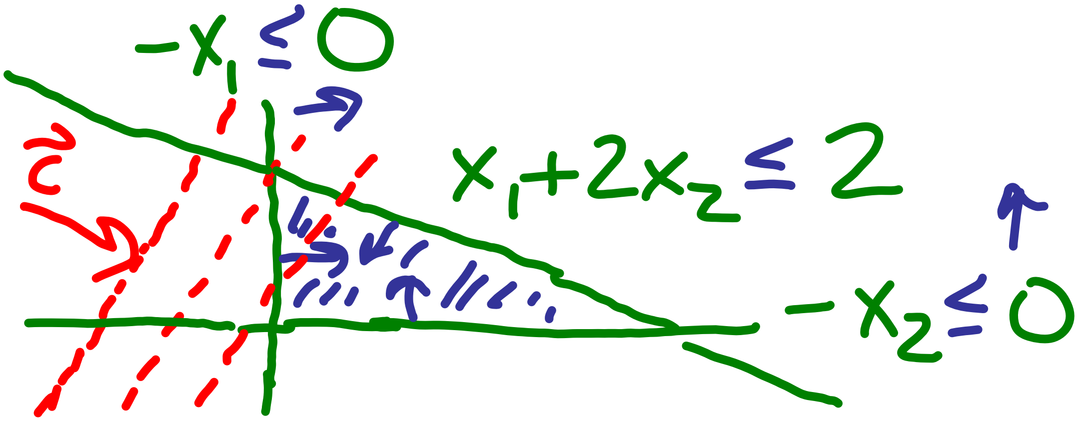
- Given a matrix A & vectors $\vec{b}, \vec{c}$ seek $\max\{\vec{c}\cdot\vec{x} \mid A\vec{x} \le \vec{b}\}$

- $\{\vec{x} \mid A\vec{x} \le \vec{b}\}$ is polytope P if set is bounded



e.g.

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}} \le \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}_{\vec{b}}$$



Solving Linear Programs via Simplex Method

- Define $G(P, \vec{c}) :=$ directed graph on 1-skeleton of P, i.e. on vertex-edge graph of P, with

$$x_1 \to x_2 \iff \vec{c} \cdot \vec{x}_1 < \vec{c} \cdot \vec{x}_2$$

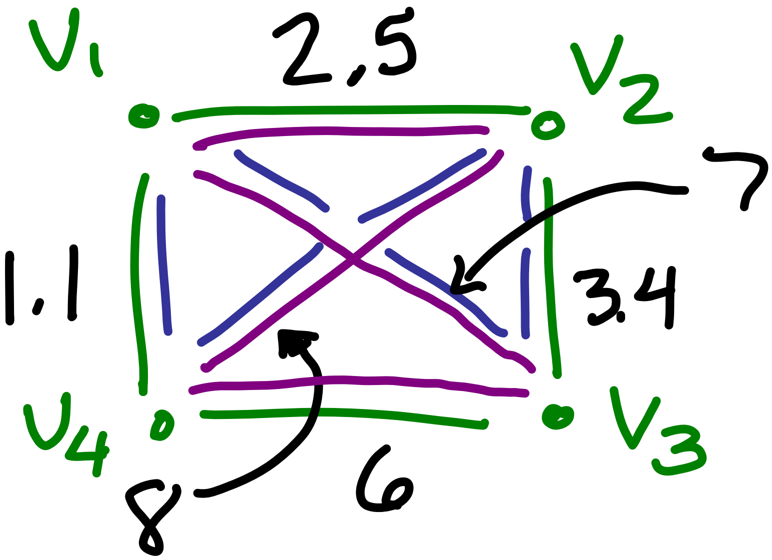
- $\max\{\vec{c} \cdot \vec{x} \mid A\vec{x} \leq \vec{b}\} =$ sink of $G(P, \vec{c})$

Simplex Method: walk from some vertex $v \in G(P, \vec{c})$ following arrows $v \to v_1 \to v_2 \to \ldots \to s$ to sink s

- also may walk backwards to source of $G(P, \vec{c})$ to minimize $\vec{c} \cdot \vec{x}$

An Example: Traveling Salesman Problem

$G =$



Polytope Vertices:

$(1, 0, 1, 1, 0, 1)$, $(1, 1, 0, 0, 1, 1)$, $(0, 1, 1, 1, 1, 0)$

e_{12} e_{14} e_{23} e_{34}

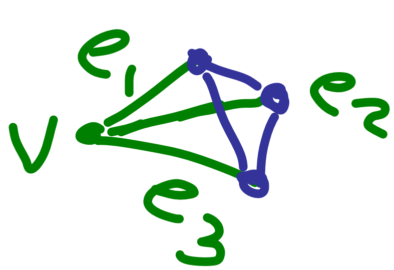
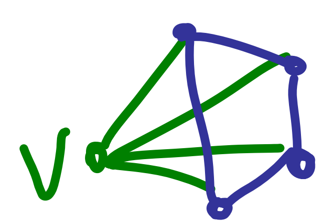
Cost Vector:

$$\vec{c} = (2.5, 7, 1.1, 3.4, 8, 6)$$

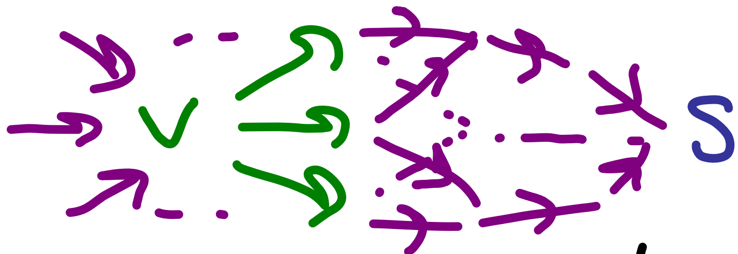
Quick Background on Polytopes

- A **polytope** in $\mathbb{R}^d$ is convex hull of finite # vertices, or equiv'ly a bounded set that is an intersection of half spaces.

- Polytope is **simple** if for each vertex v and each collection $e_1, e_2, .., e_r$ of edges emanating out from v, there is **r-dim'l face** containing all these edges.

e.g.  but not 

Pivot Rule: method to choose which out arrow



to follow from v towards sink s.

Key Questions:

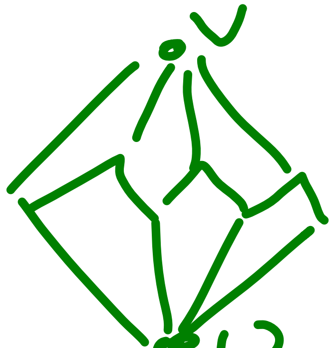
1. what is typical complexity of simplex method (path length)?

2. What is worst case? (i.e. diameter of $G(P, \vec{c})$)

Hirsch Conjecture: For d-dim'l polytopes with n facets (max'l faces), diameter of 1-skeleton graph, denoted $\Delta(d,n)$, satisfies $\Delta(d,n) \leq n-d$.

Francisco Santos: After many decades eluding many people, he constructed counterexamples

("spindles" := polytopes with vertices v, w s.t. each facet includes v or w.)



43-dim's, 86 facets, diam $\geq$ 44

Nonrevisiting path conjecture.

For each u,v in polytope P, there is path u to v not revisiting any facet it has left.

Non-Revis. Path Conj $\Rightarrow$ Hirsch Conj.

- nonrevisiting path leaves a facet at each step & still belongs to d facets at its conclusion

Strong Monotone Path Conjecture:

there exists directed path of length $\leq n-d$ from any vertex to vertex v maximizing $\vec{c} \cdot v$ with cost increasing each step.

Our Plan

Impose further conditions on P and $\vec{c}$ that will imply a corollary of the following which we hope might also hold:

For each $u, v \in P$, each directed path from u to v never revisits any facet it has left.

This property would make <u>all</u> pivot rules efficient for P and $\vec{c}$.

Remark (we'll revisit later)

another rich direction, explored by many, is topological structure of space of "monotone paths" from u to v

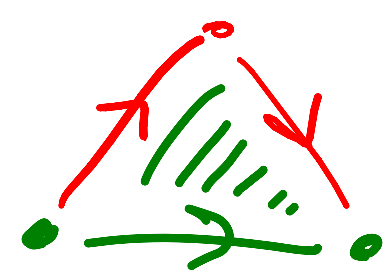
e.g. Baues Conjecture (cf. Billera-Kapranov-Sturmfels) & fiber polytopes (Billera-Sturmfels)

Baues Conjecture "arose from search for CW models for iterated loop spaces Ω^i of CW space X"

Note: One of our main results will hint at connections between diameter/nonrevisiting questions & these structural questions

Def'n (H.): $G(P,\vec{c})$ has the **Hasse diagram property** if it is Hasse diagram of finite poset, i.e. $v_1 \to v_2 \to \cdots \to v_r$ for $r \geq 3$ directed path precludes $v_1 \to v_r$

Note: precludes d-simplices as faces for $d \geq 2$

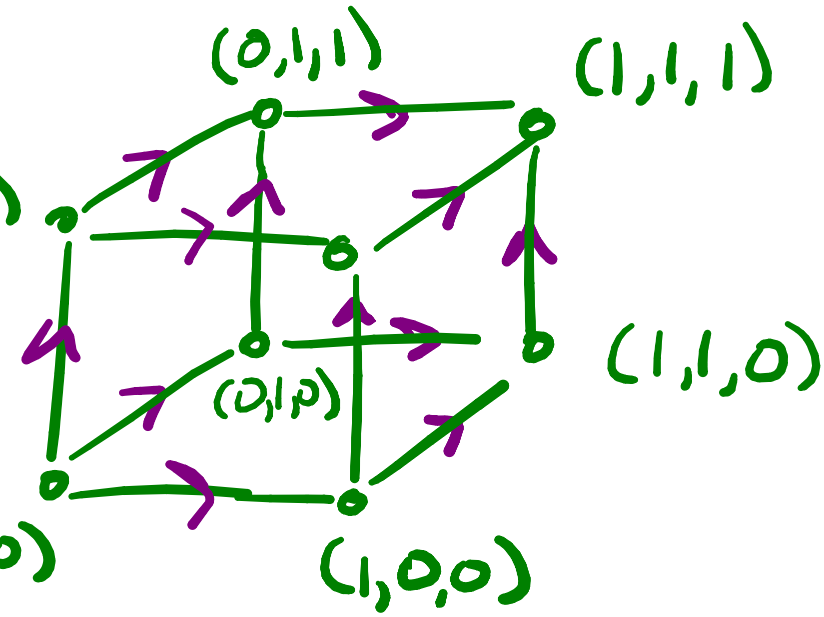


Note: Equivalent to non-revisiting of 1-dim'l faces

Note: Poset often not "atomic"

An Example: Hypercube Polytope

$P =$



$\vec{c} = (1,2,3)$

Poset with Hasse diagram above

$\parallel$

"Boolean lattice" B_3 or $B_{\{1,2,3\}}$

$\parallel$

Poset of subsets of $\{1,2,3\}$ ordered by containment i.e.

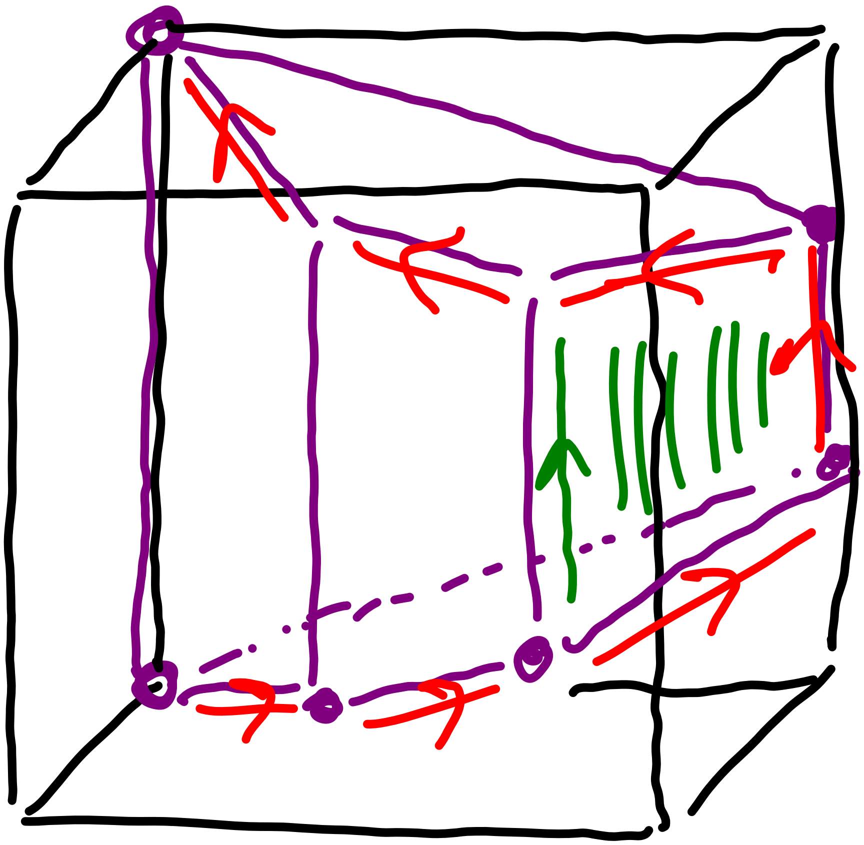
$$\begin{tikzcd}[arrows=dash]
 & \{1,2,3\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{1,2\} \arrow[d] \arrow[dr] & \{1,3\} \arrow[dl] \arrow[dr] & \{2,3\} \arrow[dl] \arrow[d] \\
\{1\} \arrow[dr] & \{2\} \arrow[d] & \{3\} \arrow[dl] \\
 & \emptyset &
\end{tikzcd}$$

Important Non-Examples:

"Klee-Minty Cubes"

e.g. $n=3$

$\vec{c}=(0,0,1)$



- path visits all vertices!
- first polytopes exhibiting inefficiency of simplex method

Figure modelled after one in Gärtner-Henk-Ziegler paper

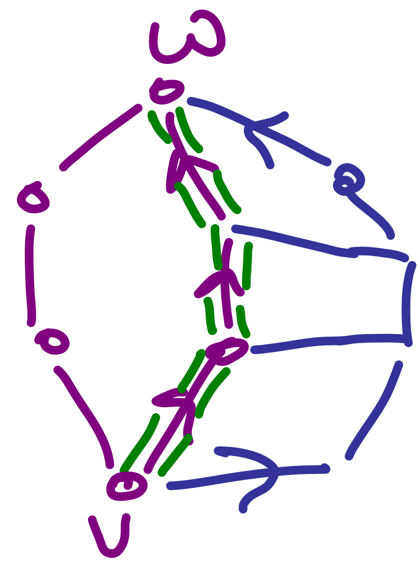
n-dimensional Klee-Minty cube

$$:= \left\{(x_1,\dots,x_n) \;\middle|\; 0\le x_1\le 1;\ \begin{array}{l}\varepsilon x_{i-1} < x_i < 1-\varepsilon x_{i-1} \\ \text{for } i>1;\ 0<\varepsilon<\frac{1}{2}\end{array}\right\}$$

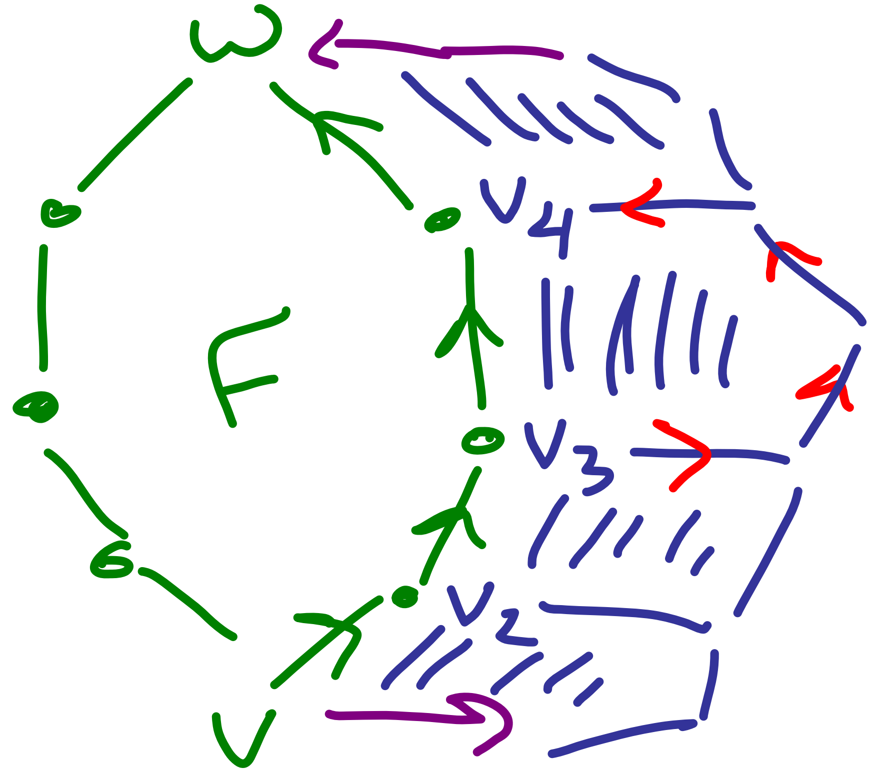
Note: Klee-Minty cubes violate Hasse diagram property in way that seems to be at the heart of what leads to existence of "long" directed path (visiting all 2^n vertices) in it

Our hope: Hasse diagram property precludes such issues.

Lemma (H.): For $F \subseteq G$ with $\dim(G) = \dim(F) + 1$ in simple polytope P & generic $\vec{c}$ s.t. $G(P, \vec{c})$ is a Hasse diagram, then there does not exist $v, w \in F$ with directed path P_F from v to w in F, outward oriented edge from v to $G \setminus F$ and inward oriented edge $G \setminus F$ to w.



Corollary: Monotonicity of out-degrees & partic. outward directions.

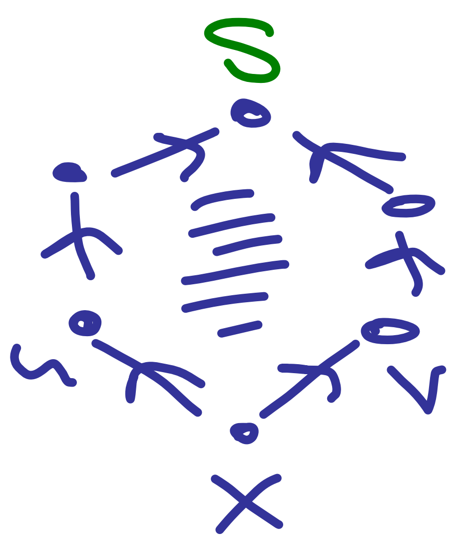


} violate Hasse diagram (at switch from out to in)

Corollary: For each face $F \subseteq P$ with $\hat{0} \in F$ or $\hat{1} \in F$, directed paths cannot revisit F after departing from it.

<u>Recall:</u> A poset L is a **lattice** if for each $u, v \in L$ there exists unique least upper bound ("join") for u and v, denoted $u \vee v$, and unique greatest lower bound for u and v ("meet"), $u \wedge v$.

Note: for P simple & $G(P, \vec{c})$ Hasse diagram, an upper bound for u, v both covering x is sink s of unique 2-face containing x, u, v.



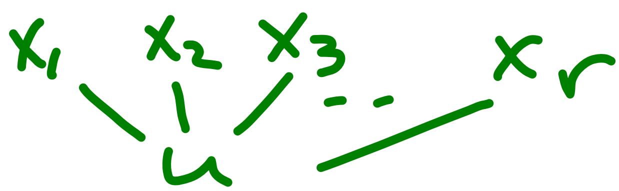
"Pseudo-joins" in a Polytope

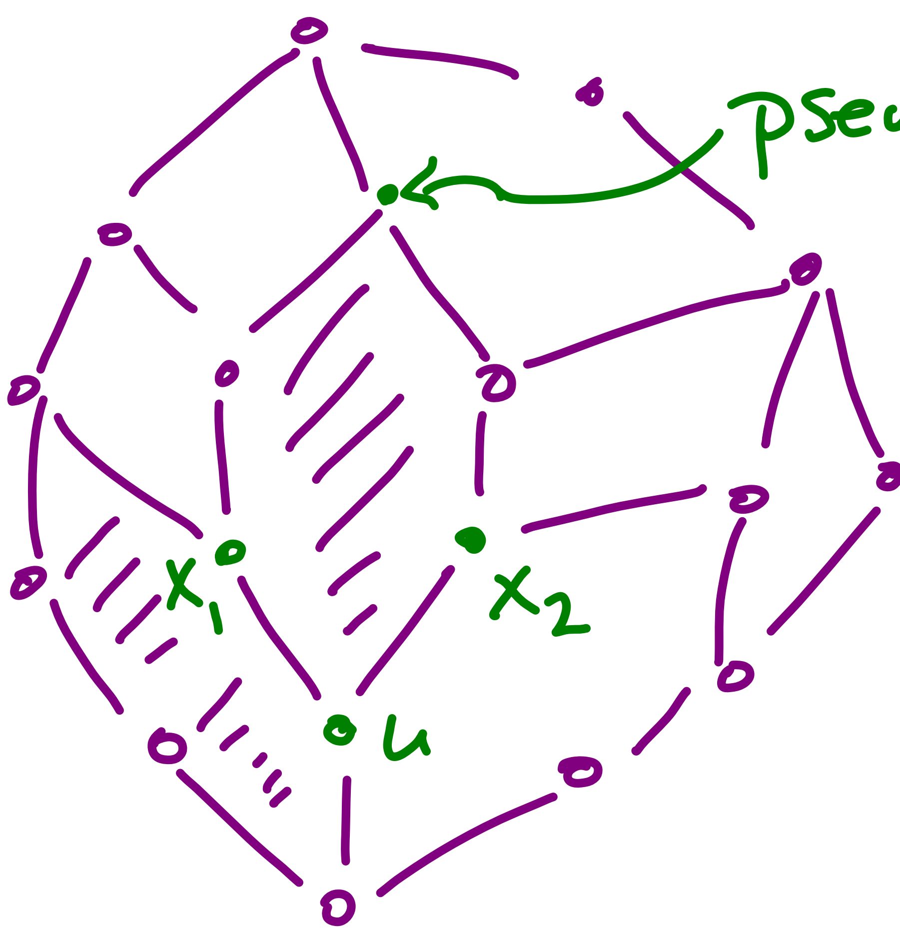
Let P be simple polytope w/ generic cost vector $\vec{c}$ such that $G(P,\vec{c})$ is Hasse diagram of poset L with $x_1, x_2, \ldots, x_r \in L$ s.t. there exists $u \in L$ with $u \lessdot x_i$ for $i = 1, 2, \ldots, r$.

Recall:

- $u \lessdot v$ means $u < v$ with no possible z s.t. $u < z < v$ "cover rel'n"
- a_i is atom of $[u, v]$ for $u \lessdot a_i$

Def'n (H.): The pseudo-join of $x_1, x_2, \ldots, x_r$ is the sink of the unique r-face of simple polytope P containing





pseudojoin of x_1 & x_2 $(r=2)$

Lemma (H.): For P a simple polytope & $\vec{c}$ generic cost vector s.t. $G(P, \vec{c})$ is Hasse diagram of poset L, let $S, T \subseteq \{a_1, \ldots, a_n\}$ be distinct sets of atoms. Then pseudojoin$(S) \neq$ pseudojoin(T).
For L a lattice, this also holds for atoms in each interval $[u,v] \subseteq L$.

Cor: Subposet of pseudojoins is isomorphic to Boolean lattice of subsets of $\{a_1, \ldots, a_n\}$.

Note: Since pseudo-join of $x_1, \dots, x_r$ is an upper bound, there exists directed path from $x_1 \vee \dots \vee x_r$ to pseudo-join$(x_1, \dots, x_r)$

Thm (H.): Let P be a simple polytope and $\vec{c}$ be generic cost vector with $G(P, \vec{c})$ Hasse diagram of finite lattice. Then

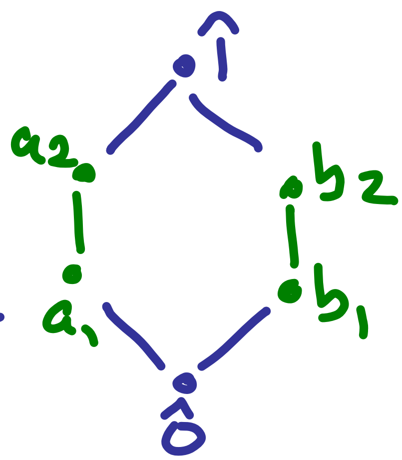
pseudo-join$(x_1, x_2, \dots, x_r) = x_1 \vee \dots \vee x_r$

Pf: induction on r with $r=2$ base case especially tricky part.

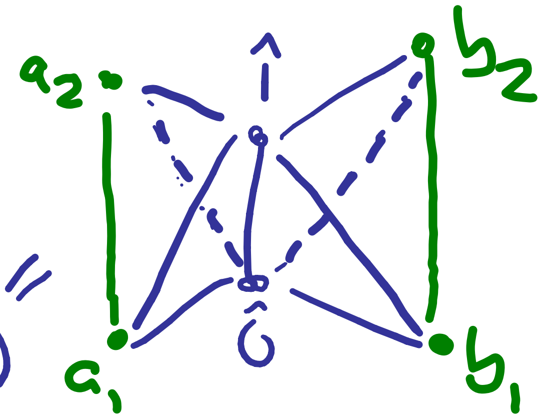
Defn: The order complex (or nerve) of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-dim'l faces are the $(i+1)$-chains $v_0 < v_1 < \dots < v_i$ in P.

e.g.

$P =$



$\Delta(P) =$



Thm (Hall; Popularized by Rota):

$$\mu_P(u,v) = \tilde{\chi}(\Delta(\underbrace{(u,v)}_{\text{subposet } \{z \in P \mid u < z < v\}}))$$

Fact: K regular CW complex $\Longrightarrow$

$$\Delta(F(K) \setminus \{\hat{0}\}) = sd(K) \cong K$$

Quillen Fiber Lemma (a.k.a. Quillen Theorem A): Given a poset map $f: P \to Q$ s.t. $q \in Q$ implies $\Delta(f^{-1}_{\leq}(q)) = \Delta(\{p \in P \mid f(p) \leq q\})$ is contractible, then $\Delta(P) \simeq \Delta(Q)$.

Remark: Used extensively in finite group theory (to characterize groups G via subgroup lattice $L(G)$) & in topological combinatorics.

e.g. **Thm** (Shareshian):
G solvable $\iff L(G)$ shellable

Thm (Stanley):
G supersolv. $\iff L(G)$ supersolv.

Thm (H.): For P a simple polytope & $\vec{c}$ a generic cost vector such that $G(P,\vec{c})$ is the Hasse diagram of a finite lattice L,

$$\Delta(\underbrace{u,v}_{= \{z \in L \mid u < z < v\}}) \simeq \text{ball or sphere } S^{|A(u,v)|-2}$$

$$A(u,v) := \{a \in L \mid u \lessdot a < v\}$$
= "atoms" of $[u,v]$

Idea: Use poset map

$$f(x) = \bigvee_{a_i \in A(u,v)} a_i = \text{pseudojoin of } \{a_i \mid a_i \in A(u,v)\}$$

by "join = pseudo-join" theorem

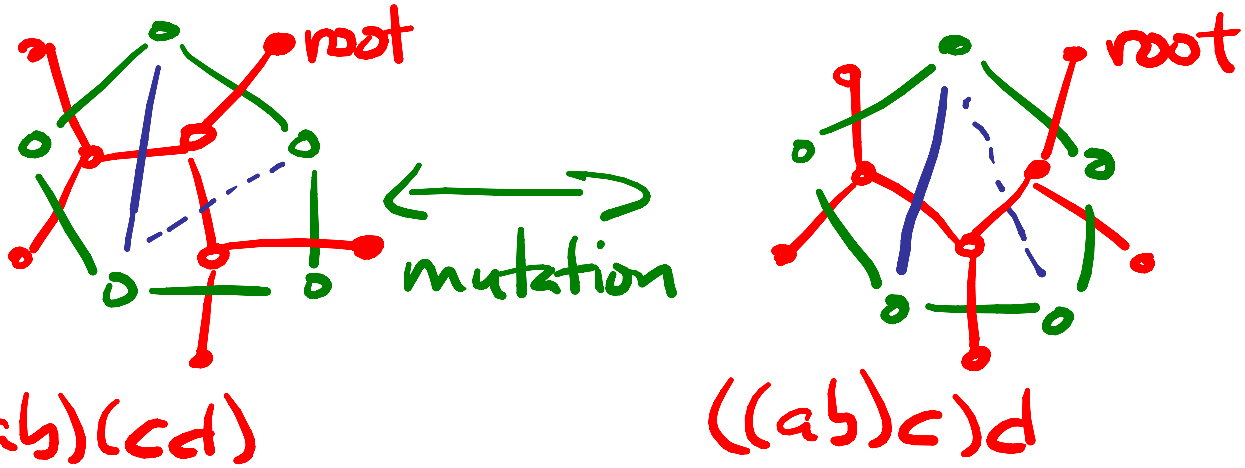
then use distinctness of pseudojoins so image = Boolean lattice w/ or without maximal element

Applications: Unified proof for

- permutahedra ⇝ weak order

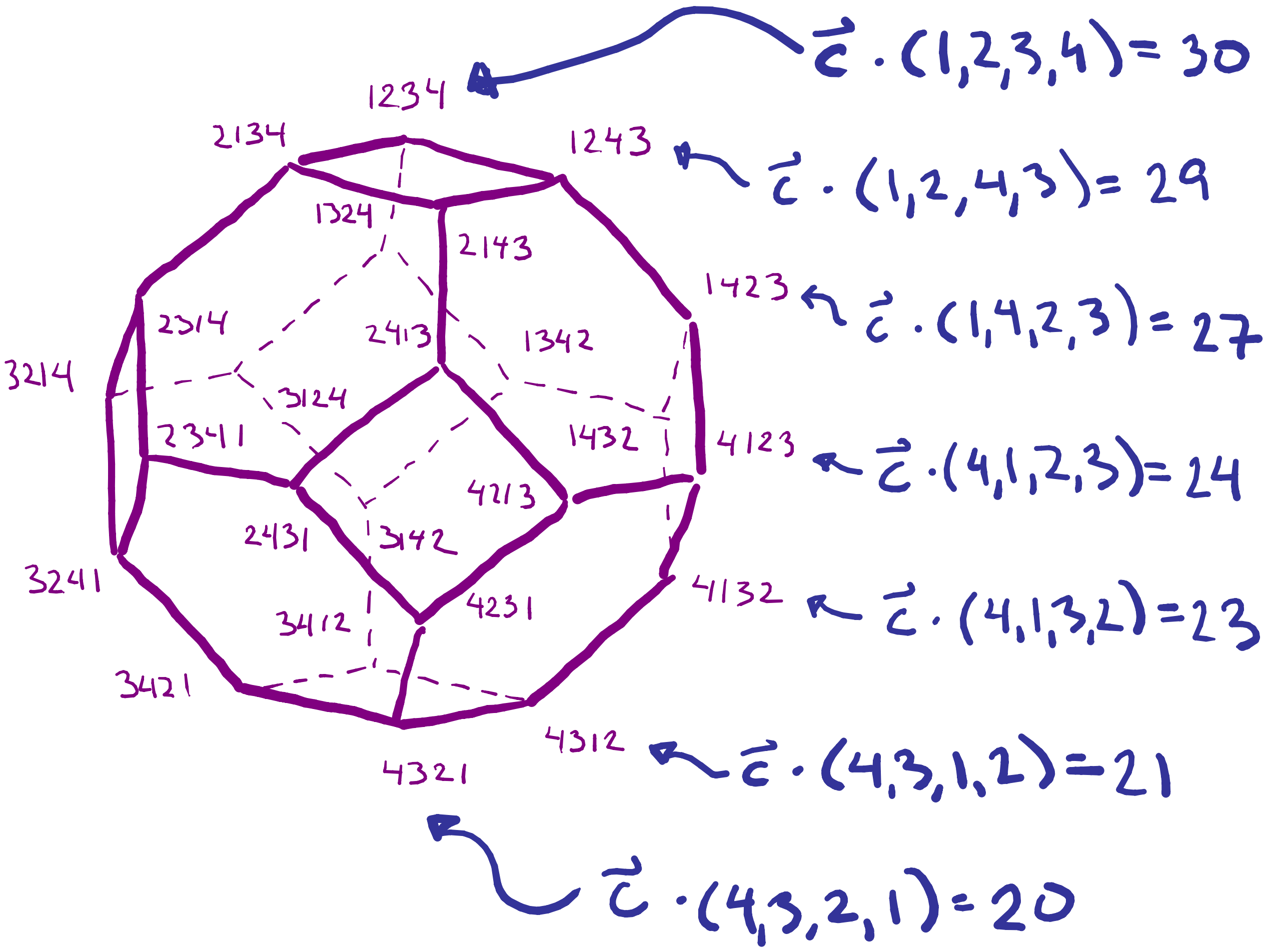
- associahedra ⇝ Tamari lattice
  (a.k.a Stasheff polytopes)

- generalized associahedra ⇝ Cambrian lattices
  (related to cluster algebras of finite type & mutation)



Permutahedron as Weak Order

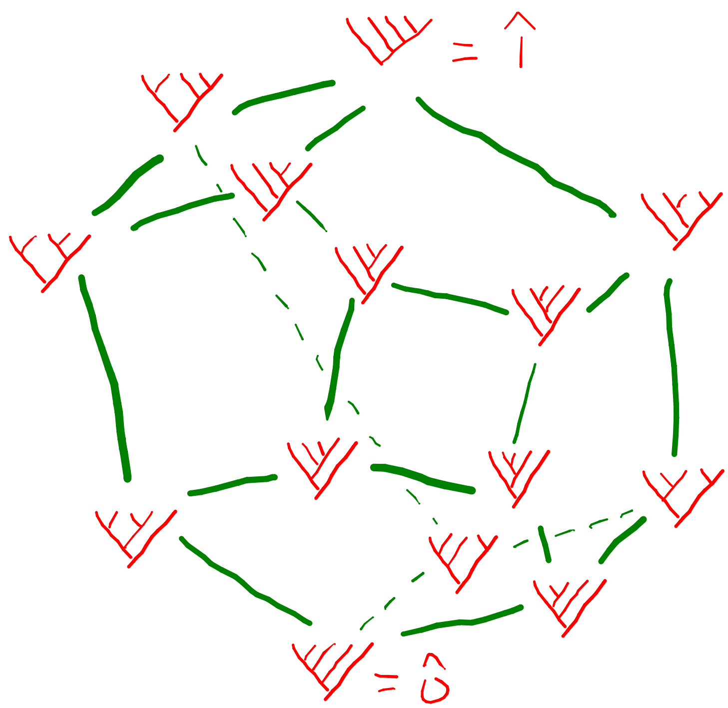
- cost vector $\vec{c}$ any strictly ascending vector such as $\vec{c} = (1,2,3,4)$.



- Homotopy type 1st due to Edelman (type A) & Björner

Associahedron as Tamari Lattice

- Use Loday's realization
- Poset of binary trees with cover relations: $((a,b),c) \lessdot (a,(b,c))$



- Homotopy type 1st due to Björner & Wachs via nonpure lexicographic shellability

Related Complexes from Alg. Topology (to $\Delta(G(P,\vec{c}))$)

Def'n: A stippling of polytope P assigns to each face F a source $n(F)$ (resp. a sink $K(F)$) s.t.

- $F \leq G$ & $n(G) \leq F$ (resp. $K(F) \leq G$) $\Rightarrow n(F) = n(G)$ (resp. $K(F) = K(G)$)

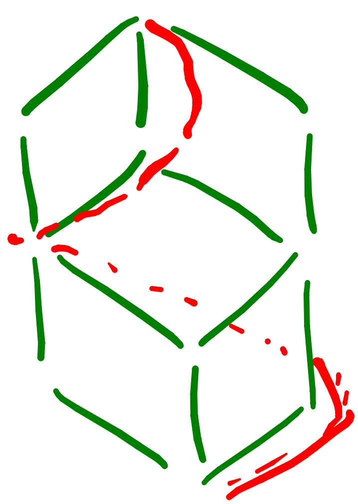
Def'n: The complex of cellular strings in P w.r.t. a stippling consists of cells $(F_1, F_2, \ldots, F_r)$ s.t. $K(F_i) = n(F_{i+1})\ \forall i$

Thm (Billera-Kapranov-Sturmfels): For stippling of P via generic cost vector $\vec{c}$, complex $\simeq S^{\dim(P)-2}$

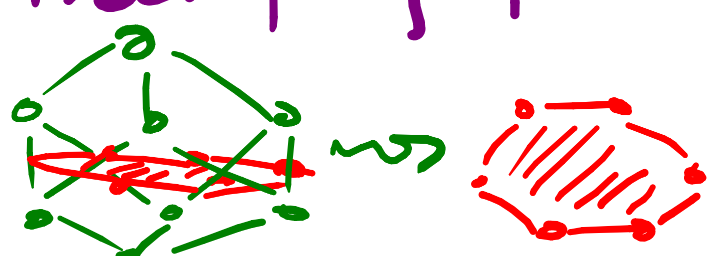
Thm (Billera-Kapranov-Sturmfels):

Subcomplex of "coherent cellular strings" $\cong S^{\dim(P)-2}$. More specifically, one obtains "monotone path polytope" (a "fiber polytope" given by map to line segment)

Loosely speaking: Coherent cellular strings are those not "wrapping around" polytope



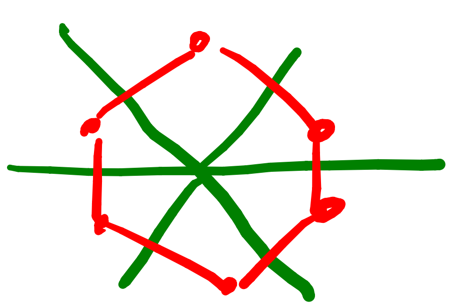
(coherent maximal stiplings $\leftrightsquigarrow$ vertices of fiber polytope)



Some Further Remarks

1. Any zonotope P & generic cost vector $\vec{c}$ yields $G(P,\vec{c})$ with non-revisiting property, hence Hasse diagram property.

Recall: A zonotope is a Minkowski sum of line segments or dual to chamber complex of hyperplane arrangement



(Minkowski sum of normal vectors to hyperplanes)

2. Given shelling on simplicial polytope X, this induces "facial order" on vertices of X^*. For $G(X^*)$ Hasse diagram of lattice, pseudo-joins equal joins, are distinct & $\Delta(u,v) \simeq$ ball or sphere.

Recall: Polytope dual has face poset upside-down

vertices $\rightsquigarrow$ facets (max'l faces)

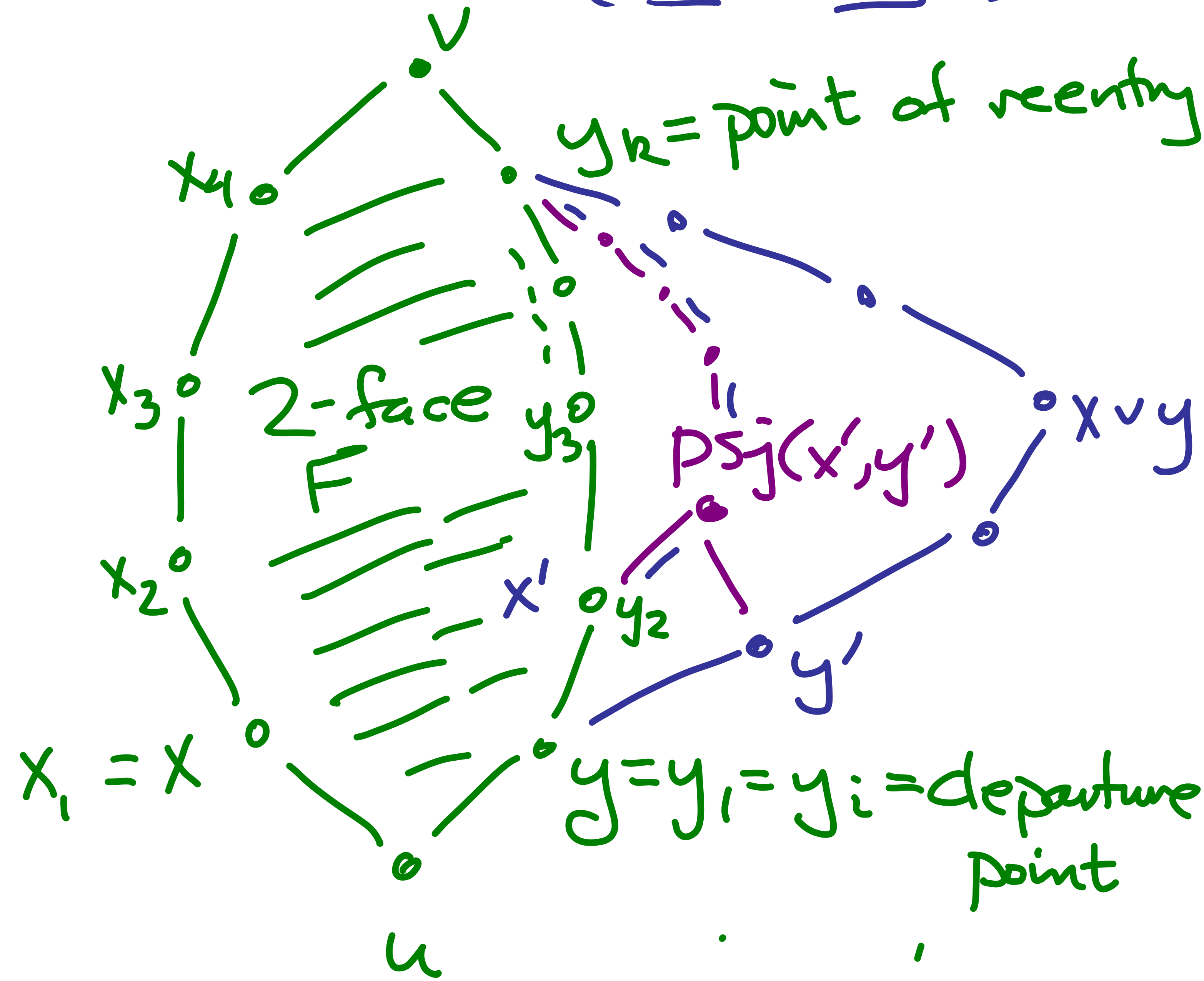
edges $\rightsquigarrow$ codim one faces

$\vdots$

3. Checking the Hasse Diagram Property (thanks to Lou Billera)

- let $A :=$ directed adjacency matrix for $G(P, \vec{c})$
- $(A^r)_{i,j} :=$ # directed paths of length r from v_i to v_j
- Calculate $A, A^2, \ldots, A^{|V(G)|-1}$
- Want $\operatorname{trace}(A^T \cdot A^r) = 0$ for $r = 2, 3, \ldots, |V(G)|-1$

Idea for "Join = Pseudo-Join" (r=2 case)



- $y' \notin F \Rightarrow \mathrm{psj}(x', y') \notin F$

$\qquad\qquad\quad \|$ ⟵ by induction on length longest path to ↑

$\qquad\quad \mathrm{join}(x', y')$

strict inequality ⇝ $\qquad \wedge|$

$\Rightarrow \exists$ smaller $k-i \Rightarrow\!\Leftarrow$ $\qquad y_k \in F$

Some Further Questions

Qn 1: Does P simple + $G(P,\vec{c})$ Hasse diagram of lattice $\Rightarrow$ no directed path can revisit face it has departed? (If not, variations?)

Qn 2: Variations on hypotheses? Non-simple polytopes? Non-lattices?

Qn 3: Structure of posets of joins/pseudo-joins for non-simple polytopes?

Qn 4: Additional interesting examples?

- graph associahedra? (nonrevisiting proven by Barnard-McConville & related to nested set complexes e.g. in work of Feichtner-Yuzvinsky)
- classes of MV polytopes?

Qn 5: Natural/nice homotopy equiv. between our order complexes & complexes of Billera-Kapranov-Sturmfels and others?

Qn 5b: Do order complexes detect nonrevisiting, or always spheres too?

Thank you!

Idea for Distinctness of Pseudo-Joins

(1) Reduce to $S \subsetneq T$ with $|T| = |S| + 1$

- $S_1 \subsetneq S_2 \subsetneq S_3$ with $psj(S_1) = psj(S_3)$
  $\Longrightarrow psj(S_2) = psj(S_3)$
- $S_1 \not\subseteq S_2$ & $S_2 \not\subseteq S_1$ then use

  $S_1 \cap S_2 \subsetneq S_1$ with $psj(S_1 \cap S_2) = psj(S_1)$

(2) Use codim one nonrevisiting lemma

(3) For $[u,v]$, use that v is an upper bound for the atoms of $[u,v]$ w/ l.u.b. in $[u,v]$

Idea for Inductive Step:

Induct on $|S|$ with $|S| = 2$ base case as just discussed.

$T \subsetneq S$

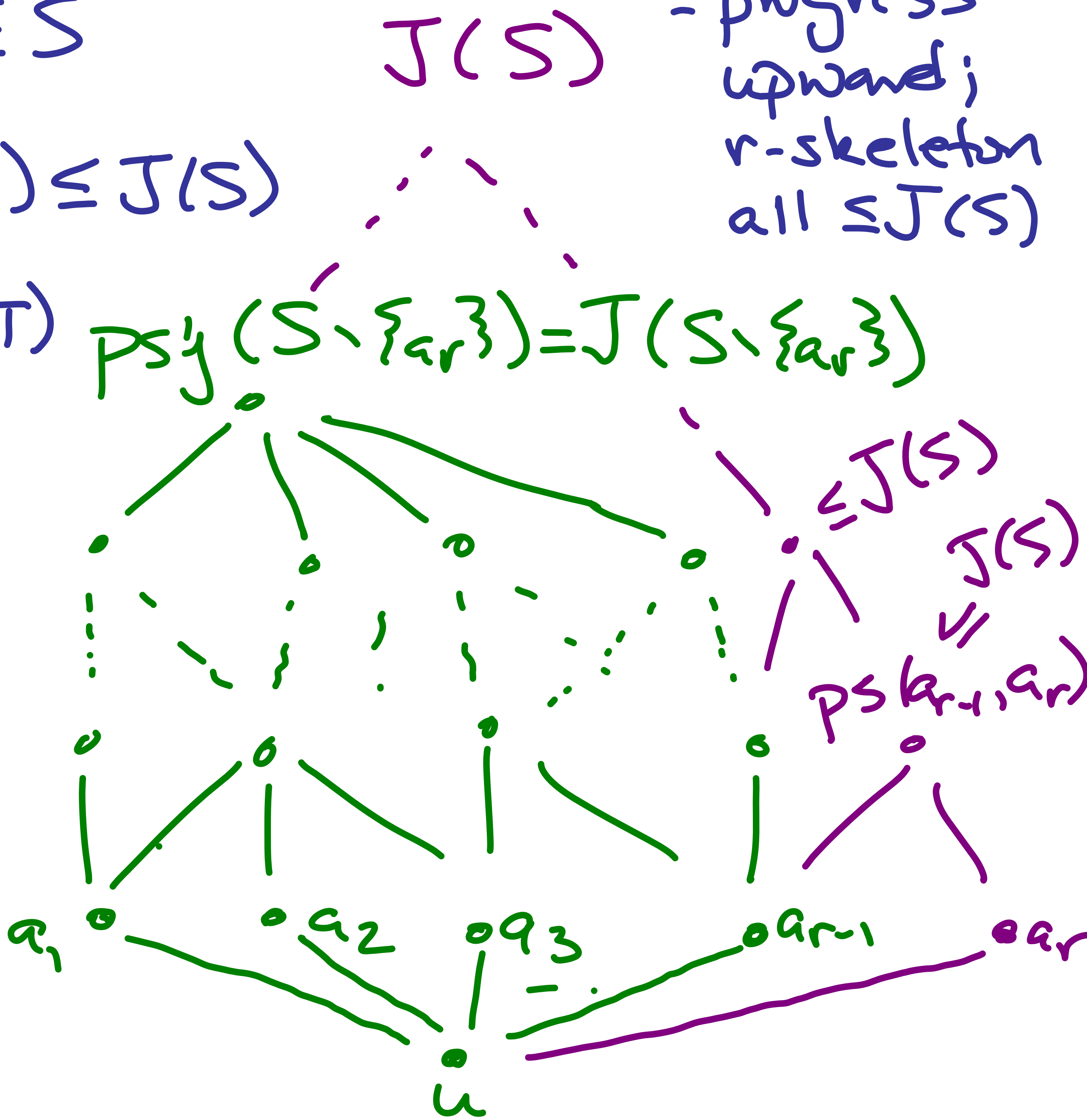
$\Downarrow$

$J(T) \leq J(S)$

$\parallel$

$ps(T)$

- progress upward; r-skeleton all $\leq J(S)$



Generalized Associahedra & Cambrian Lattices

- related to cluster algebras
- homotopy type 1st due to Nathan Reading