Posets from Polytopes

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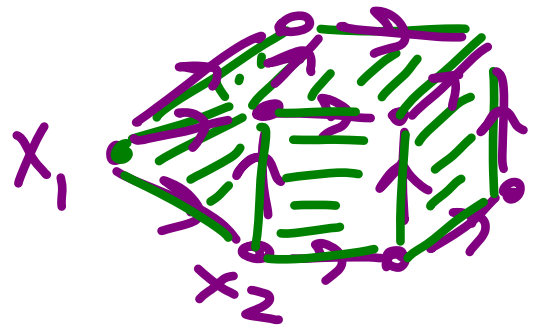
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Linear Programming

- Given a matrix A & vectors $\vec{b}, \vec{c}$ seek $\max\{\vec{c} \cdot \vec{x} \mid A\vec{x} \leq \vec{b}\}$

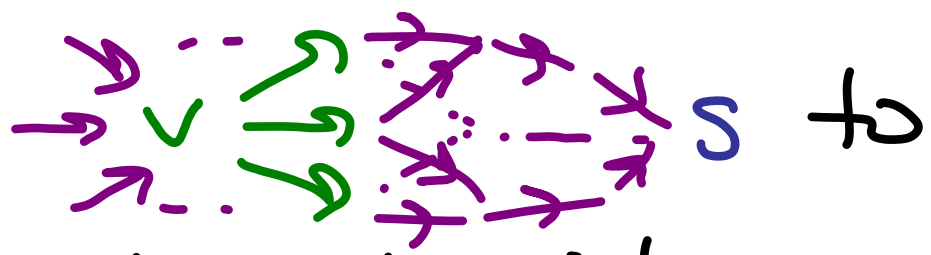- $\{\vec{x} \mid A\vec{x} \leq \vec{b}\}$ is polytope P if set is bounded

- Define $G(P, \vec{c}) :=$ directed graph on 1-skeleton of P with $x_1 < x_2 \iff \vec{c} \cdot \vec{x_1} < \vec{c} \cdot \vec{x_2}$

Solving Linear Programs via Simplex Method

- $\max \{ \vec{c} \cdot \vec{x} \mid A\vec{x} \leq \vec{b} \} = $ sink of $G(P, \vec{c})$

Simplex Method: walk from some vertex $v \in G(P, \vec{c})$ following arrows $v \to v_1 \to v_2 \to \ldots \to s$ to sink s

Pivot Rule: method to choose which out arrow s to follow from v towards sink s.

Key Questions: what is typical complexity of simplex method (path length)? What is worst case?

Hirsch Conjecture: For
d-dimensional polytope with
n facets (maximal faces),
diameter of 1-skeleton,
denoted $\Delta(d,n)$, satisfies
$\Delta(d,n) \leq n-d$.

Francisco Santos: Counterexamples!

Non-revisiting Path Conjecture:

for each $u, v \in P$, there exists
a path u to v not revisiting
any facet it has left.

Non-Revis. Conj. $\Rightarrow$ Hirsch Conjecture
(by giving short path) so Both False!

Our Plan

Impose further conditions on P and $\vec{c}$ that will imply a corollary of the following which we hope might also hold:

> For each $u, v \in P$, each directed path from u to v never revisits any facet it has left.

This property would make <u>all</u> pivot rules efficient for P and $\vec{c}$.

Quick Background on Polytopes

- A **polytope** in $\mathbb{R}^d$ is convex hull of finite # vertices, or equiv'ly a bounded set that is an intersection of half spaces.

- A polytope is **simple** if for each vertex v and each collection $e_1, e_2, .., e_r$ of edges emanating out from v there is an **r-dim'l face** containing all these edges
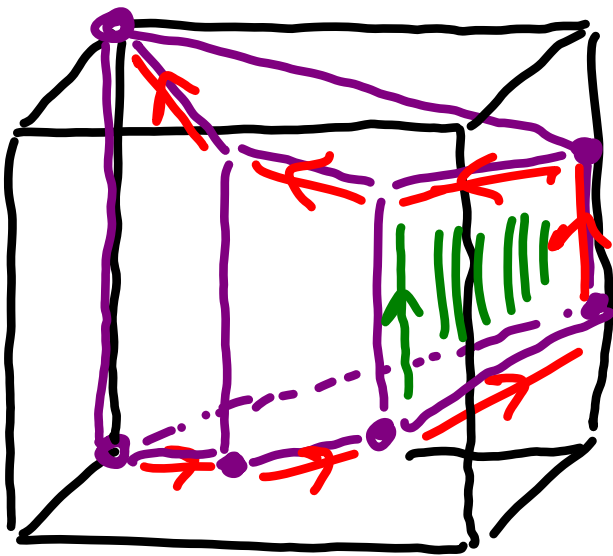
e.g. v with e_1, e_2, e_3 but <u>not</u> v
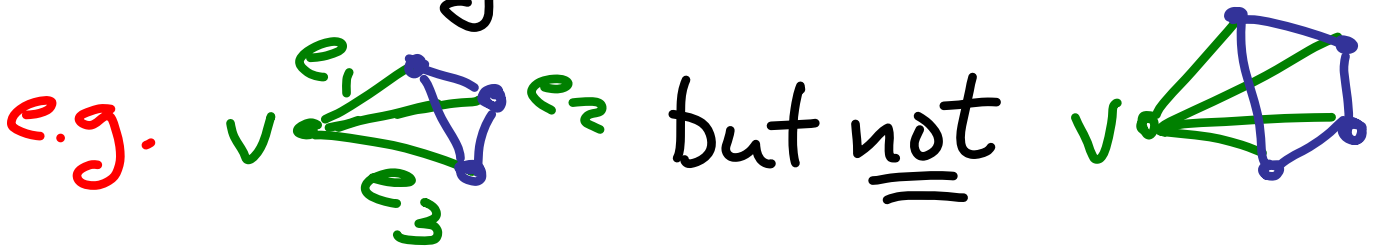
New Def'n: $G(P, \vec{c})$ has the

Hasse diagram property if it is

Hasse diagram of finite poset,

i.e. $v_1 \to v_2 \to \cdots \to v_r$ for $r \geq 3$

directed path precludes $v_1 \to v_r$

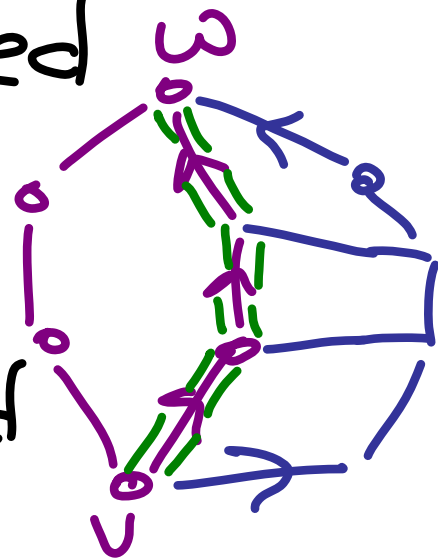Important Non-Examples:

"Klee - Minty Cubes"

e.g. $n = 3$

$\vec{c} = (0, 0, 1)$

- path visits all vertices!

n-dim'l KM-cube $= \{(x_1, \dots, x_n) \mid 0 \leq x_i \leq 1 ; \begin{array}{l} \varepsilon x_{i-1} < x_i < 1 - \varepsilon x_{i-1} \\ \text{for } i > 1 ; 0 < \varepsilon < \frac{1}{2} \end{array}\}$

Lemma: Given $F \subseteq G$ with $\dim(G) = \dim(F) + 1$ in simple polytope P w/ generic $\vec{c}$ s.t. $G(P, \vec{c})$ is a Hasse diagram, then there does not exist $v, w \in F$ with directed path P_F from v to w in F, outward oriented edge v to $G \setminus F$ and inward oriented edge $G \setminus F$ to w.

Corollary: Monotonicity of out-degrees & partic. outward directions.

} violate Hasse diagram

(at switch from out to in)
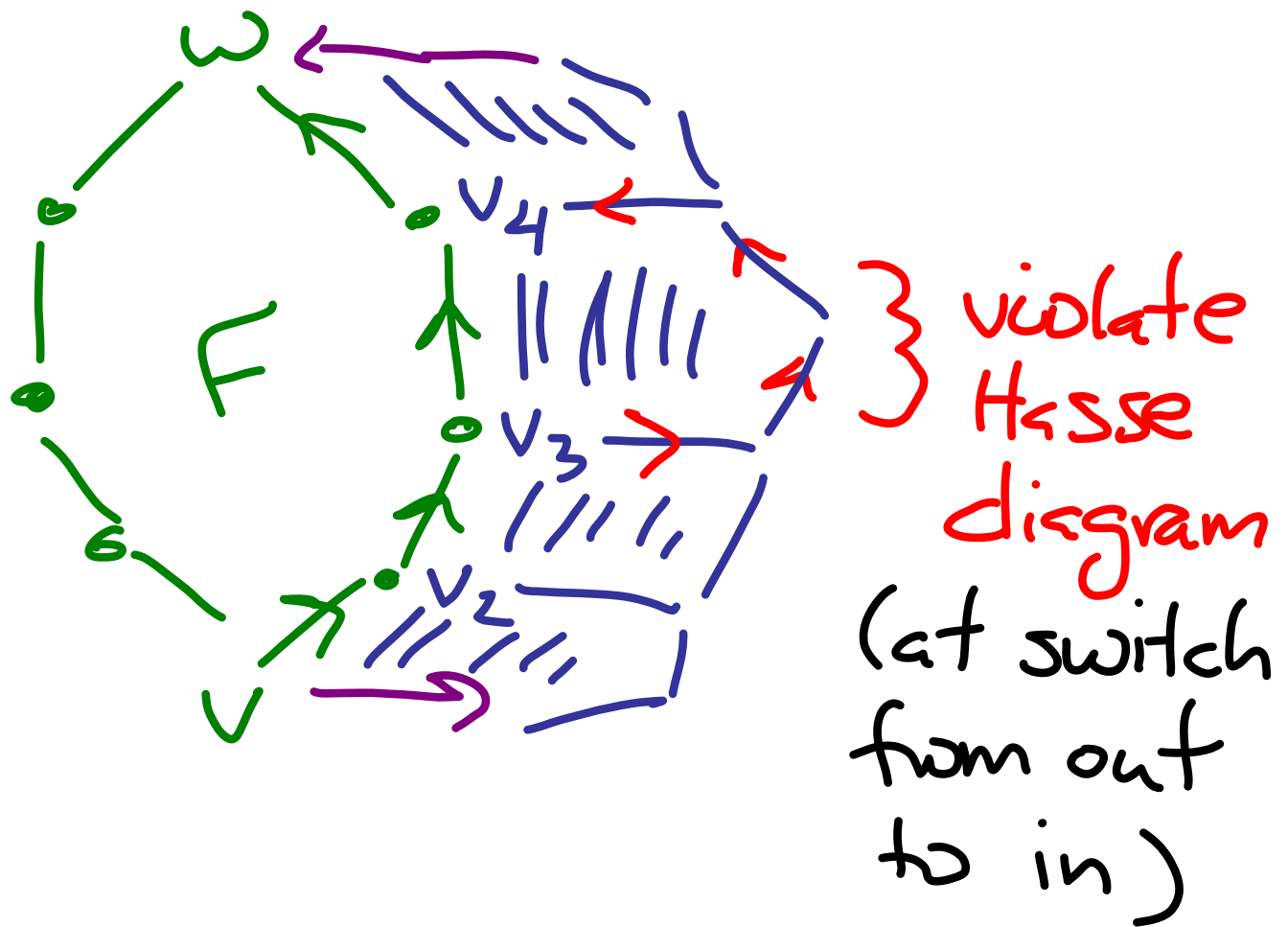
Corollary: for each face $F \leq P$ with $\hat{0} \in F$ or $\hat{1} \in F$, directed paths cannot revisit F after departing from it.

<u>Recall</u>: A poset L is a lattice if for each $u, v \in L$ there exists unique least upper bound ("join") for u and v, denoted $u \vee v$ and unique greatest lower bound for u and v ("meet"), $u \wedge v$.

<u>Note</u>: for P simple & $G(P, \vec{c})$ Hasse diagram, an upper bound for u, v both covering x is snk s of unique 2-face containing x, u, v.
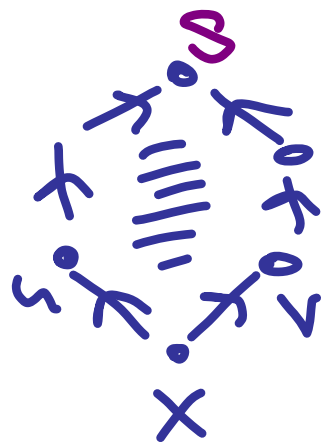
"Pseudo-joins" in a Polytope

Let P be simple polytope w/ generic cost vector $\vec{c}$ such that $G(P, \vec{c})$ is Hasse diagram of poset L with $x_1, x_2, .., x_r \in L$ s.t. there exists $u \in L$ with $u \lessdot x_i$ for $i = 1, 2, .., r$. Define pseudo-join of $x_1, x_2, .., x_r$ as sink of unique r-face of P containing

$$x_1 \quad x_2 \quad x_3 \quad\quad x_r$$
$$\searrow \quad | \quad \swarrow \cdots \underline{\quad\quad}$$
$$u$$

Lemma: $S \neq T \Rightarrow \text{pseudo}(S) \neq \text{pseudo}(T)$.

Note: Since pseudo-join of $x_1, \ldots, x_r$ is an upper bound, there exists directed path from $x_1 \vee \cdots \vee x_r$ to pseudo-join$(x_1, \ldots, x_r)$

Thm: Let P be a simple polytope and $\vec{c}$ be generic cost vector with $G(P, \vec{c})$ Hasse diagram of finite lattice. Then

$$\text{pseudo-join}(x_1, x_2, \ldots, x_r) = x_1 \vee \cdots \vee x_r$$

Pf: induction on r with $r = 2$ base case especially tricky part.
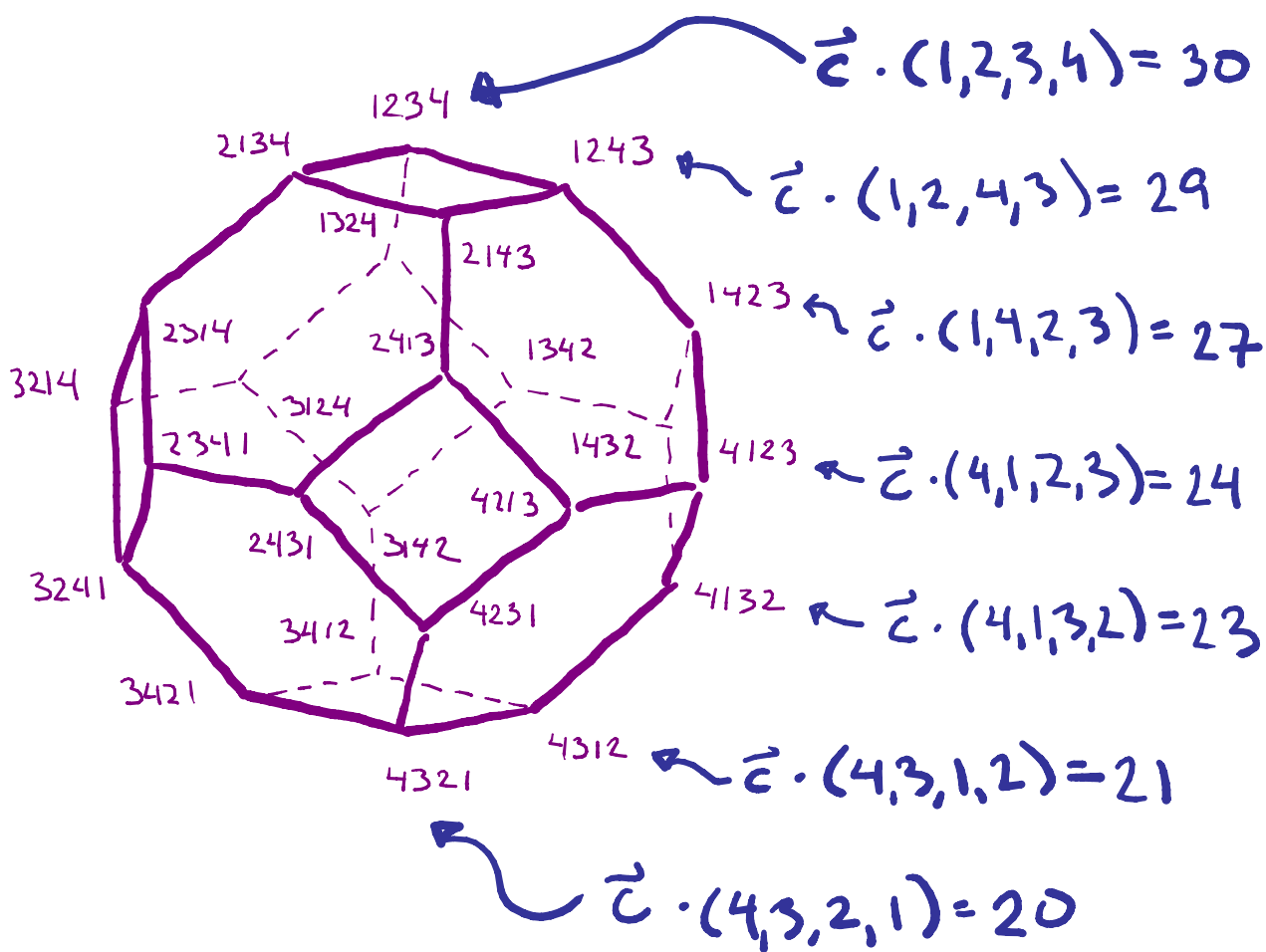
Thm: Let P be a simple polytope with generic cost vector $\vec{c}$ such that $G(P,\vec{c})$ is the Hasse diagram of a finite lattice L. Then each open interval $(u,v)=\{z\in L\,|\,u<z<v\}$ has order complex homotopy equivalent to a ball or a sphere

Applications:

- Permutahedra $\rightsquigarrow$ weak order

- associahedra $\rightsquigarrow$ Tamari lattice

- generalized associahedra $\rightsquigarrow$ Cambrian lattices
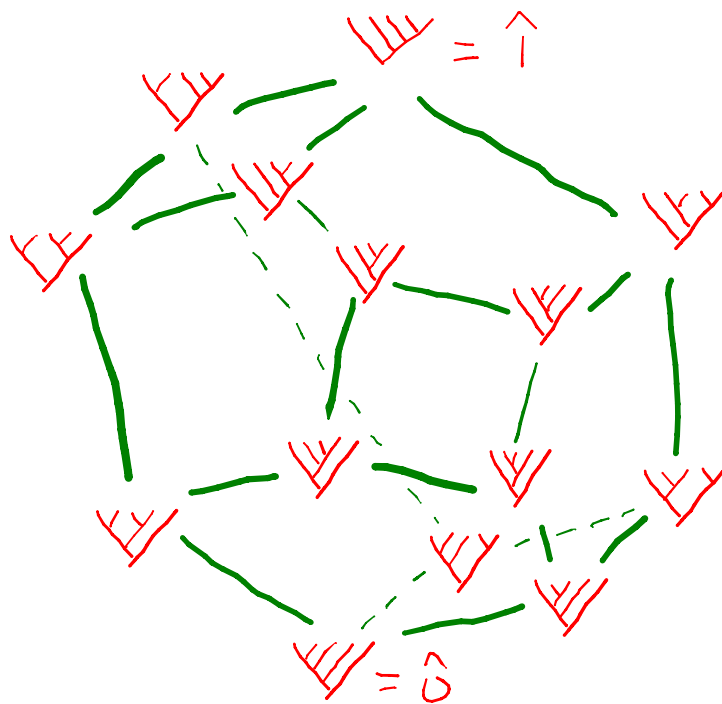
Permutahedron as Weak Order

- cost vector $\vec{c}$ any strictly ascending vector such as

$$\vec{c} = (1, 2, 3, 4).$$

$\vec{c} \cdot (1,2,3,4) = 30$

$\vec{c} \cdot (1,2,4,3) = 29$

$\vec{c} \cdot (1,4,2,3) = 27$

$\vec{c} \cdot (4,1,2,3) = 24$

$\vec{c} \cdot (4,1,3,2) = 23$

$\vec{c} \cdot (4,3,1,2) = 21$

$\vec{c} \cdot (4,3,2,1) = 20$

- Homotopy type 1st due to Edelman (type A) & Björner

Associahedron ≅ Tamari Lattice

- Use Loday's realization
- Poset of binary trees with cover relations : $\curlyvee$ ⋖ $\curlyvee$

$$((a,b),c) \qquad (a,(b,c))$$

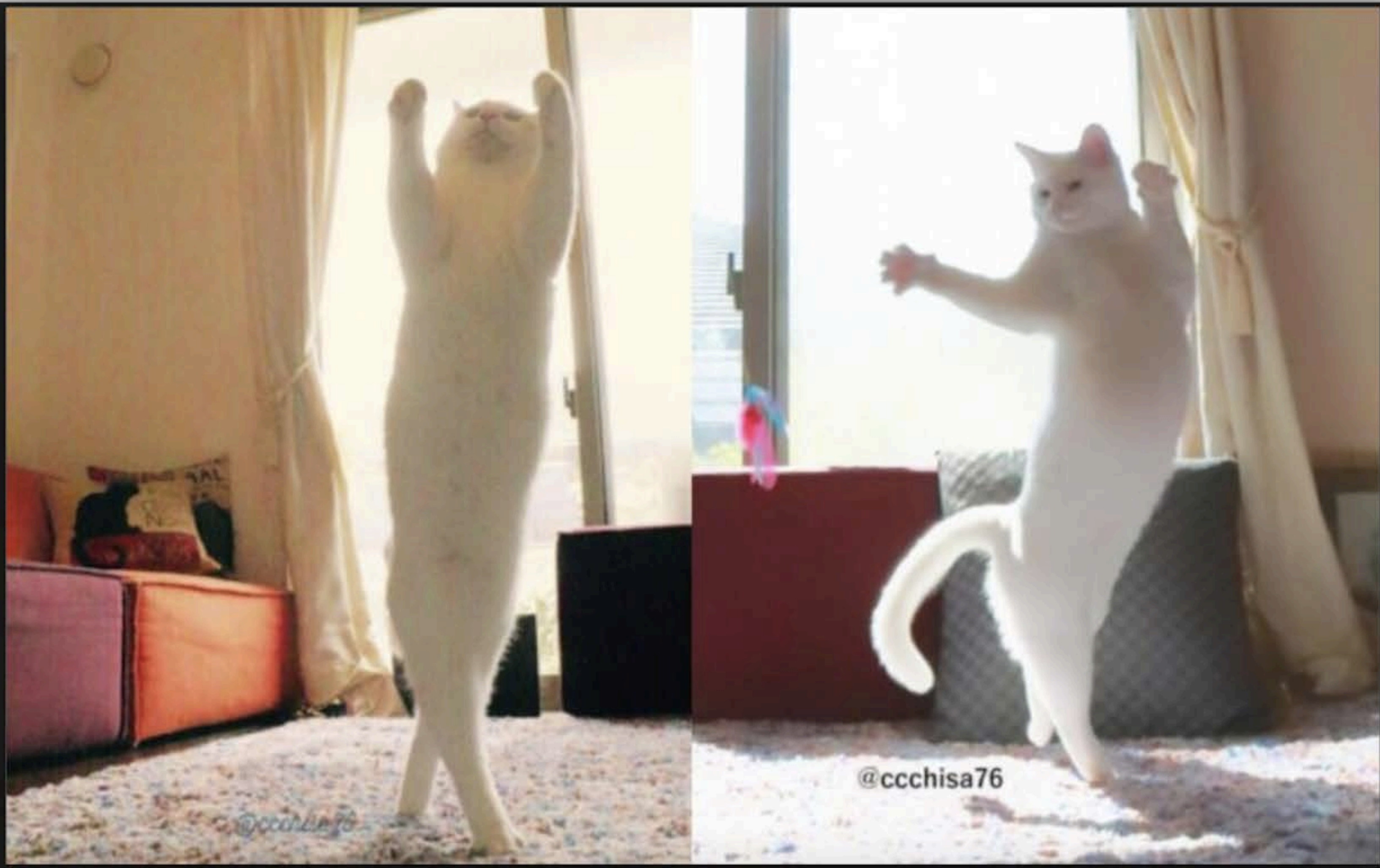- Homotopy type 1st due to Björner & Wachs via nonpure lexicographic shellability

Further Questions

Qn 1: More examples?

Qn 2: Does P simple polytope + $G(P, \geq)$ Hasse diagram of lattice imply no directed path u to v can revisit facet it has departed?

Qn 3: What do you get when you google cat ballet?

@ccchisa76