Fibers of Maps to Totally Nonnegative Spaces

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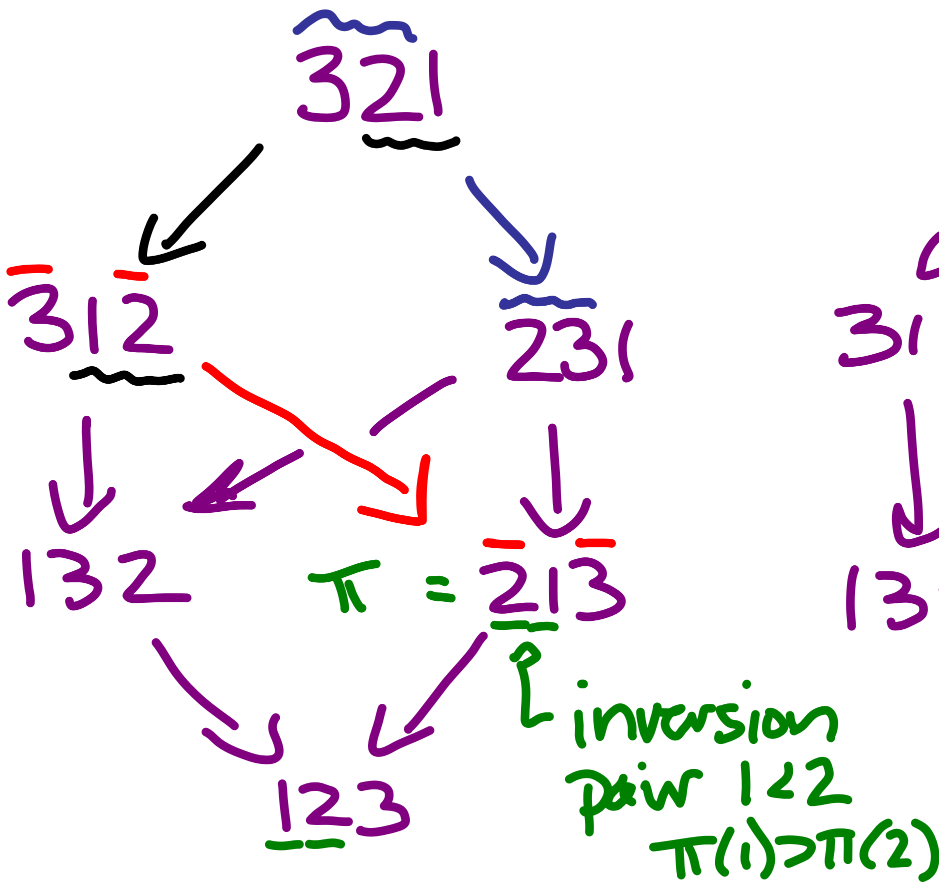
Outline for Talk

1. Background & Motivations
2. Fibers $f^{-1}_{(i_1,\ldots,i_d)}(M)$ of $f_{(i_1,\ldots,i_d)}$ for fixed matrix M (joint work with Jim Davis & Ezra Miller)
3. Maps with Potentially Analogous Structure & Further Questions

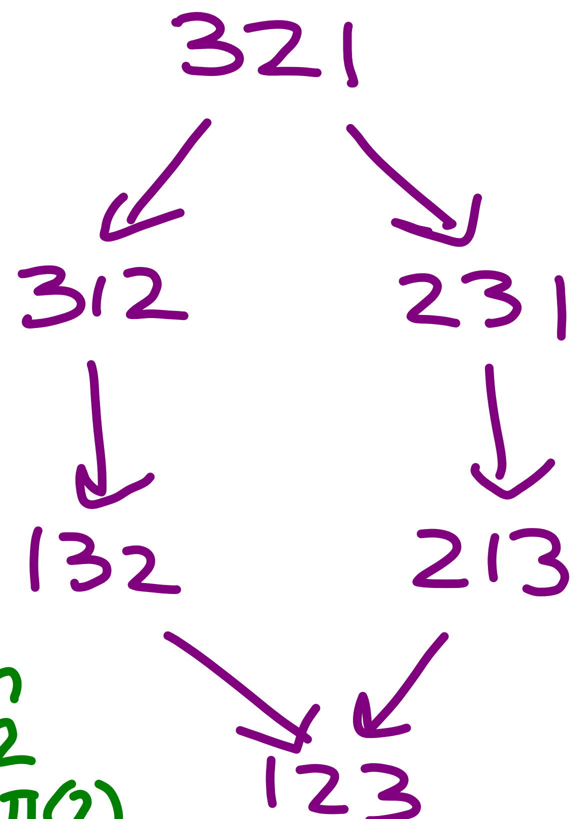
e.g. $$f_{(1,2,1)}(\underbrace{t_1,t_2,t_3}_{\in\, \mathbb{R}^3_{\geq 0}}) = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ & 1 & t_2 \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Partially Ordered Sets (Posets)
Describing Structure in Sorting



"Bruhat order"



"weak order"

(bubble sorting)

(sorting by swapping pairs to eliminate "inversion pairs, i.e. $i<j$ s.t. $\pi(i) > \pi(j)$)

Coxeter Groups (Generalizing S_n)

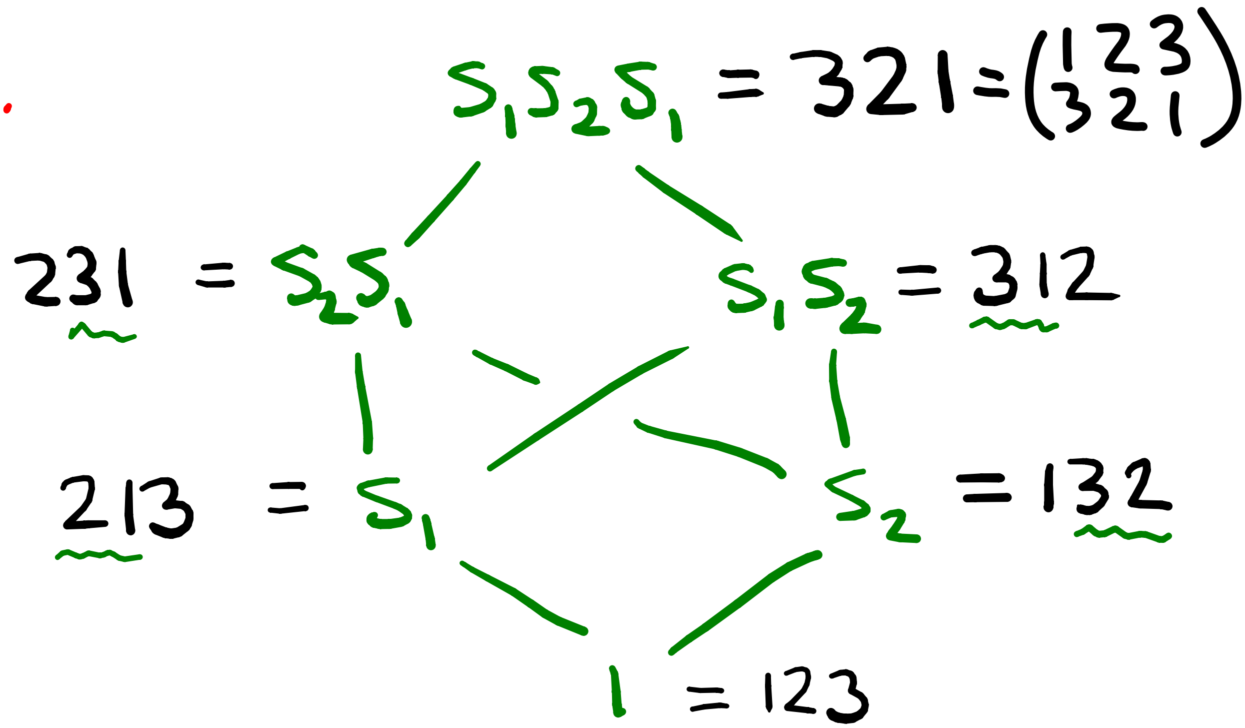
- $s_i := (i, i+1)$ = adjacent transposition
  ($\Uparrow$ S_n) a.k.a. **simple** "type A" **reflection** (in W)
- $s_{i_1} \cdots s_{i_d}$ is **reduced expression** for $w \in W$ if $w = s_{i_1} \cdots s_{i_d}$ for d as small as possible, and $(i_1, \ldots, i_d)$ is its **reduced word**.
- **length** of w, "$\ell(w)$" := # inversion pairs in w
  $\|$
  this smallest d

  e.g. $s_1 s_2 s_1 = s_2 s_1 s_2$ has length 3

  $$\begin{array}{c} \| \\ 321 \xrightarrow{s_1} 231 \xrightarrow{s_2} 213 \xrightarrow{s_1} 123 \end{array}$$

Bruhat Order for Coxeter Groups

e.g.

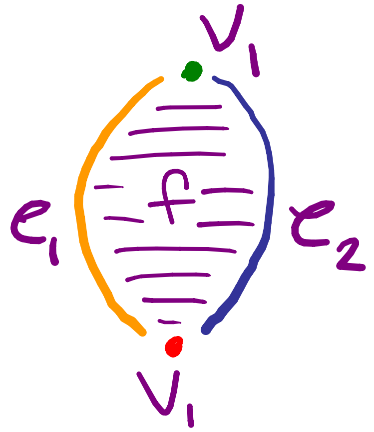


Order Relations:

$u \leq v$ for $u, v \in W$ <=> any reduced expression for v has subexpression that is reduced expression for u.

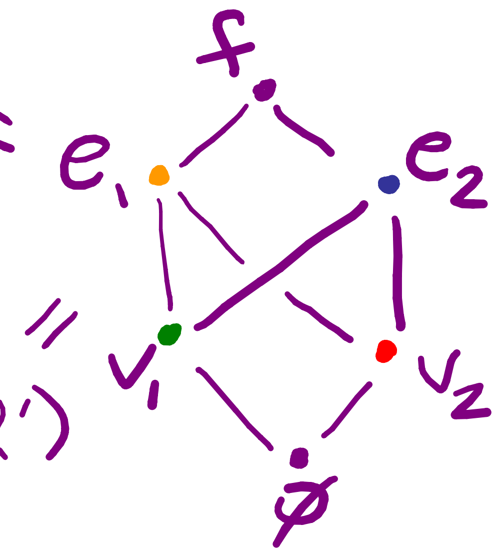
CW Complexes & their Face Posets

e.g.

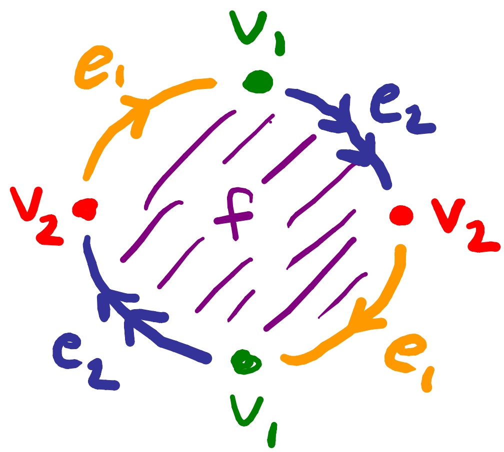
$K =$ 

$K \cong$ ball

$\to$

$F(K) =$ 

$= F(K')$

$K' =$ 

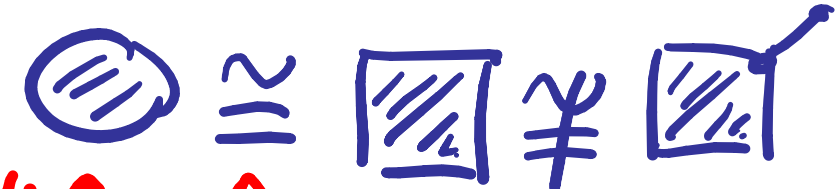
$K' = \mathbb{RP}^2$

$\to$ "face poset"

$(u \le v \iff u \le \bar{v})$

Recall: A CW complex: cells $e_\alpha \cong \mathbb{R}^{\dim(\alpha)}$, characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \to \bigcup_{\beta \le \bar{\alpha}} e_\beta$

& attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

Recall: 

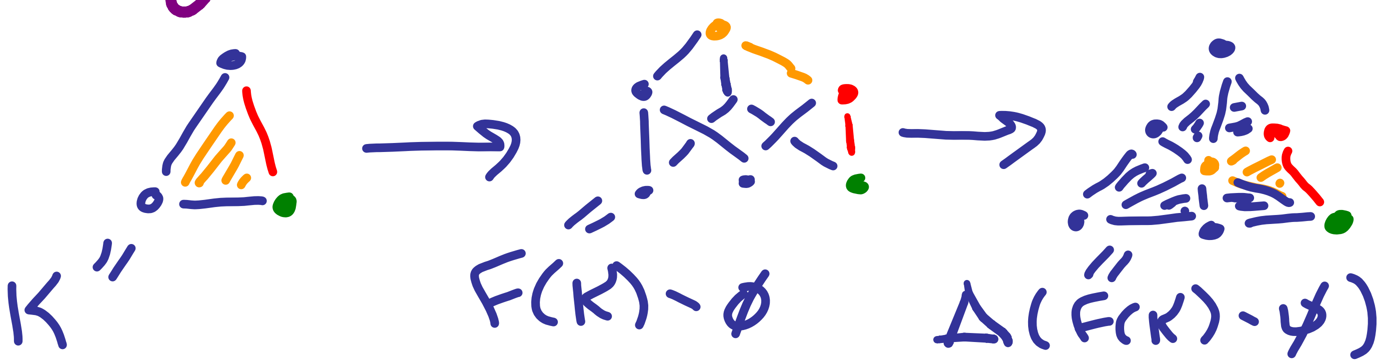
closed ball ↗ ↖ homeomorphic

Recall: CW complex is regular if each f_α is homeomorphism.

- $\Delta(P)$ = "nerve" or "order complex" of P

  = abstract simplicial complex s.t. i-dim'l faces are the poset chains $v_0 < v_1 < \ldots < v_i$

- K regular $\Rightarrow K \cong \Delta(F(K) \setminus \emptyset) = sd(K)$



Totally Nonnegative Parts of Stratified Spaces from Representation Theory

- Lusztig (94), Fomin–M. Shapiro (00) ... initiated study of "totally nonnegative", real part of spaces of matrices, spaces of flags (i.e. GL_n/B) and beyond...

- totally nonnegative := all minors nonnegative
  (minors: $i \times i$ submatrices)

- "flag": $\langle \vec{v}_1 \rangle \subseteq \langle \vec{v}_1, \vec{v}_2 \rangle \subseteq \ldots \subseteq \mathbb{R}^n$

- These stratified spaces are conjecturally (in some cases) & provably (in other cases) regular CW complexes homeomorphic to closed balls
- Proving by studying map fibers
  - imposes restrictions on rel'n's amongst exp'd Chevalley gen's
  - reveals structure in Lusztig's canonical bases

Aside: Sharply contrasts entire flag variety, Grassmannian, etc. having rich topology (topic of Schubert calculus)

Totally Nonnegative, Real Part of "Unipotent Radical" (a Space of Matrices)

- $x_i(t) = I_n + tE_{i,i+1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$ ← row i; column $i+1$

  $x_i(t) = \exp(te_i)$ (general finite type, expon'd Chevalley generator)

  $I_n + tE_{i,i+1}$ (type A)

- $f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \in \mathbb{R}^{n^2}$

  $(i_1,\dots,i_d)$: reduced word

  $(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$

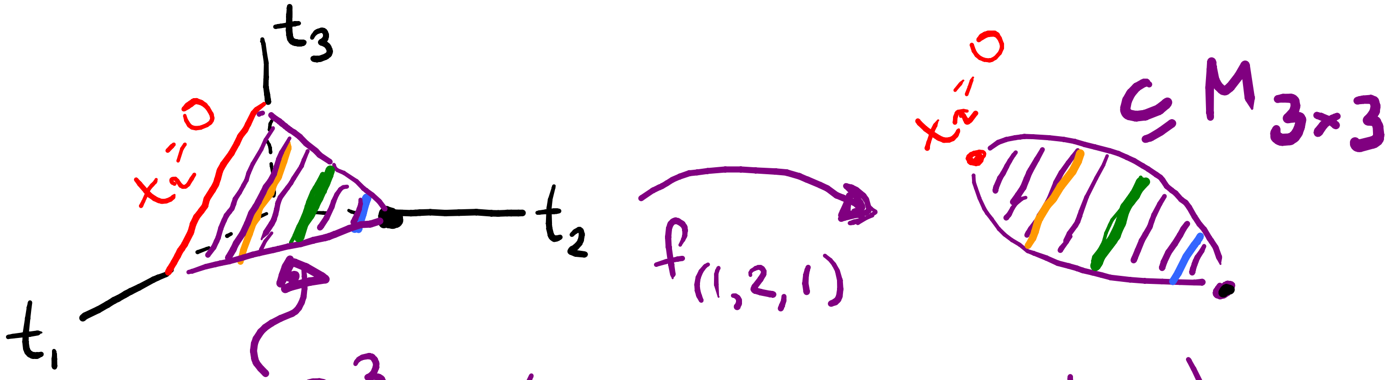
e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

w_0 case:

$$\left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \middle| \begin{matrix}\text{tot.} \\ \text{nonneg}\end{matrix} \right\} = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map $f_{(i_1, \ldots, i_d)}$



$\mathbb{R}^3_{\geq 0} \cap (\sum t_i = K$ hyperplane$)$

K: positive constant

$$f_{(1,2,1)}(t_1, t_2, t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$\downarrow t_2 = 0$

$$x_1(t_1) \cdot x_1(t_3)$$

$$f_{(1,2,1)}(t_1, 0, t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1 + t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1 + t_3)$$

simplex faces w/ same image "$x_1^2 = x_1$"

e.g. $\{x_1(t) \mid t > 0\} = \{x_1(t_1)x_1(t_2) \mid t_1, t_2 > 0\}$

Question (Björner, Bernstein)

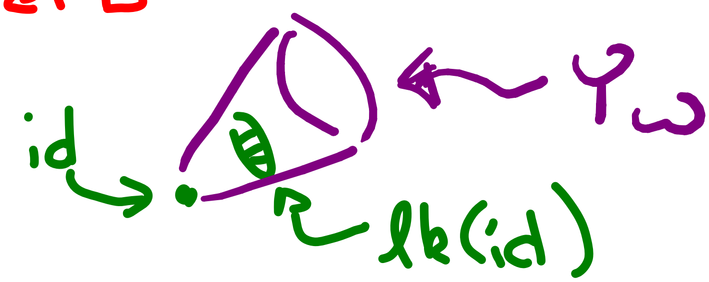
Find naturally arising, rep'n theoretic regular CW complexes with Bruhat order as face poset.

Fomin-Shapiro Conjecture: The Bruhat stratification of lk(id) in totally nonneg. real part of unipotent radical in Borel in algebraic group is regular CW complex homeomorphic to closed ball (w/ Bruhat order as face poset).

$$Y_w = \left[B^- w B^- \cap \left(\begin{matrix}\text{unipotent} \\ \text{subgp of } B\end{matrix}\right)\right]_{\geq 0} = \text{im}\left(f_{(i_1, \dots, i_d)}\right)$$

- B^-: lower triangular opposite Borel B^-
- w: permutation w
- unipotent subgp of B: upper triang. w/ 1's on diagonal
- ≥ 0: totally nonneg. part

and



Concrete Realization: Products $x_{i_1}(t_1) \cdots x_{i_d}(t_d)$ of elementary matrices, by Whitney, Loewner & Lusztig.

Theorem (H., 2014, Inventiones): Fomin-Shapiro Conjecture holds.

A Key Idea: 0-Hecke Algebra of W to Describe Strata

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

$\Downarrow$ suppress parameters

$x_i x_i = x_i$

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2 t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1 t_2}{t_1+t_3}\right)$ ← for $t_1, t_2, t_3 > 0$

$\Downarrow$ (type A)

$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$

(& analogous relations outside type A)

Upshot: $\mathrm{im}(f_1) = \mathrm{im}(f_2) \Leftrightarrow x(f_1) = x(f_2)$ — equal as 0-Hecke algebra elements

Thm (Lusztig): If $(i_1, \ldots, i_d)$ is reduced, then $f_{(i_1,\ldots,i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

A Motivation for Nonnegative Real Part of Unipotent Radical

- Given quantized env. alg. $U = U^- \otimes_{\mathbb{Q}(v)} U^0 \otimes_{\mathbb{Q}(v)} U^+$ of Kac-Moody alg. (e.g. affine Lie alg.), then canonical basis is a basis B for U^- such that highest weight module with highest weight vector v_λ has basis $\{v_\lambda b \mid b \in B, v_\lambda b \neq 0\}$ for each λ.

- $f^{-1}_{(i_1, \ldots, i_d)}(p) \to f^{-1}_{(j_1, \ldots, j_d)}(p)$ coordinate change

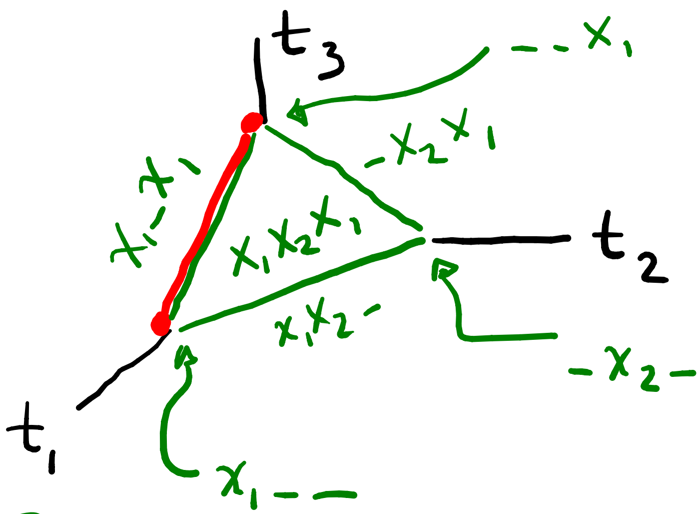
$$(t_1, t_2, t_3) \longmapsto \left(\frac{t_2 t_3}{t_1 + t_3}, t_1 + t_3, \frac{t_1 t_2}{t_1 + t_3}\right)$$

tropicalizes to coordinate change:

$$(a, b, c) \mapsto (b + c - \min(a, c), \min(a, c), a + b - \min(a, c))$$

for canonical bases w/ same braid move

Faces of (Preimage) Simplex as Subexpressions in 0-Hecke Algebra



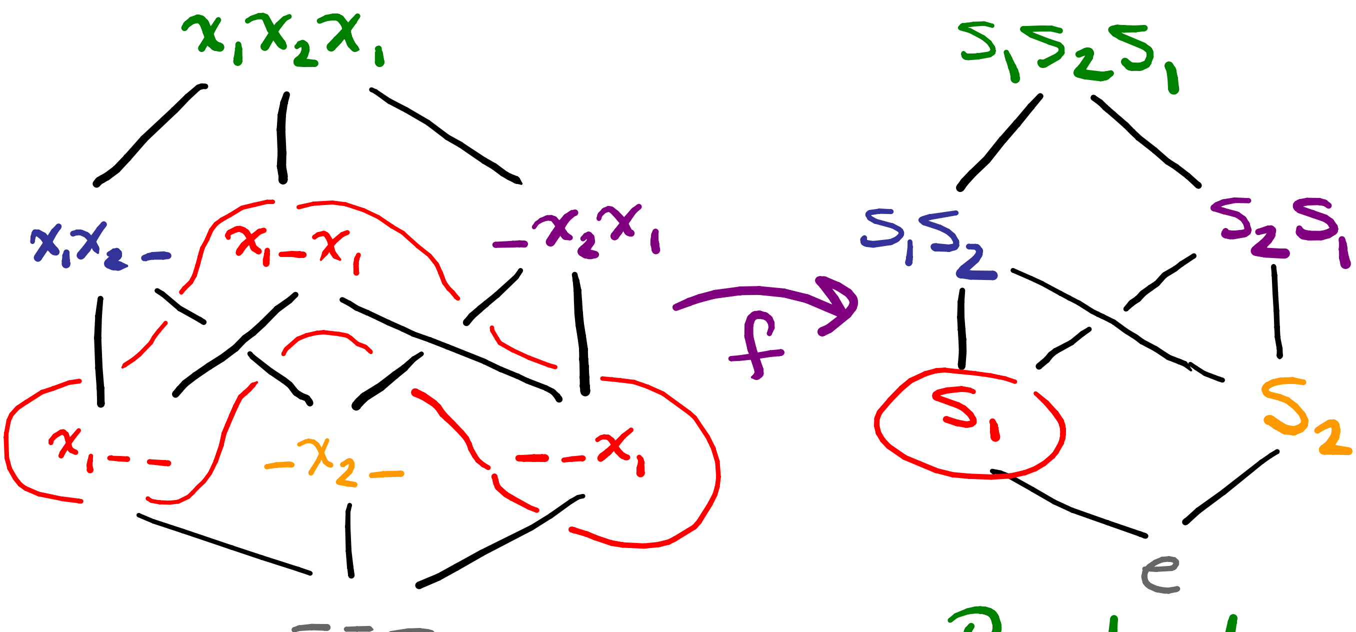
- let Y_w^o = open cell in $\mathrm{im}(f_{(i_1 \ldots i_d)})$ indexed by $w \in W$

- let $\delta(x_{i_1} \ldots x_{i_d})$ denote (unsigned) 0-Hecke algebra product (a.k.a. "Demazure product")

e.g. $\delta(\underbrace{x_1 x_2 x_1}_{x_2 x_1 x_2} x_2 x_1) = \delta(x_2 x_1 x_2 x_1) = s_1 s_2 s_1$

↑ since $x_2 x_2 = x_2$

Map of Face Posets Induced by Map $f_{(i_1,\dots,i_d)}$ of Spaces

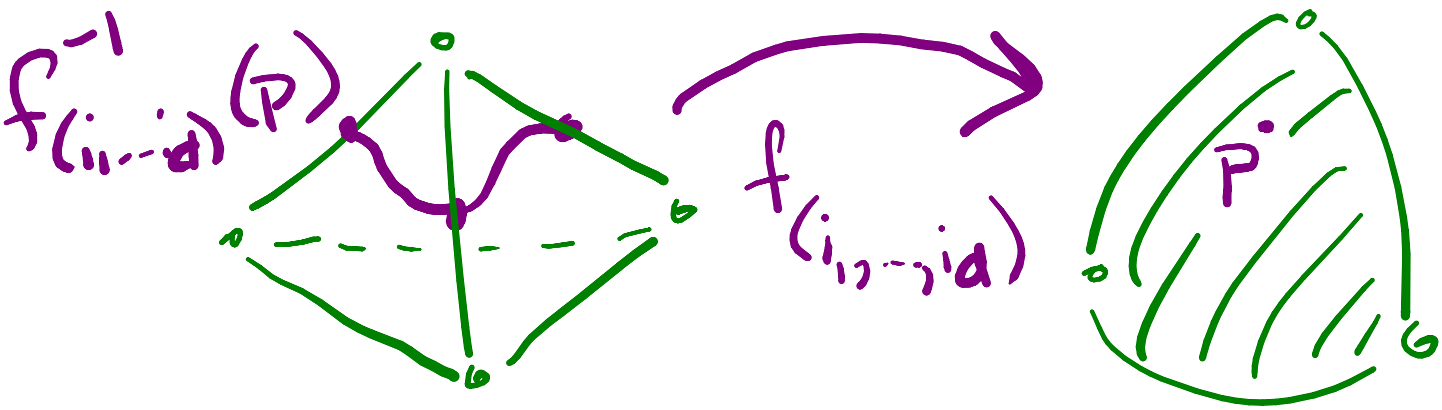


Boolean lattice B_n = Face poset of simplex

Bruhat order = $F(\mathrm{im}(f_{(i_1,\dots,i_d)}))$

$$f : x_{i_{j_1}} \cdots x_{i_{j_r}} \longmapsto \delta(x_{i_{j_1}} \cdots x_{i_{j_r}})$$

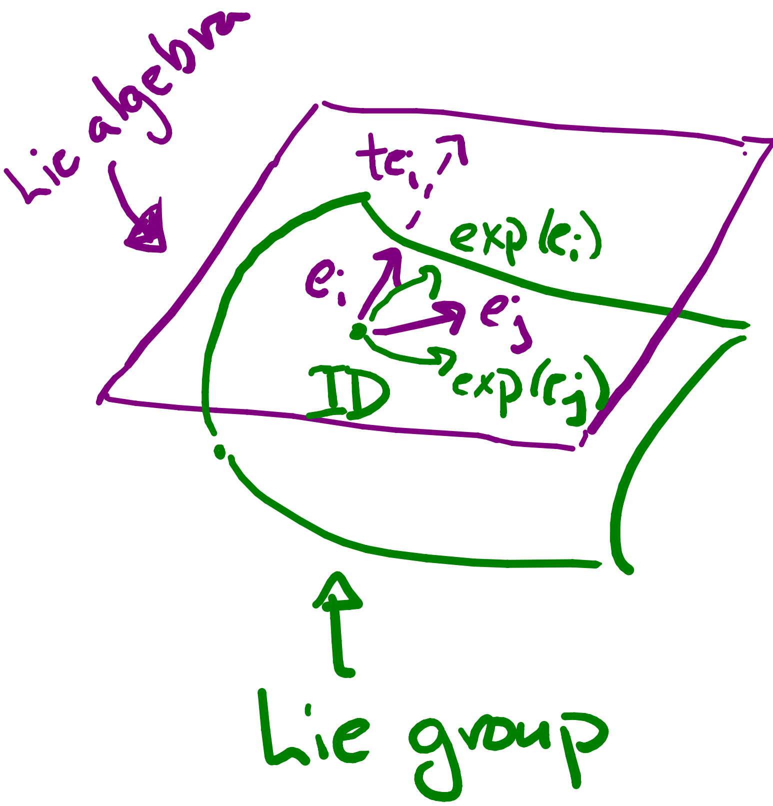
Conjecture (Davis-H-Miller): $f^{-1}_{(i_1,\ldots,i_d)}(p)$ is regular CW complex homeomorphic to interior dual block complex of subword complex $\Delta((i_1,\ldots,i_d),w)$ for $p \in Y_w^\circ$.



Thm (DHM): $f^{-1}_{(i_1,\ldots,i_d)}(p)$ has cell decomposition & "correct" face poset.

Thm (DHM): Interior dual block complex of $\Delta((i_1,\ldots,i_d),w)$ is contractible.

A Motivation to Study Fibers: Relations Among (Exponentiated) Chevalley Generators



$$te_i = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Big\downarrow \exp(\_)$$

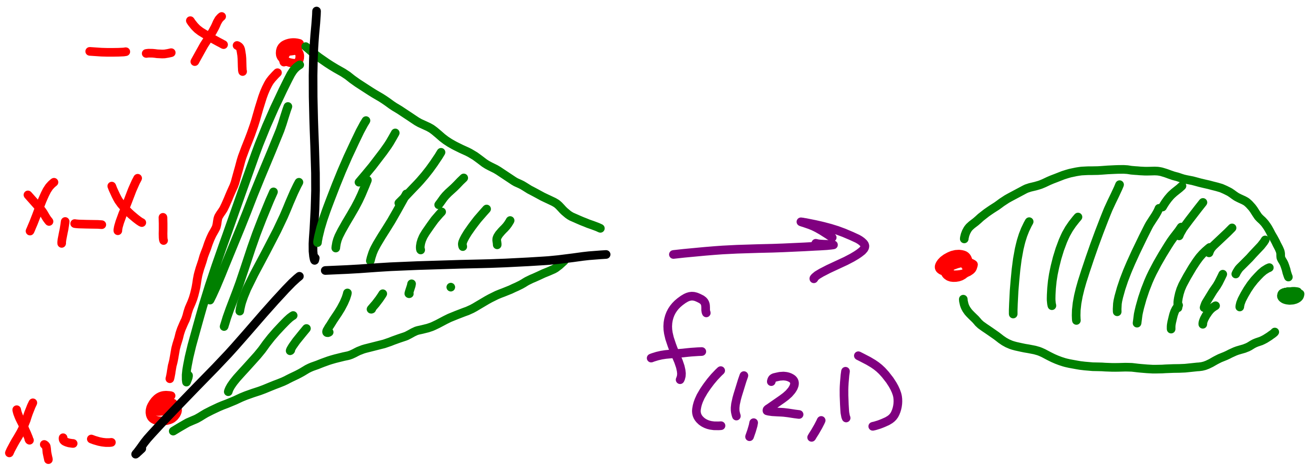
$$\begin{pmatrix} 1 & t & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\exp(t_1 e_i)\exp(t_2 e_j) = f_{(i,j)}(t_1, t_2) = x_i(t_1)\, x_j(t_2)$$

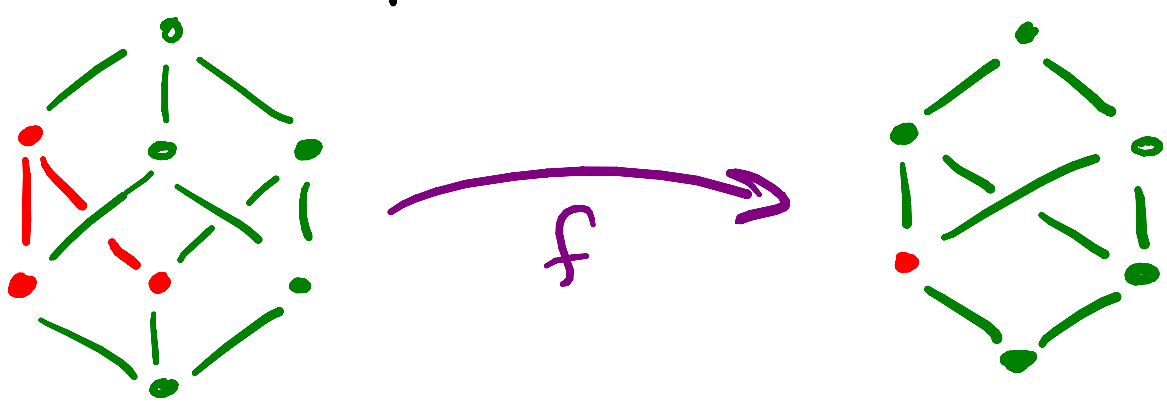
$$\exp(te_i) = \boxed{ID + te_i} + \underbrace{\frac{t^2 e_i^2}{2}}_{=0} + \underbrace{\frac{t^3 e_i^3}{6}}_{=0} + \dots$$

Rel'ns $\leftrightsquigarrow$ Elts in same fiber of $f_{(i_1, \dots, i_d)}$

Combinatorics of Fibers



Induced map of face posets:



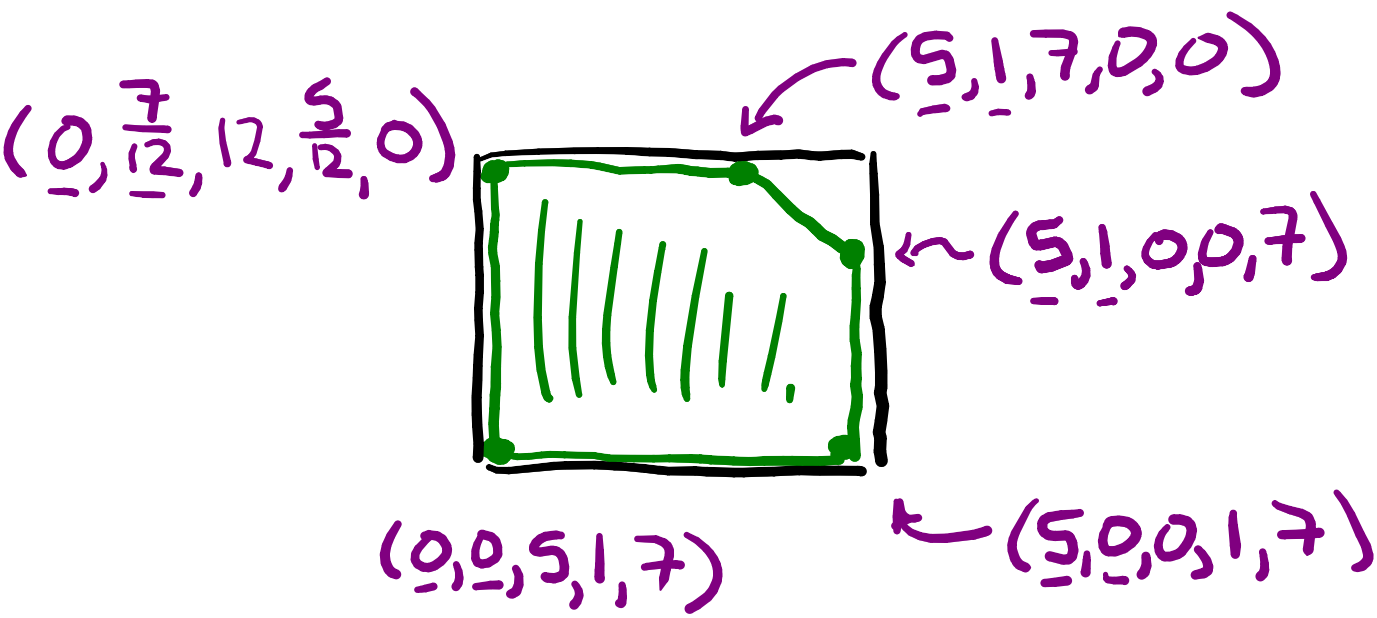
Thm (Armstrong-H., 2011): For each $u \in W$, $f^{-1}_{\geq}(u) = \{x \in B_n \mid f(x) \geq u\}$ is dual (i.e. upside-down) to face poset for subword complex $\Delta((i_1, \ldots, i_\ell), u)$.

Thm (DHM, 2018): $f_{=}^{-1}(u)$ is face poset of interior dual block complex for subword complex $\Delta((i_1,\dots,i_d),u)$, for f induced by $f_{(i_1,\dots,i_d)}$.

e.g. $f^{-1}_{(1,2,1,2,1)}(M)$ for $M \in Y^{o}_{s_1 s_2 s_1}$

$M = x_1(5)x_2(1)x_1(7)$

$f^{-1}_{(1,2,1,2,1)}(M) =$

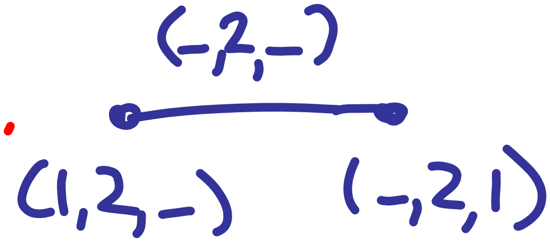


Subword Complexes

- introduced by Allen Knutson & Ezra Miller to serve as Stanley-Reisner complexes of initial ideals of coordinate rings of matrix Schubert varieties.

$\Delta(Q, w)$ = simplicial complex of subwords Q' of Q s.t. $Q \setminus Q'$ contains a reduced word for w.

- Q: reduced or nonreduced word
- w: Coxeter group element

e.g. 

$= \Delta((1,2,1), s_1)$

Thm (Knutson-Miller): $\Delta(Q,\omega)$ is shellable.

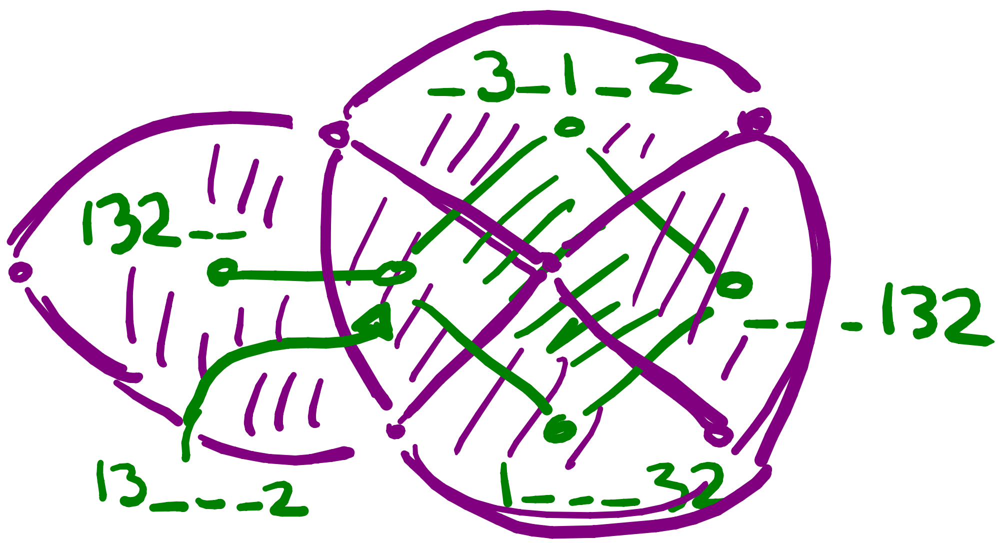
Thm (Knutson-Miller): $\Delta(Q,\omega)$ is homeomorphic to ball or sphere.

Example of Subword Complex & its Interior Dual Block Complex

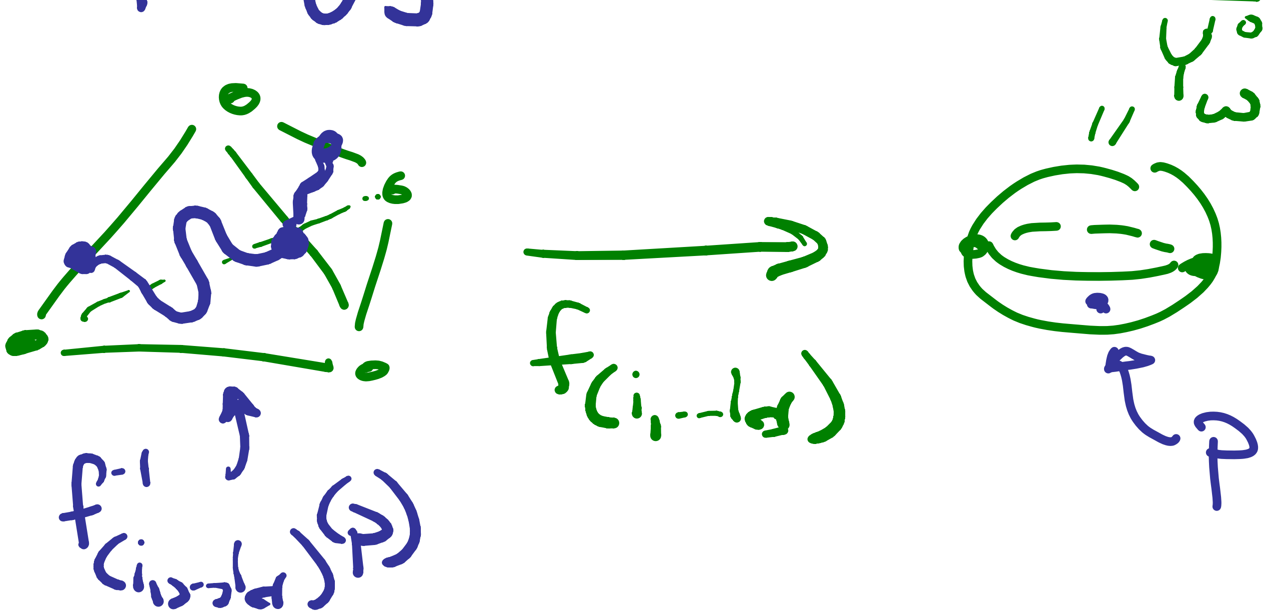
$\Delta(Q,\omega) =$

$Q = 132132$

$\omega = s_1 s_3 s_2$



Topology of Fibers



Thm (Davis-H-Miller): Each fiber $f^{-1}_{(i_1,\dots,i_d)}(p)$ admits a cell decomposition induced by the natural cell decomposition of the simplex Δ_{d-1}.

Proof: Parametrization + continuity lemmas

Parametrization of Open Cells

e.g. $\delta(3,1,2,1,2,1,2,1) = s_3 s_1 s_2 s_1 = \omega$

$f^{-1}_{(3,1,2,1,2,1,2,1)}(p)$, $p \in Y^\circ_\omega$

rightmost subword for ω

$$x_3(t_1)\,x_1(t_2)\,x_2(t_3)\,x_1(t_4)\,x_2(t_5)\,x_1(t_6)\,x_2(t_7)\,x_1(t_8)$$

"free" parameters: t_2, t_3, t_4, t_5

"dependant" parameters: t_1, t_6, t_7, t_8

<u>Thm</u> (Davis-H-Miller):

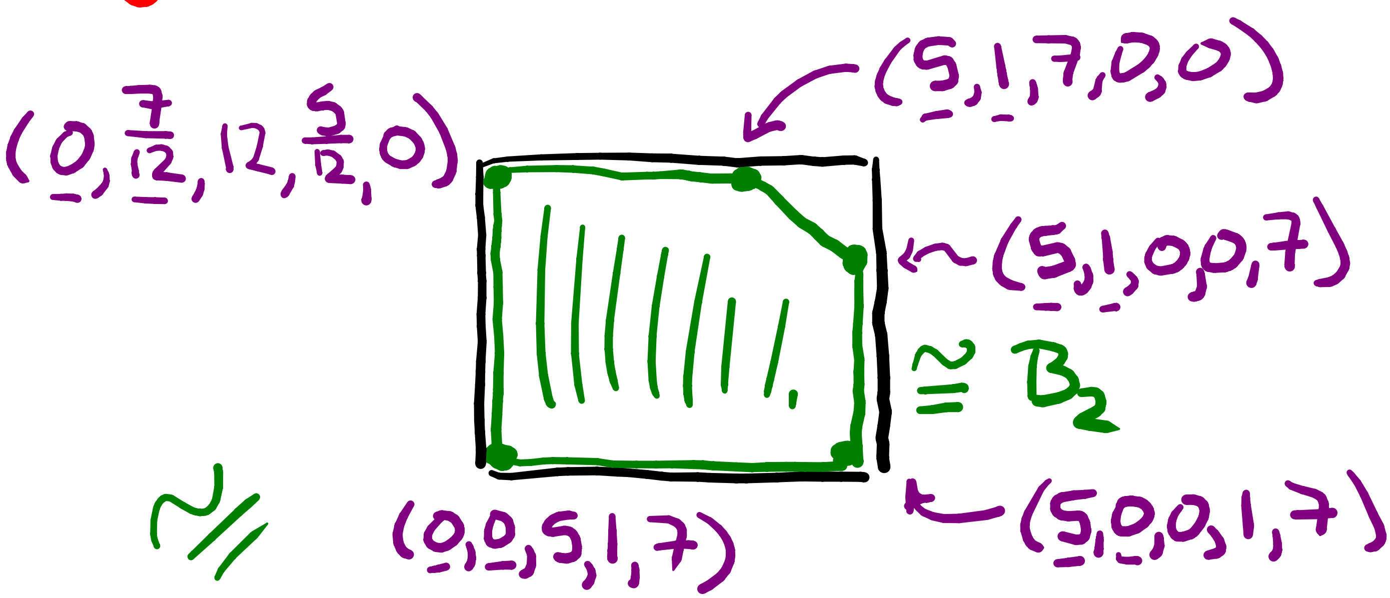
$$[0,1)^{\dim(\sigma)} \cong \bigcup_{v \le \bar{\tau} \le \bar{\sigma}} \tau$$

v ← source vertex of $\bar{\sigma}$

Examples of Fibers:

(from vantage point of cell parametrization)

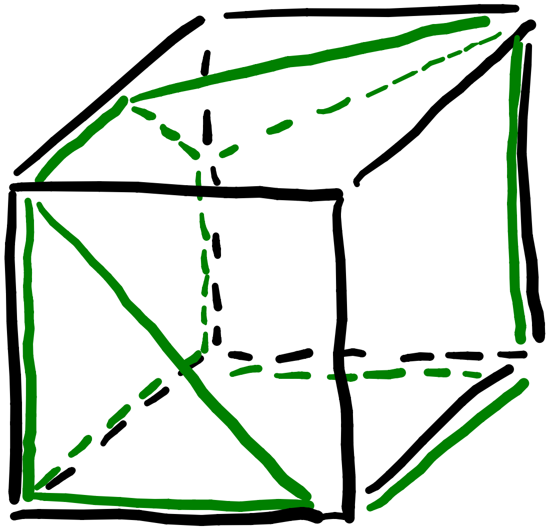
e.g.



$f^{-1}_{(1,2,1,2,1)}(M)$ for $M = x_1(5)x_2(1)x_1(7)$

$\cap$

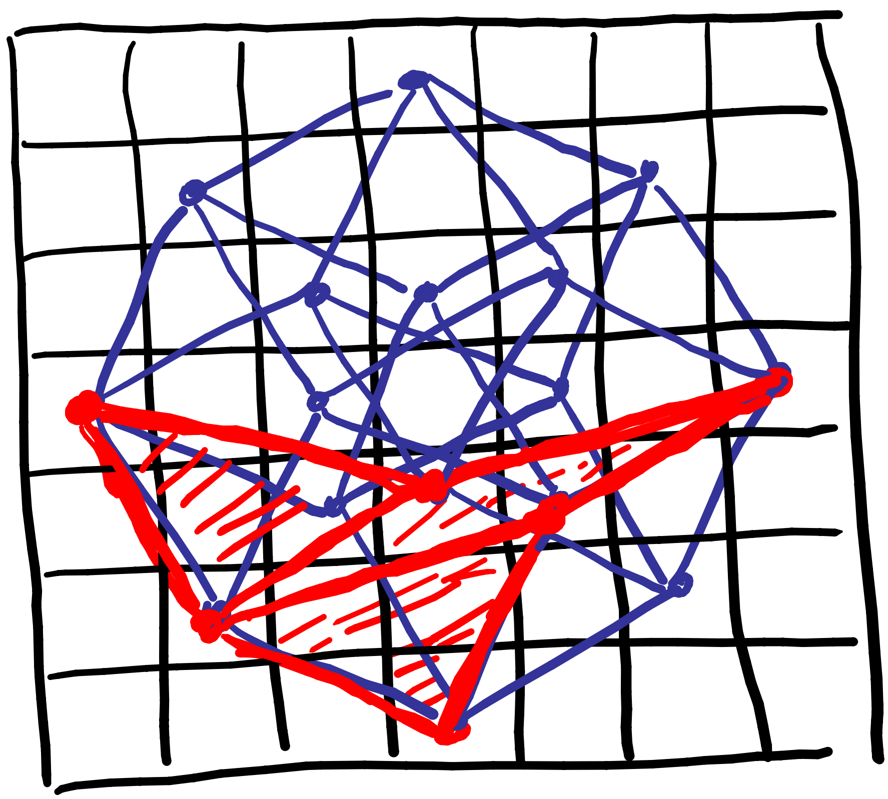
$U^{0}_{s_1 s_2 s_1}$



$\cong B_3$

$\cong$

$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^{\circ}_{s_1 s_2 s_1}$



$\cong$

$f^{-1}_{(1,2,1,2,1,2)}(M)$ for $M \in Y^{\circ}_{(1,2)}$

Thm (DHM, 2018): Interior dual block complex of $\Delta((i_1,\dots,i_d),u)$ is collapsible, hence contractible.

Pf: Discrete Morse theory

Observation: DHM Conjecture $\Rightarrow$ $f^{-1}_{(i_1,\dots,i_d)}(M)$ is contractible.

Idea: DHM Conjecture says $f^{-1}_{(i_1,\dots,i_d)}(p) \cong$ interior dual block complex of $\Delta((i_1,\dots,i_d),u)$ for $p \in Y_u^\circ$, which is contractible.

Thm (DHM): DHM Conjecture would imply new proof of FS-Conjecture.

Idea: Use Topological Relationship Fibers to Image: Let $g: B \to Z$ be continuous surjection from ball B to Hausdorff space Z whose restriction to $\mathrm{int}(B)$ is an embedding. Suppose also:

(1) $g(\partial B) \cong \partial B \cong S^n$

(2) $g(\partial B) \cap g(\mathrm{int}(B)) = \emptyset$

(3) $g^{-1}(p)$ is contractible $\forall p \in g(\partial B)$

Then $Z \cong B$.

- Shellability of Bruhat order $\Rightarrow \partial B \cong S^n$ requirement by induction on dimension.

Maps/Spaces with Seemingly Analogous Structure

1. Totally nonnegative real part of Grassmannian: $Gr_{\geq 0}(k,n) = (GL_n/P)_{\geq 0}$

Postnikov: polytope of "plabic graphs" w/ "measurement map" to $Gr_{\geq 0}(k,n)$ + theory of (reduced) plabic graphs

Postnikov-Speyer-Williams: $Gr_{\geq 0}(k,n)$ is CW complex (via attaching maps that are not homeomorphisms)

Galashin-Karp-Lam 2017 preprint: $Gr_{\geq}(k,n)$ is homeom. to closed ball

2. Totally nonneg. real part of Flag variety: $\mathcal{Fl}_{\geq 0} = (GL/B)_{\geq 0}$

Rietsch: poset of closure reln's. Cells $R^{\circ}_{u,v}$ given by $u \leq v$ in Bruhat order

Marsh-Rietsch: parametrization for $R^{\circ}_{u,v}$

Williams: poset is CW poset

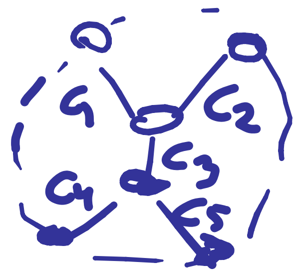
Rietsch-Williams: (1) CW complex w/ attaching maps via canonical bases.
(2) Contractibility of each cell closure

Galashin-Karp-Lam 2018 preprint:
Homeomorphism type for total space.

3. Map to Stratified Space E_n of Electrical Networks

(Curtis-Ingerman-Morrow, Kenyon-Wilson, ...)

- arises as image of:

Resp : 

$\longmapsto$ response matrix

Lam:

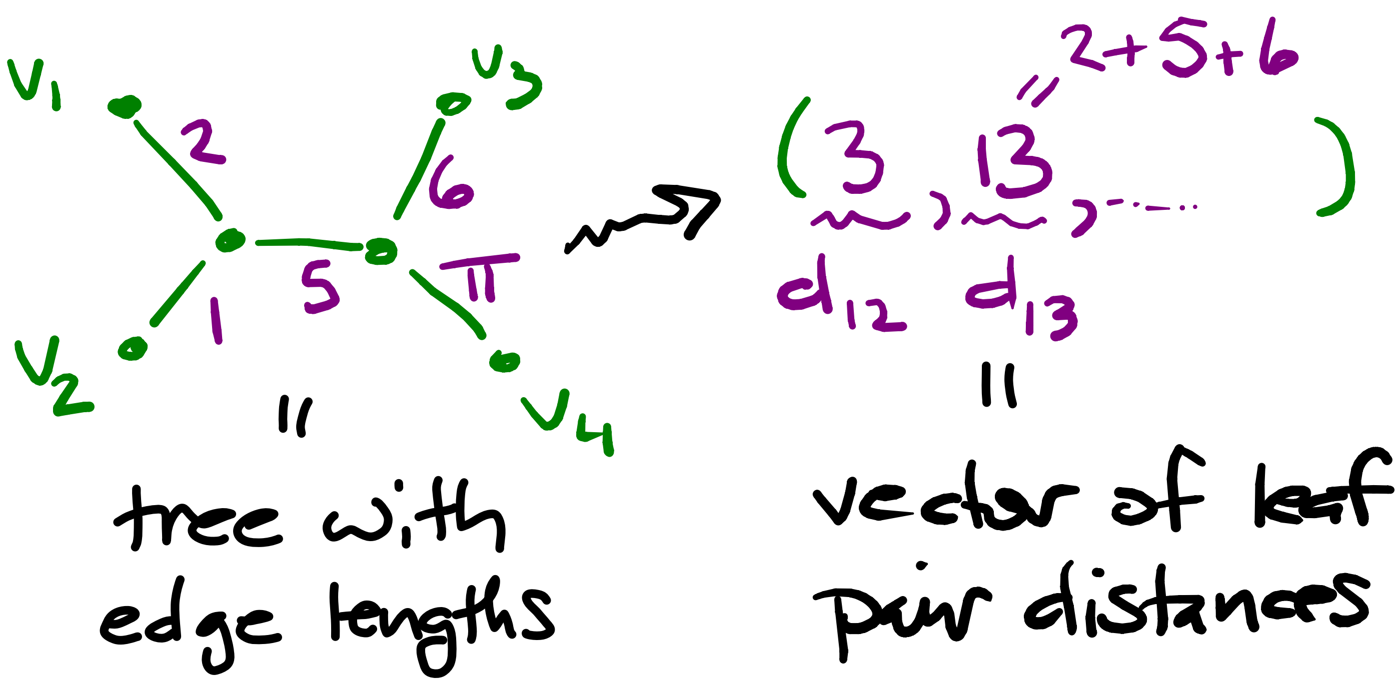
- $F(E_n)$ is "Eulerian", i.e. $\mu(u,v) = (-1)^{rk(v)-rk(u)}$ "$= \tilde{\chi}(u,v)$"

H-Kenyon:

- $F(E_n)$ shelling & hence $F(E_n)$ is a CW poset (i.e. face poset of regular CW complex)

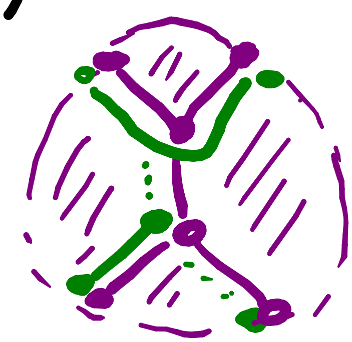
H-Kenyon Corollary: shelling for each [u,v] in face poset for "edge product space of phylogenetic trees", hence face poset is CW poset.

(builds upon Gill-Linusson-Moulton-Steel)



Some Further Questions

1. Subword complex analogues for fibers of other maps?
   e.g. H-Kenyon: $\Delta(G, elec(H))$ for planar electrical networks (for H a minor of G)



2. Fiber Structure for: electrical networks, $(GL_n)_{\geq 0}$, $(\mathcal{F}\ell_n)_{\geq 0}$, $(GL_n/P)_{\geq 0}$, ...?

3. Proof of DHM Conjecture?

Thanks!