Representation Stability in Configuration Spaces via Whitney Homology of the Partition Lattice

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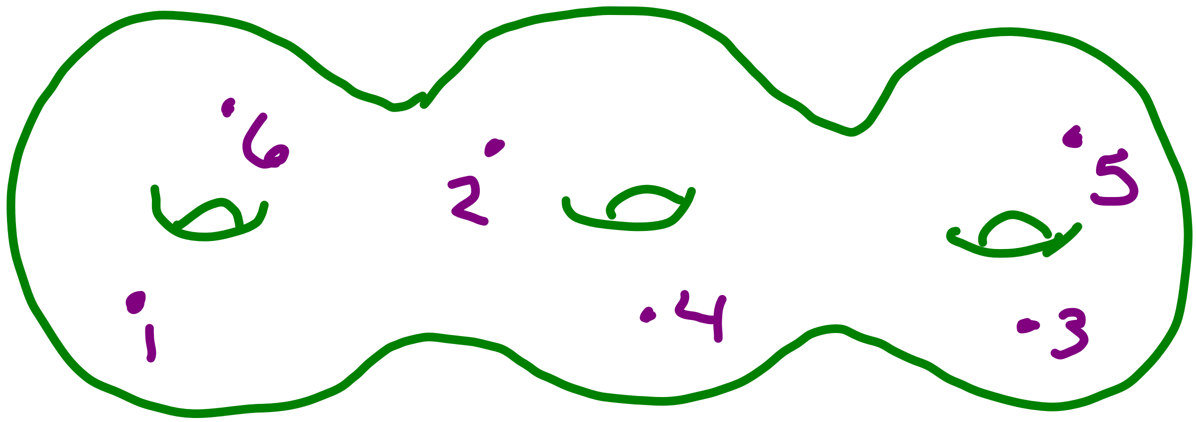
(based on paper to appear in IMRN)

A "Point" in a Configuration Space with S_n-rep'ns on Cohomology



- Manifold = 3-holed torus
- $n = 6 =$ # distinct labeled points
- S_n acts freely on configuration space by permuting pt. labels, inducing rep'n on each cohomology group

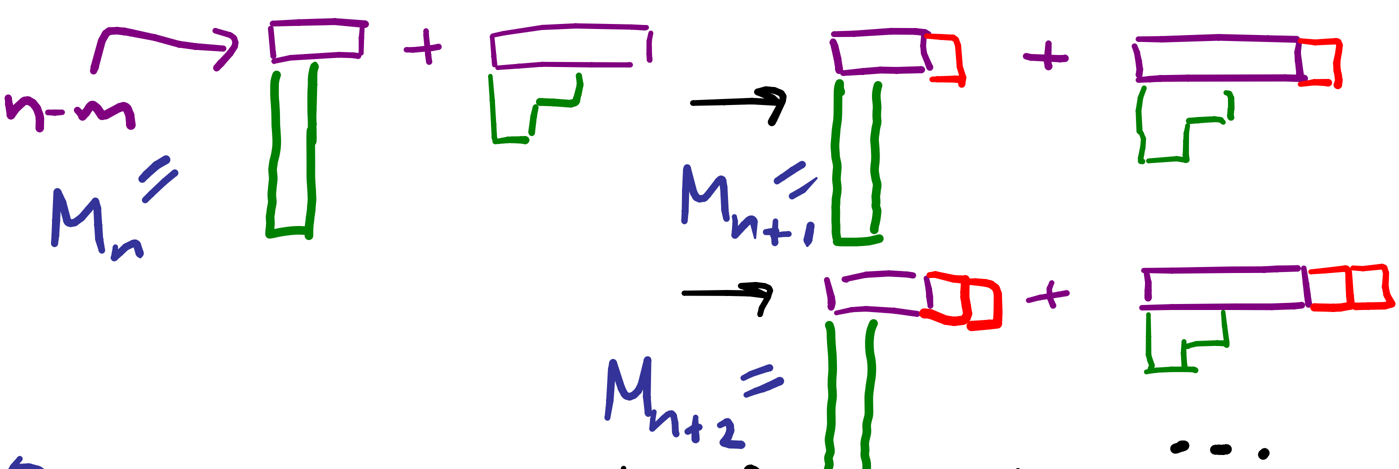
Representation Theoretic Stability

Defn (Church, Farb): A series of S_n-modules $M_1, M_2, \ldots$ for $n = 1, 2, \ldots$ **stabilizes at $B > 0$** if for each $n > B$, we have

$$M_n = \sum_{\lambda \vdash m \leq B} c_\lambda V(\lambda) \quad \text{where } V(\lambda) \cong S^{(n-m,\lambda)}$$

and where c_λ does not depend on n

e.g.



Our Focus: S_n-rep'ns from partition lattice

Our Starting Point:

Thm (Church-Farb): $H^i(M_n, \mathbb{Q})$ stablilizes for $n \geq 4i$ where M_n is configuration space of n distinct points in plane & i is held fixed.

Thm (Church-Farb): More generally, letting M_n^d be the configuration space of n distinct labeled points on connected orientable d-manifold, $H^i(M_n^d, \mathbb{Q})$ stabilizes for

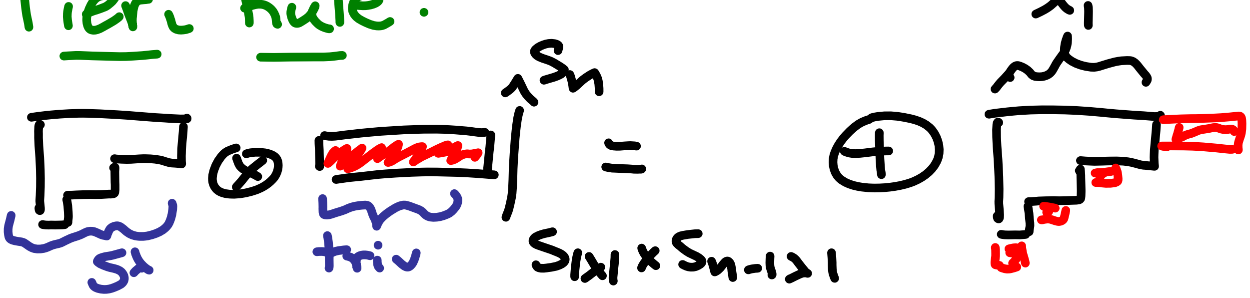
$$\begin{cases} n \geq 4i & \text{if } d = 2 \\ n \geq 2i & \text{if } d > 2 \end{cases}$$

Our First Objective: Sharpen these bounds for $M^d = \mathbb{R}^d$

How Representation Stability Typically Arises

- Finite number of irred. rep'ns S^λ; S^λ 1st appearing in $M_{|\lambda|}$
- Each M_n with $n \geq |\lambda|$ likewise includes $S^\lambda \otimes \mathrm{triv}_{n-|\lambda|} \big\uparrow_{S_{|\lambda|} \times S_{n-|\lambda|}}^{S_n}$
- Church-Ellenberg-Farb prove stability bounds of $n = 2\max|\lambda|$
- H-Reiner prove sharp stability bds for $\mathrm{PConf}(\mathbb{R}^d)$ at $n = \max(|\lambda| + \lambda_1)$

Pieri Rule:



Church-Farb Method for Orientable Manifolds

- Use Totaro's E_2-page of Leray spectral sequence (showing cohom. of config. space of n distinct pts on manifold M is determined by cohom. of M + $H^i(M_n(\mathbb{R}^d))$) to deduce stability of each page from previous page:

$$E_2^{p,(d-1)q} = \bigoplus_{\substack{S \text{ with} \\ |S| = n-q}} H^{q(d-1)}(\underbrace{C_S(\mathbb{R}^d)}_{\text{product of subspace arrangement complements}}) \otimes H^p(M^S)$$

for set partition S with $|S|$ parts

e.g. for $S = \{1,3\}\{2,4,5\}$

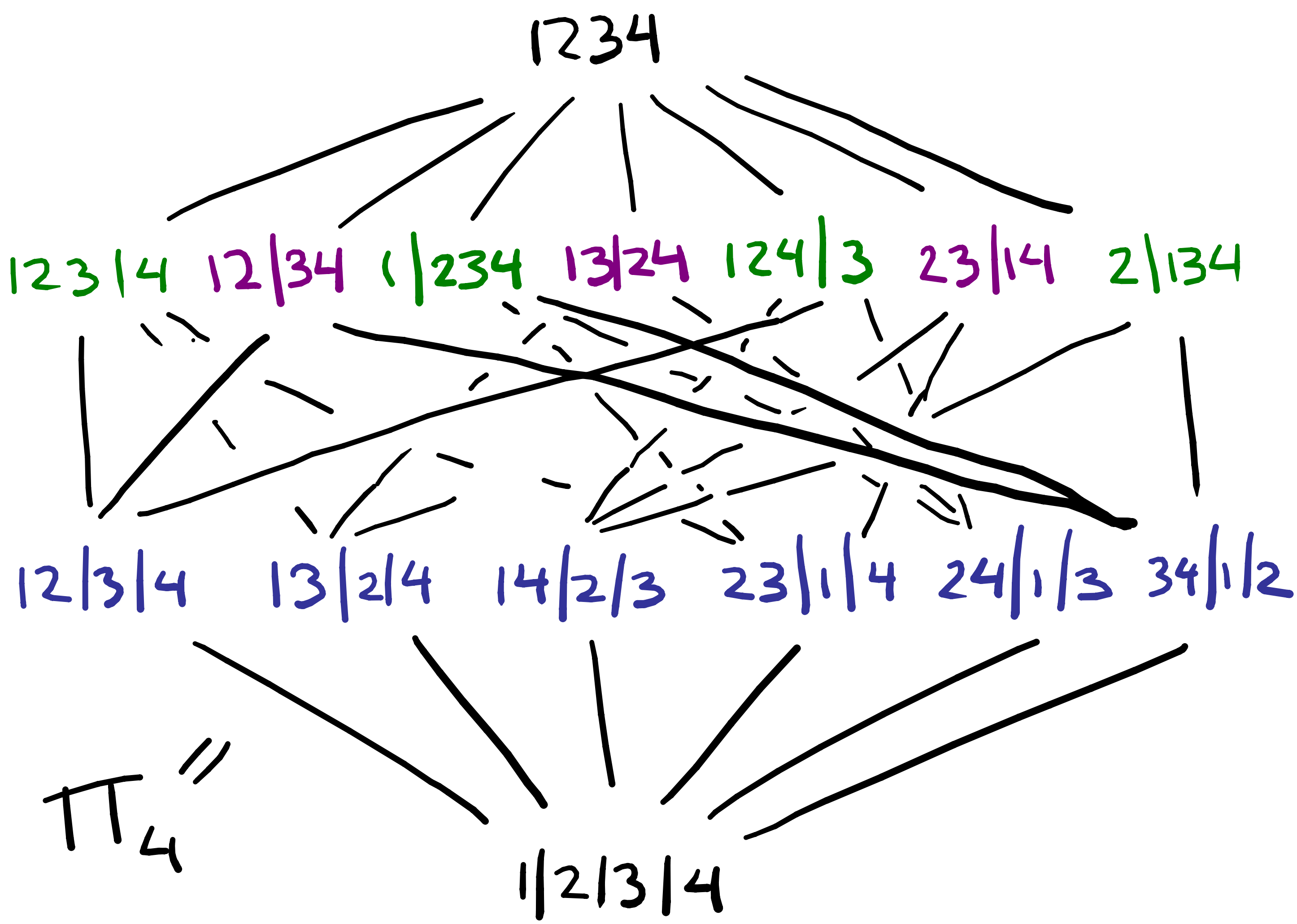
- $C_S(\mathbb{R}^d) := \{\underline{x} \in (\mathbb{R}^d)^5 \mid x_1 \neq x_3;\ x_2 \neq x_4 \neq x_5\}$

$$= C_{\{1,3\}}((\mathbb{R}^d)^2) \times C_{\{2,4,5\}}((\mathbb{R}^d)^3)$$

- $M^S := \{\underline{x} \in M^5 \mid x_1 = x_3;\ x_2 = x_4 = x_5\}$

& $E_2^{p,q} = 0$ for $d-1 \nmid q$

Partition Lattice Π_n & its S_n-representions



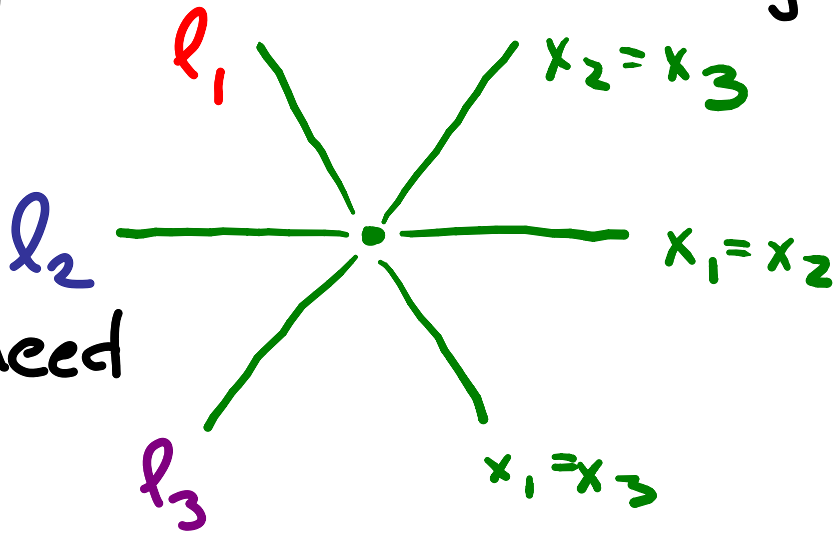
$\Pi_4 =$

- S_n acts by permuting values

e.g. $(13)[12|3|45] = 32|1|45$

Reinterpreting via Subspace Arrangement Complements

- M_n = complement of type A (complex) braid arr't $\{x_i = x_j \mid 1 \le i < j \le n\}$



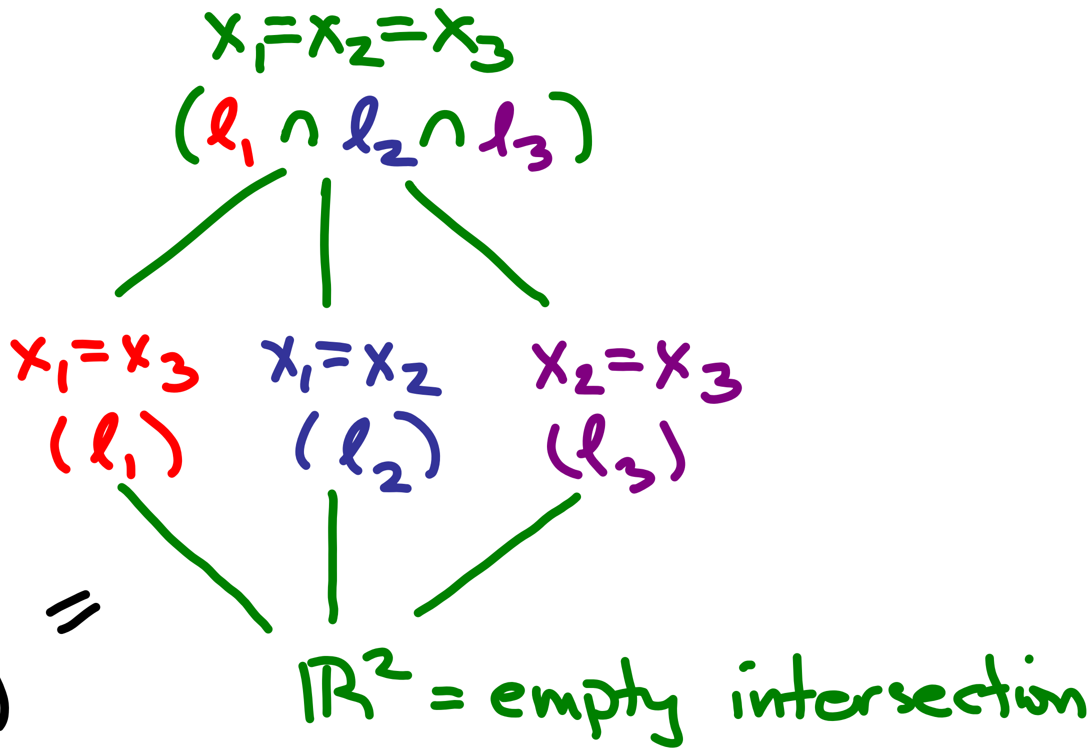
Warning: figure is $\mathbb{R}$-picture, need $\mathbb{C}$-picture

(Config space pt $p_i \leftrightarrow x_i \in \mathbb{C}$)

- Π_n = intersection poset $\mathcal{L}(A_{n-1})$

- S_n-module structure for $H^i(M_n)$ will translate to "Whitney homology" in Π_n, "$WH_i(\Pi_n)$"

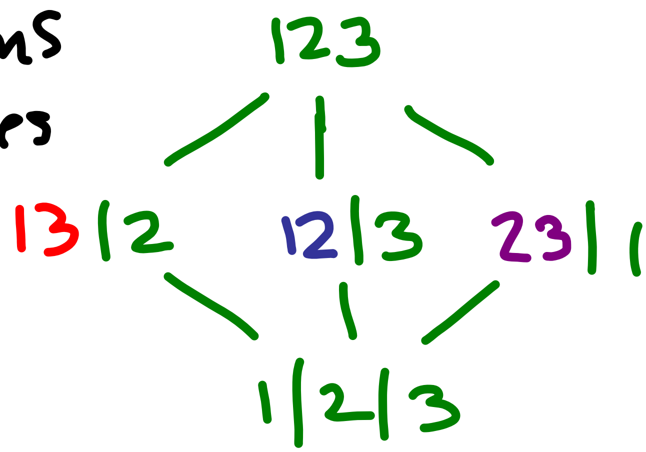
$\mathcal{L}(A_2) =$



$\cong$

$=$ poset of intersections of subspaces

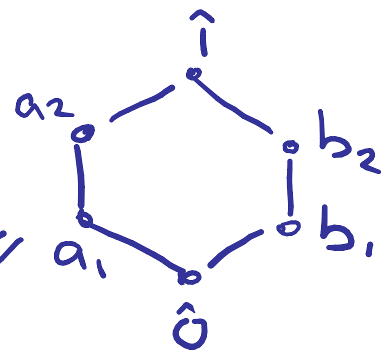
$\Pi_3 =$

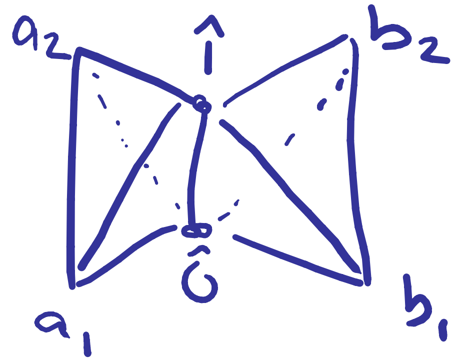


$=$ lattice of set partitions

$x_i = x_j \iff i,j$ in same block

Def'n: The order complex of a finite poset P is the simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-chains in P.

e.g. $P =$ 

$\Delta(P) =$ 

- Let $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$ e.g. for Π_n

Convention: When we speak of topological properties (homology, etc.) of poset P, we mean $\Delta(P)$ or $\Delta(\bar{P})$.

Poset rank := # steps from bottom

Goresky-MacPherson formula

(for cohomology of subspace arr't)

$$\tilde{H}^i(M_{\mathcal{A}}) \cong \bigoplus_{x \in L_{\mathcal{A}}^{>0}} \tilde{H}_{\operatorname{codim}(x)-2-i}(\hat{0}, x)$$

subspace arr't complement

as groups

intersection lattice

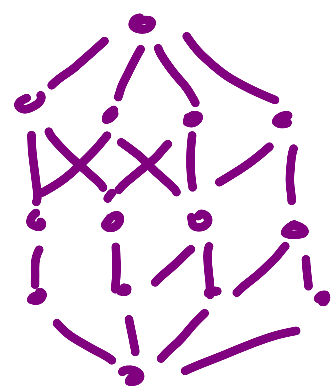
Plan: Apply to braid arrangement using upcoming S_n-equivariant version due to Sundaram-Welker, yielding Whitney homology. (See also Blagojević, Lück, Ziegler for more general versions)

S_n-Representations on Chains (i.e on Faces) & on Homology

- S_n action on set partitions is order-preserving & rank-preserving

(Recall P is graded if for each $u<v$ all saturated chains u to v have same length)

e.g.



- Hence, induces S_n-action on

$$\{\text{chains } u_1 < u_2 < \ldots < u_j\}$$

$$\Updownarrow$$

$$\{\text{faces of } \Delta(\overline{\Pi}_n)\}$$

- S_n-action on chains commutes with simplicial boundary map

$$d(u_0 < \cdots < u_r) = \sum_{0 \le i \le r} (-1)^i (u_0 < \cdots < \hat{u}_i < \cdots < u_r)$$

- Thus, S_n-action on i-faces (i^{th} chain gp) induces rep'n on i^{th} homology

- But homology of Π_n is concentrated in top degree due to EL-shellability of Π_n (since shellable $\Rightarrow$ homotopy equivalent to wedge of spheres)

G-Equivariant Enrichment of Goresky-MacPherson Formula

Thm (Sundaram-Welker): Let A be a G-arrangement of $\mathbb{C}$-linear subspaces in $\mathbb{C}^n$ for G a finite subgroup of $GL_n(\mathbb{C})$. Then

$$\tilde{H}^i(M_A) \cong_G \bigoplus_{x \in (L_A^{>0})/G} \mathrm{Ind}_{\mathrm{Stab}(x)}^{G} \tilde{H}_{\mathrm{codim}(x) - i - 2}(\hat{0}, x)$$

(in our case) $= $ "$WH_i(\underbrace{L_{A_n}}_{\Pi_n})$"

Note: there are numerous variations, e.g. allowing us also to handle config. spaces in $\mathbb{R}^{2d+1}$!

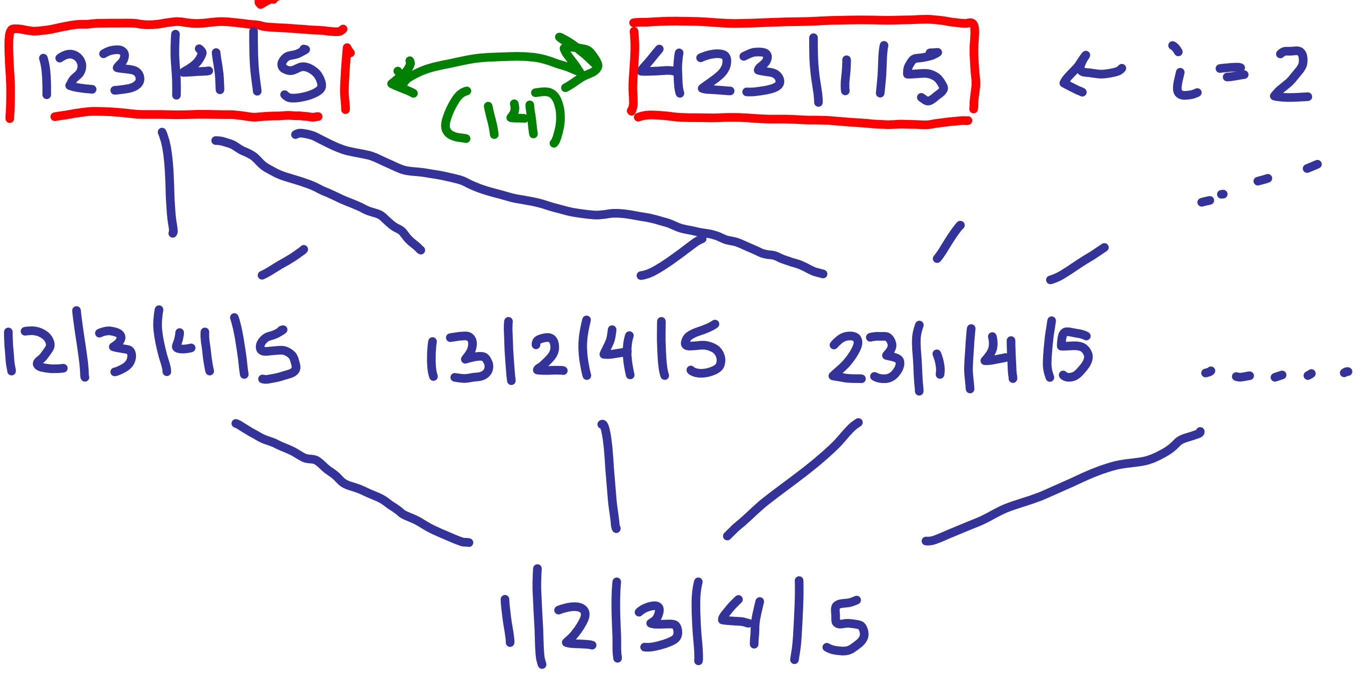
Whitney Homology (for Graded Posets)

$WH_i(P) :=$ "i-th Whitney homology of P"

$$= \bigoplus_{rk(u)=i} \tilde{H}_{i-2}(\hat{0},u) = \bigoplus_{\lambda \text{ has } n-i \text{ blocks}} WH_\lambda(P)$$

$$WH_\lambda(P) := \bigoplus_{\substack{u \in P \\ type(u)=\lambda}} \tilde{H}_{top}(\hat{0},u)$$

$\lambda = (3,1,1) =$ list of block sizes



Aside: $WH_i(P) \cong_{S_n} \beta_{\{1,\ldots,i\}}(P) \oplus \beta_{\{1,\ldots,i-1\}}(P)$

Thm (H-Reiner): Let M_n^d = config. space of n distinct pts in $\mathbb{R}^d$. Then $H^i(M_n^2)$ stabilizes sharply at $3i+1$.
More generally, $H^i(M_n^{2d})$ stabilizes sharply for $n \geq 3\frac{i}{2d-1}+1$ & $H^i(M_n^{2d+1})$ stabilizes sharply for $n \geq 3\frac{i}{2d}$.

Idea: Determine stability of $\widehat{WH}_i$ & "$\widehat{Lie}_i$"

Thm (H-Reiner): $\langle H^i(M_n^d), S^{(n-|\nu|,\nu)}\rangle$ vanishes for $|\nu| \leq 2i$ and becomes constant for $n \geq n_0 := \begin{cases} |\nu|+i & \text{for } d \text{ odd} \\ |\nu|+i+1 & \text{for } d \text{ even} \end{cases}$

Proof Techniques & Results We'll Use

Thm (Hanlon-Stanley): $\Pi_n \cong \mathrm{sgn} \otimes (\xi_n \uparrow_{C_n}^{S_n})$

Thm (Joyal): $\mathrm{lie}_n \cong \xi_n \uparrow_{C_n}^{S_n}$

Cor: $\Pi_n \cong \mathrm{lie}_n \otimes \mathrm{sgn}$

Thm (Kraskiewicz & Weyman):

$$\mathrm{lie}_n \cong \bigoplus_{\substack{T \text{ SYT} \\ \text{w/ } \mathrm{maj}(T) \equiv 1 \pmod n}} S^{\lambda(T)}$$

Thm (Sundaram):

$$ch(WH_\lambda) = \prod_{j \text{ odd}} h_{m_j}[\pi_j] \prod_{j \text{ even}} e_{m_j}[\pi_j]$$

$$= (h_{m_1}) \Big(\prod_{\substack{j \text{ odd} \\ j > 1}} h_{m_j}[\pi_j] \Big) \Big(\prod_{j \text{ even}} e_{m_j}[\pi_j] \Big)$$

Thm (Sundaram):

S_j-rep'n on top homology of Π_j

$$ch(WH_\lambda) = \prod_{j \text{ odd}} h_{m_j}[\Pi_j] \prod_{j \text{ even}} e_{m_j}[\Pi_j]$$

$$= \underbrace{(h_{m_1})}_{\substack{\| \\ ch(\text{triv}_{m_1})}} \underbrace{\left(\prod_{\substack{j \text{ odd} \\ j>1}} h_{m_j}[\Pi_j]\right)\left(\prod_{j \text{ even}} e_{m_j}[\Pi_j]\right)}_{\text{"}\widehat{Wh_\lambda}\text{" has degree} \leq 2i \text{ by } \star}$$

where ch = "Frobenius characteristic" isom.

$ch(f) = \sum_\mu f(\mu)\frac{p_\mu}{z_\mu}$ from S_n class functions to ring of symmetric fn's

$$h_n := \sum_{1 \leq i_1 \leq i_2 \leq \cdots} x_{i_1} x_{i_2} \cdots x_{i_n} = ch(\text{trivial rep'n})$$

$$e_n := \sum_{1 \leq i_1 < i_2 < \cdots} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_n} = ch(\text{sgn rep'n})$$

Obs: Π_n has 1st row upper bd $n-1$ for $n>2$ &
$e_m[\Pi_2] = e_m[h_2]$ has 1st row upper bd $m+1$

★ **Key Fact for Stability:** $u \in \Pi_n$ of rank i has at most $2i$ letters in nontrivial blocks

Significance: Gives upper bound of $2i$ on $|\lambda|$, where sharp stability bound is $\max\{|\lambda|+\lambda_1\}$

12|34|56|78 ← max # letters in nontriv. blocks $=$ 2-rank $= 2i$

$\lambda = (2,2,2,2)$

12|34|56|7|8

$\lambda = (2,2,2,1,1)$

12|34|5|6|7|8

123|4|5|6|7|8

$\lambda = (3,1,1,1,1,1)$

12|3|4|5|6|7|8

1|2|3|4|5|6|7|8

Thm (Sundaram):

$$ch(WH_\lambda) = \prod_{j \text{ odd}} h_{m_j}[\overset{S_j\text{-rep'n on top homology of } \Pi_j}{\Pi_j}] \prod_{j \text{ even}} e_{m_j}[\Pi_j]$$

$$= (\underbrace{h_{m_1}}_{ch(\mathrm{triv}_{m_1})})\underbrace{\left(\prod_{\substack{j \text{ odd} \\ j>1}} h_{m_j}[\Pi_j]\right)\left(\prod_{j \text{ even}} e_{m_j}[\Pi_j]\right)}_{\text{" } \widehat{Wh_\lambda} \text{" has degree} \leq 2i \text{ by } *}$$

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Obs: Π_n has 1st row upper bd $n-1$ for $n>2$ & $e_m[\Pi_2] = e_m[h_2]$ has 1st row upper bd $m+1$

Key Properties of Symmetric Functions

- $S^\lambda \overset{ch}{\longleftrightarrow}$ schur fn $s_\lambda = \sum_{\substack{T \text{ SSYT} \\ \text{shape } \lambda}} x^T$

  "Frobenius charact." isom.

$$T = \begin{array}{|c|c|c|c|} \multicolumn{4}{c}{\overbrace{\hphantom{1\;1\;2\;2}}^{\lambda_1}} \\ \hline 1 & 1 & 2 & 2 \\ \hline 3 & 4 & \multicolumn{2}{c}{\le} \\ \cline{1-2} \multicolumn{1}{c}{\le} & \multicolumn{1}{c}{\le} \end{array} \quad \rightsquigarrow \quad \underbrace{x_1^2 x_2^2 x_3 x_4}_{x^T}$$

$\wedge$

$\Longrightarrow$ $\boxed{s_\lambda \text{ includes monomial divisible by } x_1^{\lambda_1} \text{ but not } x_1^{\lambda_1+1}.}$

- Wreath product of rep'ns $\longleftrightarrow$ plethysm of symmetric functions

$\Longrightarrow$ $\boxed{\begin{array}{l} f \text{ includes } x_1^a \text{ \& } g \text{ includes } x_1^b \\ \text{then } f \cdot g \text{ includes } x_1^{a+b} \text{ while} \\ f[g] \text{ cannot include } x_1^{(\deg f)b+1} \end{array}}$

Wiltshire-Gordon Conjectures & Related Results

Defn (Wiltshire-Gordon):

$$V_n^k = \bigoplus_{\substack{|\lambda|=n \\ \ell(\lambda)=n-k \\ \lambda \text{ has no parts of size } 1}} WH_\lambda(\Pi_n)$$

strips away tensoring w/ trivial rep'n

Thm (H-Reiner):

$$\mathrm{Ind}\left(\mathrm{Res}\left(V_n^k\right) \oplus V_{n-1}^k\right) \cong \mathrm{Res}\left(V_{n+1}^{k+1}\right)$$

(conjectured by Wiltshire-Gordon)

e.g. $n+1=5$ & $k+1=3$ dimension formula:

$$4\cdot\left(\binom{4}{2} + (3-1)!\right) = \binom{5}{3}\cdot(3-1)!\,(2-1)! = 20$$

Key Question: Decompose V_n^k into irreducible rep'ns, since this would exactly give the S^λ irrep's yielding $S^\lambda \otimes triv_{n-|\lambda|}$ rep'ns comprising k-th cohomology for config. space of n distinct, labeled pts in $\mathbb{R}^2$.

Progress (Next Theorem): Answer instead for $\bigoplus_k V_n^k$.

Open Qn: Analogous results for $\mathbb{R}^d$ for $d>2$?

Thm (H-Reiner):

$$V_n = ch\left(\bigoplus_k V_n^k\right) \cong \bigoplus S^{\lambda(T)}$$

T is "Whitney generating" SYT

where T is **Whitney generating** if either

(1) $T = \emptyset$ or $\boxed{1}\boxed{2}$ or $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$

or

(2) $T\big|_{\{1,2,3,4\}}$ is one of the four shapes:

$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} 4 & \\ \cline{1-1}\end{array}$ $\quad T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & & \\ \cline{1-1}\end{array}$

$$T_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad T_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

with the following further restrictions:

(a) If T_3, then the first ascent k with $k \geq 4$ is odd

(b) If T_4, then the first ascent k with $k \geq 4$ is even

ascent := i such that $i+1$ in weakly higher row

Idea: Both sides satisfy same recurrence: categorified $d_n = n d_{n-1} + (-1)^n$

$$\widehat{WH}_n = \widehat{WH}_{n-1} \uparrow_{S_{n-1}}^{S_n} + (-1)^n \tau_n$$

for $\tau_n = \chi^{(3,1^{n-3})} - \chi^{(2,2,1^{n-4})}$ for $n \geq 4$

Motivations from Number Theory for Rep'n Stability for $PConf(\mathbb{R}^d)$

- Church-Ellenberg-Farb & Matchett-Wood-Vakil, & others:

$$\langle H^i(PConf_n(\mathbb{C})), V\rangle_{S_n} = \dim_{\mathbb{Q}_\ell} H^i_{et}(Conf_n; V)$$

coefs twisted by V

yielding various counting formulas over finite field via "Grothendieck-Lefschetz formula" & counting fixed pts of Frobenius map

e.g. $\lim_{n\to\infty}(\#\ \square\text{-free degree } n \text{ polys}) = q^n - q^{n-1}$

Remarks: Applications to number theory focus on $M=\mathbb{R}^2$ case

- We improve error bound in these limits

Translating "Polynomial Characters" into Symmetric fns (to get Improved "Power Saving Bounds")

- Any polynomial $P(x_1, x_2, x_3, \dots)$ gives a class fn for S_n by letting x_i = # i-cycles in conjugacy class
- The elements $\binom{X}{\lambda} = \binom{x_1}{m_1}\binom{x_2}{m_2}\cdots$ where λ has m_i parts of size i form a basis for $\mathbb{Q}[x_1, x_2, x_3, \dots]$

Prop'n (H-Reiner): $ch(\chi_P) = \begin{cases} \frac{p_\lambda}{z_\lambda} h_{n-|\lambda|} & \text{for } n \geq |\lambda| \\ 0 & \text{otherwise} \end{cases}$

for $P = \binom{X}{\lambda} = \binom{x_1}{m_1}\binom{x_2}{m_2}\cdots$

Combining with Earlier Results...

- guarantees for all $P \in \mathbb{Q}[x_1, x_2, \dots]$, $\chi_P = M\left(\sum_\mu c_\mu x^\mu\right)$ s.t. $|\mu| \leq \deg(P)$ $\forall \mu$.

- analyze $\langle \chi_P, H^i(M_n^{2d}) \rangle$ via:

Thm (H-Reiner): $\langle H^i(M_n^{2d}), S^{(n-|\nu|,\nu)} \rangle$ vanishes for $|\nu| \leq 2i$ and becomes constant for $n \geq n_0 := \begin{cases} |\nu| + i & \text{for } d \text{ odd} \\ |\nu| + i + 1 & \text{for } d \text{ even} \end{cases}$

Upshot: $\langle \chi_P, H^i(\mathrm{PConf}(\mathbb{C})) \rangle_{S_n}$ is constant for $n \geq \max\{2\deg(P), \deg(P) + i + 1\}$.

Thm: $\langle \beta_S(\Pi_n), \mathrm{triv}\rangle$ is constant for $n \geq 2\max(S) - \left(\frac{|S|-1}{2}\right)$.

Note: This follows from partitioning of $\Delta(\Pi_n)/S_n$ giving combinatorial interpretation for $\langle \beta_S(\Pi_n), \mathrm{triv}\rangle$ (i.e. from 2003 result of H.), our point of entry to this topic.

Conjecture (H-Reiner): for fixed $S \subseteq \{1, 2, \ldots, n-2\}$ with $i = \max(S)$, the rank-selected homology $\beta_S(\Pi_n)$ stabilizes sharply at $n = 4i - |S| + 1$.