Representation Stability & S_n-module Structure in the Partition Lattice

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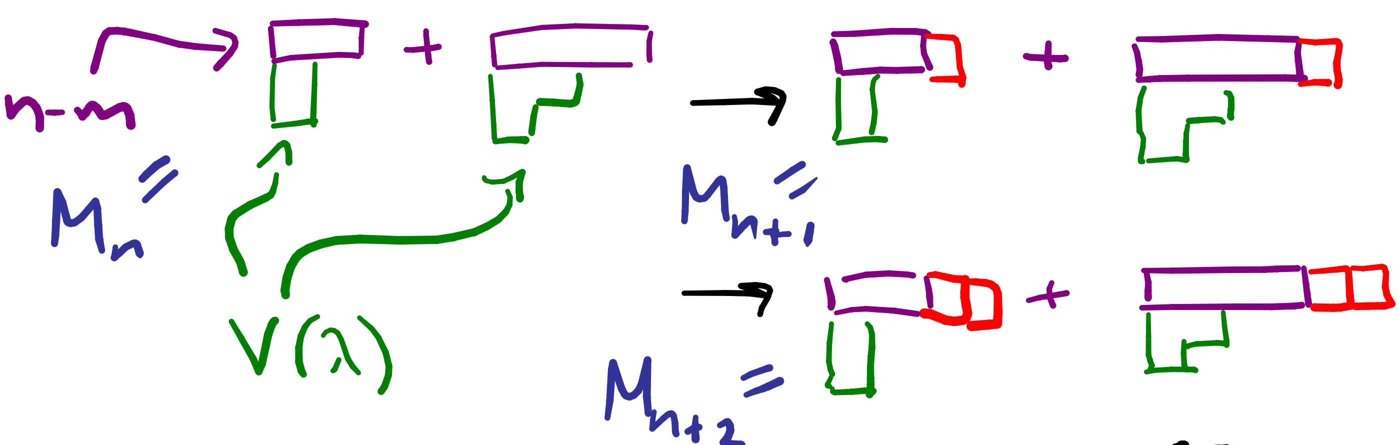
Representation Theoretic Stability

Defn (Church, Farb): A series of S_n-modules $M_1, M_2, \ldots$ stabilizes at $B>0$ if for each $n>B$, we have

$$M_n = \sum_{\lambda \vdash m \le B} c_\lambda V(\lambda) \quad \text{where} \quad V(\lambda) \cong S^{(n-m,\lambda)}$$

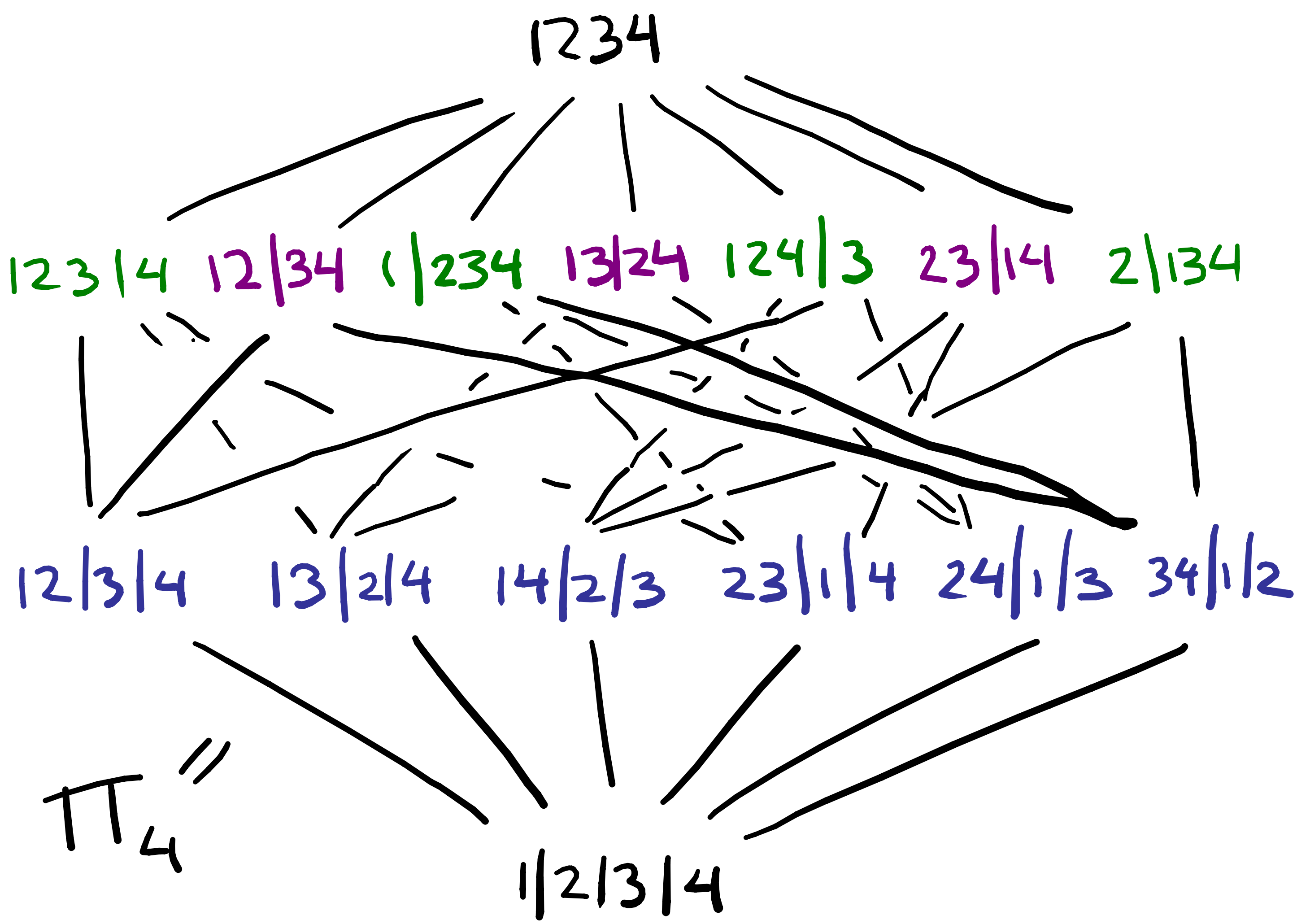
and where c_λ does not depend on n

e.g.



Our Focus: S_n-rep'ns from partition lattice

Partition Lattice Π_n & its S_n-representions



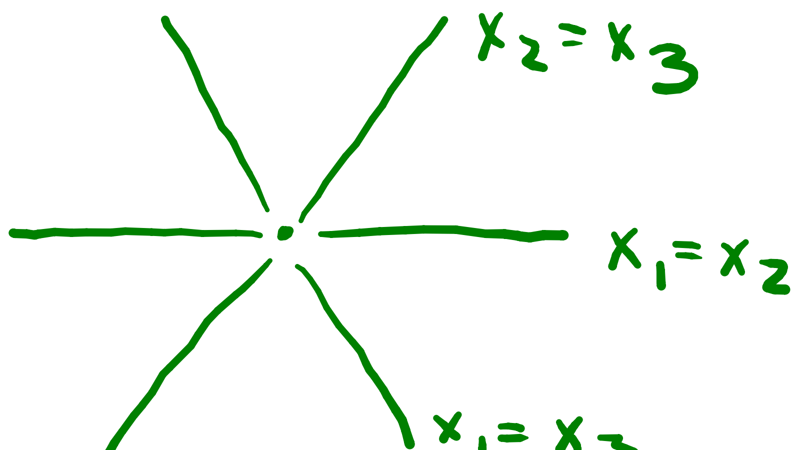
- S_n acts by permuting values

e.g. $(13)[12|3|45] = 32|1|45$

Our Starting Point:

Thm (Church-Farb): $H^i(M_n)$ stablizes for $n \geq 4i$ where M_n is configuration space of n distinct points in plane & i is held fixed.

- M_n = complement of type A (complex) braid arrangement $A_n = \{x_i = x_j \mid 1 \leq i < j \leq n\}$



- Π_n = intersection lattice of A_n

- $H^i(M_n)$ = i^{th} graded piece in Orlik-Solomon algebra of type A braid arr't

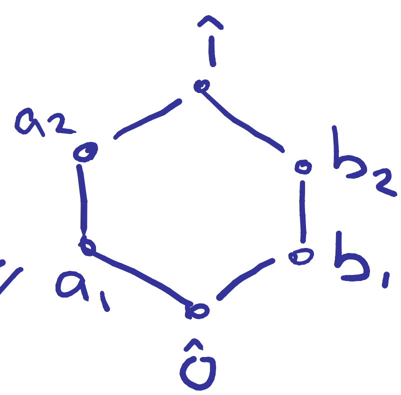
- S_n-module structure for $H^i(M_n)$ translates (via Goresky-MacPherson formula) to "Whitney homology" in Π_n.

Note: Church, Ellenberg & Farb theory of FI-modules for rep'n stability

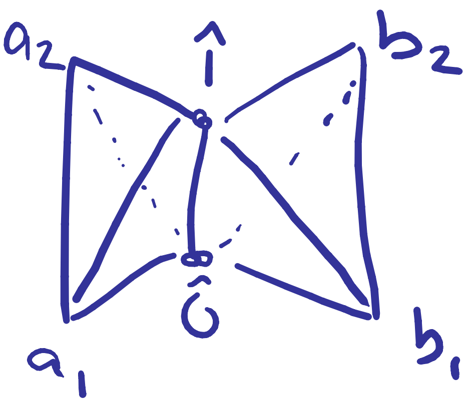
- applic's of rep'n stability in number theory (Matchett-Wood, Vakil, ...)

Def'n: The order complex (or nerve) of a finite poset P is the simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-chains $v_0 < \dots < v_i$ in P

e.g. $P =$



$\Delta(P) =$



Notations: $u \lessdot v$, "u is covered by v" in a poset P $\iff$ $u < v$ & $\{z \mid u < z < v\} = \emptyset$

- $u_0 < u_1 < \dots < u_i$ is saturated chain of P if it is a maximal chain $\iff$ gives rise to maximal face (facet) of $\Delta(P \setminus \{\hat{0}, \hat{1}\})$

S_n-Representations on Chains (i.e on Faces) & on Homology

- S_n action on set partitions is order-preserving & rank-preserving
- Thus, it induces S_n-action α_S on chains $u_1 < u_2 < \dots < u_j$ with u_r of rank i_r for $1 \le r \le j$ and $S = \{i_1, \dots, i_j\}$, in other words on faces of $\Delta(\Pi_n)$ with vertices colored S, where vertices in $\Delta(\Pi_n)$ colored by poset rank.

- S_n-action on chains commutes with simplicial boundary map

$$d(u_0 < \cdots < u_r) = \sum_{0 \le i \le r} (-1)^i (u_0 < \cdots < \hat{u}_i < \cdots < u_r)$$

- Thus, S_n-action on i-faces induces S_n-rep'n on i^{th} homology

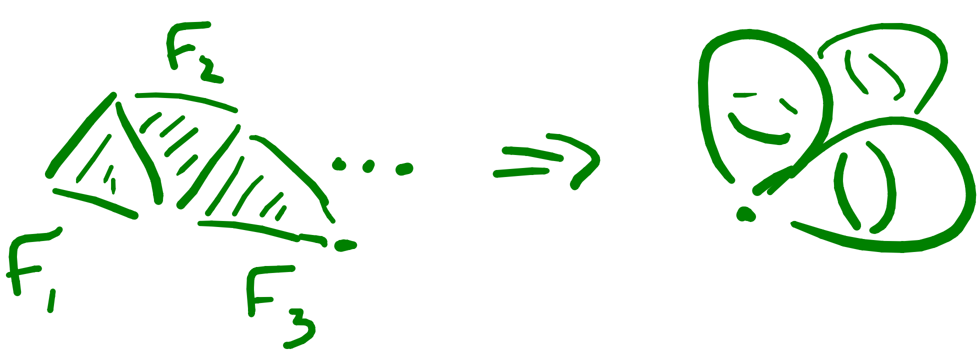
- But homology of Π_n is concentrated in top degree due to shellability

- Likewise, homology of $\Pi_n^S = \{u \in \Pi_n \mid rk(u) \in S\}$ also concentrated in top degree

- The virtual rep'n $\beta_S := \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T$ is actual S_n-rep'n on top homology of $\Pi_n^S := \{u \in \Pi_n \mid rk(u) \in S\}$ (since lower homology vanishes in Π_n^S)

<u>Note:</u> Π_n is "shellable", implying:

- $\Delta(\Pi_n \setminus \{\hat{0}, \hat{1}\})$ is homotopy equivalent to a wedge of $(n-1)!$ top-dim'l spheres
- Π_n^S is also shellable



Rank-Selected Homology & Whitney Homology

$\beta_S :=$ "rank-selected homology" for rank set S

$WH_i(P) :=$ "i-th Whitney homology of P"

$$= \bigoplus_{rk(u)=i} H_i(\hat{0}, u) = \bigoplus_{\lambda \text{ has } n-i \text{ blocks}} WH_\lambda(P)$$

$$WH_\lambda(P) := \bigoplus_{\substack{u \in P \\ type(u)=\lambda}} H_i(\hat{0}, u)$$

Thm (Sundaram): $WH_i(P) \cong \beta_{1\ldots i}(P) + \beta_{1\ldots i-1}(P)$

Observation: This implies WH_i stabilizes at same bound as $\beta_{1\ldots i}$

- β_S is subrepresentation of α_S & will stabilize at least as fast as α_S.

Goresky-MacPherson Formula

(for cohomology of subspace arr't)

$$\tilde{H}^i(M_{\mathcal{A}}) \cong \bigoplus_{x \in L_{\mathcal{A}}^{>0}} \tilde{H}_{\operatorname{codim}(x)-2-i}(\hat{0}, x)$$

$M_{\mathcal{A}}$: subspace arr't complement

$\cong$: as groups

$L_{\mathcal{A}}$: intersection lattice

(OS-Algebra = presentation of cohomology ring for complex hyperplane arr't complement)

S_n-Equivariant Enrichment

(Sundaram & Welker) as it specializes to complex hyperplane arrangements:

$$\tilde{H}^i(M_{\mathcal{A}}) \cong_G \bigoplus_{x \in (L_{\mathcal{A}}^{>0})/G} \operatorname{Ind}_{\operatorname{Stab}(x)}^{G} \tilde{H}_{\operatorname{codim}(x)-i-2}(\hat{0}, x)$$

$$= WH_i(L_{A_n}) = WH_i(\Pi_n)$$

Upshot for Stability:

- $\beta_{1,\ldots,i}(\Pi_n)$ stabilizes at $B>0$

  $\iff WH_i(\Pi_n)$ stabilizes at $B>0$

  $\iff H^i(M_n)$ stabilizes at $B>0$

Past Results on Π_n:

Thm (Hanlon–Stanley): $\Pi_n \cong \text{sgn} \otimes (\zeta_n \uparrow_{C_n}^{S_n})$

Method: Calculate $\mu_{\Pi_n^g}(\hat{0},\hat{1}) = \chi_{\Pi_n}(g)$

Thm (Joyal): $\text{lie}_n \cong \zeta_n \uparrow_{C_n}^{S_n}$

Thm (Barcelo): Explicit S_n-equivariant bijection yielding $\Pi_n \cong \text{sgn} \otimes \text{lie}_n$

Thm (Kraskiewicz & Weyman):

$$\text{lie}_n \cong \bigoplus_{\substack{T \text{ SYT} \\ \text{w/ } \text{maj}(T) \equiv 1 \pmod{n}}} S^{\lambda(T)}$$

✱ **Key Fact for Stability:** $u \in \Pi_n$ of rank i has at most $2i$ letters in nontrivial blocks

Thm (Sundaram):

S_j-rep'n on top homology of Π_j

$$\mathrm{ch}(WH_\lambda) = \prod_{j \text{ odd}} h_{m_j}[\pi_j] \prod_{j \text{ even}} e_{m_j}[\pi_j]$$

$$= (h_{m_1}) \Big(\prod_{\substack{j \text{ odd} \\ j>1}} h_{m_j}[\pi_j] \Big) \Big(\prod_{j \text{ even}} e_{m_j}[\pi_j] \Big)$$

j=1 part

" $\widehat{Wh_\lambda}$ " has degree $\leq 2i$ by ✱

where ch = "Frobenius characteristic" isom from ring of S_n-repn's to ring of symmetric functions

h_n := complete symmetric fn

$$= \sum_{1 \leq i_1 \leq i_2 \leq \cdots} x_{i_1} x_{i_2} \cdots x_{i_n} = \mathrm{ch}(\mathrm{triv})$$

e_n := elementary symmetric fn

$$= \sum_{1 \leq i_1 < i_2 < \cdots} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_n} = \mathrm{ch}(\mathrm{sgn})$$

Facts about Symmetric Functions

- ring of symmetric fns $\cong$ ring of S_n-rep'n's with $\rho_1 \otimes \rho_2 \uparrow_{S_m \times S_n}^{S_{m+n}}$ as multiplication

- $S^\lambda \longleftrightarrow$ schur fn $s_\lambda = \sum\limits_{\substack{T \text{ SSYT} \\ \text{shape } \lambda}} x^T$

for SSYT = semistandard Young tableaux

$T =$ (rows weakly increasing $\leq$, columns strictly increasing $<$)

1	1	2	2
3	4		

$\rightsquigarrow x_1^2 x_2^2 x_3 x_4 = x^T$

- wreath product of rep'ns $\longleftrightarrow$ plethysm of symmetric fns

Thm (H-Reiner): Holding i fixed & letting n grow, $\beta_{1-i}(\Pi_n)$, $WH_i(\Pi_n)$ & $H^i(M_n)$ stabilize as S_n-rep'ns at $n \geq 3i+1$.

Idea: $\widehat{WH}_i = \bigoplus_{\lambda \vdash n} \widehat{WH}_\lambda(\Pi_n)$ stabilizes at $n=2i$ since at most $2i$ letters in nontrivial blocks. Obtain upper bound of $i+1$ on length of 1st row in S^λ in $\widehat{WH}_i$. Pieri Rule lower bd on j s.t. multiplying $\widehat{WH}_i = \bigoplus c_\lambda S^\lambda$ by h_j stabilizes.

Pieri Rule:

$$h_n s_\lambda = \sum s_\mu$$

Thm (H-Reiner): $\beta_S(\Pi_n)$ for any fixed S stabilizes for $n \geq 4\max S$.

Idea: Show $ch(\alpha_S(\Pi_n))$ has upper bound of $2\max(S)$ on length of 1st row in "h-free" part of symmetric fn f (analogue of $\hat{WH}_i(n)$).

- This gives stability bound $n \geq 4\max S$ for $\alpha_S(n)$.

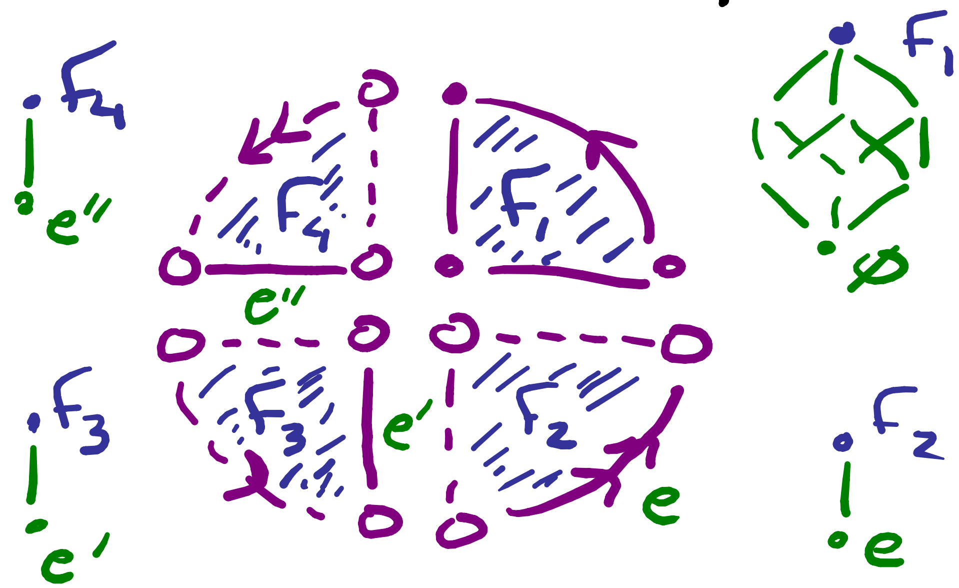
- Deduce same bound for $\beta_S(n)$ using that $\beta_S(n)$ is subrep'n of $\alpha_S(n)$

Stability for the Multiplicity "$b_S(n)$" of Trivial Rep'n in $\beta_S(n)$

Thm (H–Reiner): $b_S(n) = \langle 1, \beta_S(\Pi_n) \rangle$ stabilizes at $n \geq 2\max S - (|S|-1)/2$

Partitioning: Δ is **partitionable** if face poset $F(\Delta)$ decomposes into disjoint union of boolean algebras w/ facets as top elements

e.g.



- Δ is pure of dimension d if all maximal faces are d-dim'l
- Such Δ is balanced if vertices can be colored w/ d+1 colors s.t. no two vertices in a face have same color.
- If Δ is balanced & partitionable, then $h_S(\Delta)$ = # facets with restriction face (in min'l face in Boolean algebra) colored S.

Key Idea:

$$f_S(\Delta(\Pi_n)/S_n) = \#\ S_n\text{-orbits of faces with color set } S$$

$$= \langle 1, \alpha_S(\Pi_n) \rangle$$

$$\therefore\ h_S(\Delta(\Pi_n)/S_n) = \sum_{T \subseteq S} (-1)^{|S-T|} f_T(\Delta(\Pi_n)/S_n)$$

$$= \sum_{T \subseteq S} (-1)^{|S-T|} \langle 1, \alpha_T(\Pi_n) \rangle$$

$$= \langle 1, \beta_S \rangle$$

<u>Thm</u> (H., 2003): $\Delta(\Pi_n)/S_n$ is partitionable, giving combinatorial interpretation for

$h_S(\Delta(\Pi_n)/S_n) = \langle 1, \beta_S(\Pi_n) \rangle$

as # saturated chain orbits with "topological descent set" S.

Thm (H.-Reiner): $\langle 1, \beta_S(\Pi_n)\rangle$ stabilizes for $n \geq 2\max S - \binom{|S|-1}{2}$

Idea: Use partitioning for $\Delta(\Pi_n)/S_n$ and consequent combinatorial interpretation for $\langle 1, \beta_S(\Pi_n)\rangle$.

- Injection $\varphi_n : \left\{ \begin{array}{c} \text{facets} \\ \text{contrib.} \\ \text{to } b_S(n) \end{array} \right\} \to \left\{ \begin{array}{c} \text{facets} \\ \text{contrib} \\ \text{to} \\ b_S(n+1) \end{array} \right\}$ eventually also a surjection.

Rk: This is sharp for $S = \{i\}$ but not for every single choice of S.

e.g. (Hanlon): $\langle 1, \beta_{\{1,\ldots,i\}}(\Pi_n)\rangle = 0$ for $n > 2$.

Wiltshire-Gordon Conjectures & Related Results

Defn (Wiltshire-Gordon):

$$V_n^k = \bigoplus_{\substack{|\lambda|=n \\ \ell(\lambda)=n-k \\ \lambda \text{ has no parts of size } 1}} WH_\lambda(\Pi_n)$$

Thm (H-Reiner):

$$\operatorname{Ind}\left(\operatorname{Res}(V_n^k) \oplus V_{n-1}^k\right) \cong \operatorname{Res}(V_{n+1}^{k+1})$$

(conjectured by Wiltshire-Gordon)

Idea: Symmetric fns & generating fns

Thm (H-Reiner):

$$V_n = ch\left(\bigoplus_k V_n^k\right) \cong \bigoplus S^{\lambda(T)}$$

T is "Whitney generating" Standard Young tableau (SYT)

where T is **Whitney generating** if

- 1 and 2 both appear in 1st row
- if 3 in 1st row, then 1st "ascent" $k>2$ is even (or there is no ascent & n is even)
- if 3 & 4 in 2nd row, then 1st ascent odd (or none exists & n is odd)

ascent := i such that $i+1$ in higher row

Idea: Both sides satisfy same recurrence.

Conjecture (H-Reiner): The multiplicity of $S^{(n-k,\lambda)}$ within $H^i(M_n)$ stabilizes at $n \geq i+k+1$.

Intermediate Question:

(work in progress)

Find basis for $a_T b_T V_{\alpha_S}$ & $a_T b_T V_{\beta_S}$.

$\underbrace{a_T b_T}$ Young symmetrizer

for $a_T = \sum_{\sigma \in R(T)} \sigma$ & $b_T = \sum_{\tau \in C(T)} \mathrm{sgn}(\tau)\tau$

(since $\langle S^\lambda, V \rangle = \dim(a_T b_T V)$ for T of shape λ)

Some Further Questions

1. (Farb) How fast does the multiplicity of any particular $V(\lambda)$ stabilize within M_n?

2. (H-Reiner) How fast does the multiplicity of $V(\lambda)$ stabilize within $\beta_S(\Pi_n)$ as S is held fixed & n grows?

(Note: Qn 1 is special case of Qn 2 with $S = \{1, 2, \ldots, i\}$)

3. (Farb) What rep'ns do we get after stabilization occurs?