# Regular Cell Complexes in Total Positivity

Patricia Hersh

North Carolina State University

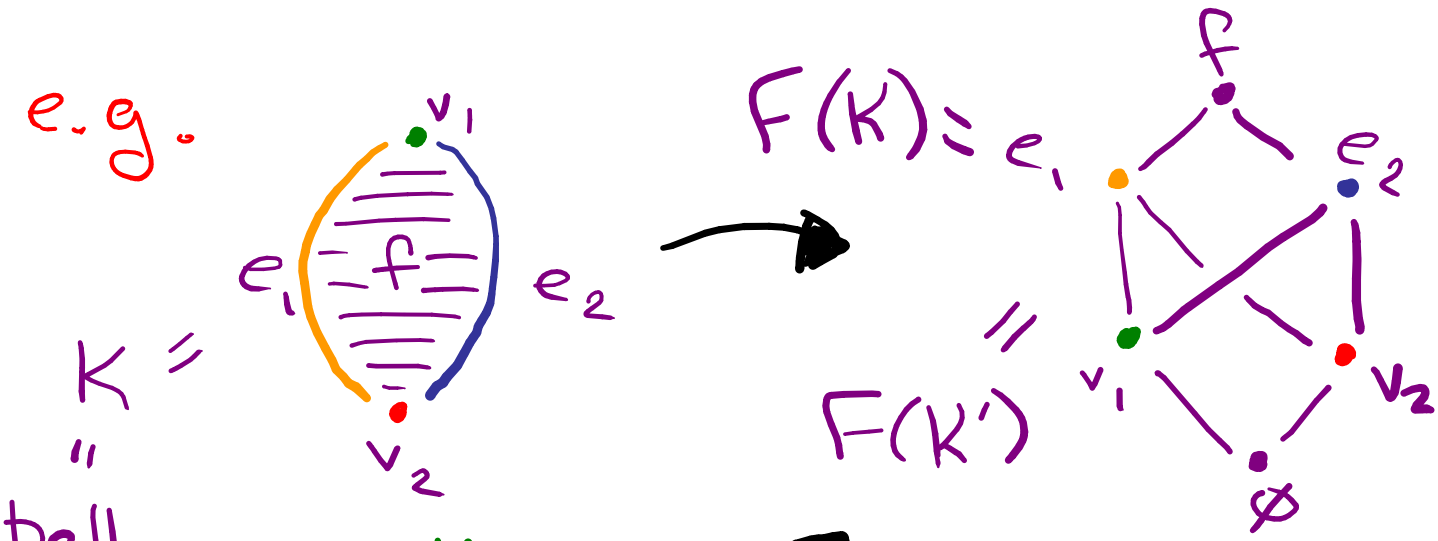
(Paper with these results by same title, to appear in Inventiones Mathematicae, 60 pages)

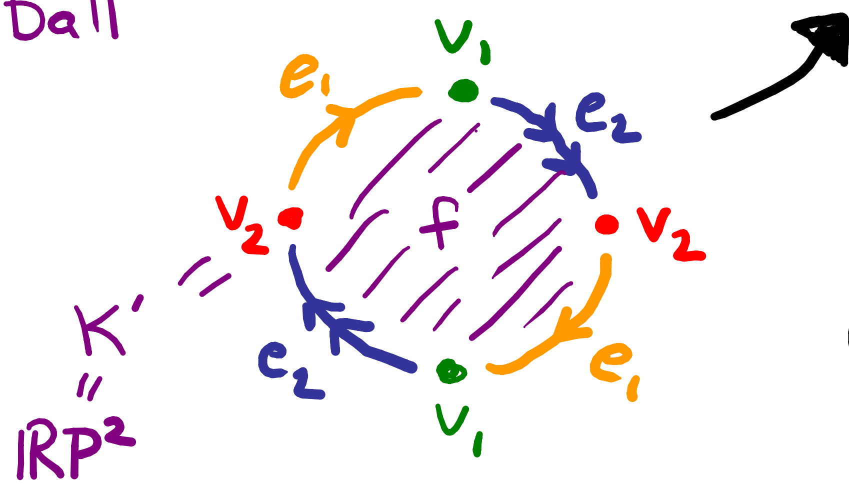
(See http://www4.ncsu.edu/~plhersh for slides, including appendix with more details)

# Topological Aspects of Total Positivity

- Lusztig initiated study of Totally nonnegative, real part of (matrix) Schubert varieties
  (i.e. part with minors all nonnegative in spaces of matrices or of flags)
- Topology (homeomorphism type) is conjecturally/provably trivial.
- Puts restrictions on possible relations among (exponentiated) Chevalley generators.
- Reveals structure in canonical bases; a motivation for cluster algebras
- Main Result of Talk: Proof of Fomin-Shapiro Conjecture via new tools exploiting interplay of combinatorial data & topological data

# Background: CW Complexes and their Closure Posets




"closure poset"
or "face poset"

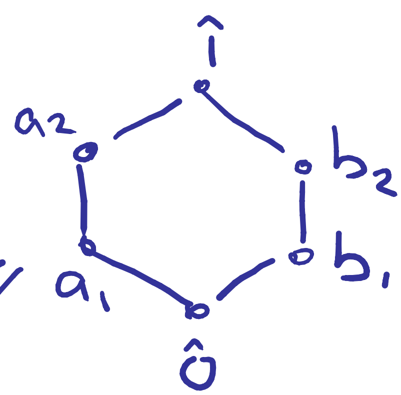
$(u \le v \iff u \le \bar{v})$

CW complexes: comprised of pieces called cells each homeomorphic to an open ball

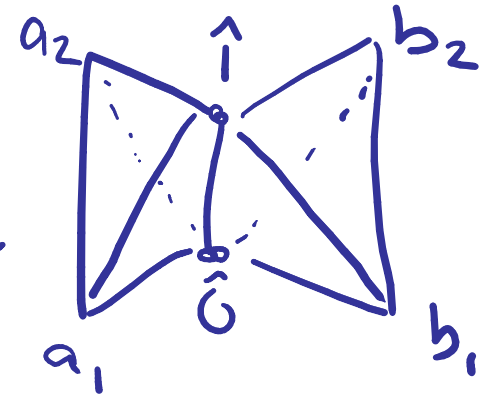
- higher dimensional cells glued to unions of lower dimensional ones by attaching maps.

**Def'n**: The order complex (or nerve) of a poset $P$ is the simplicial complex $\Delta(P)$ whose $i$-dimensional faces are the $(i+1)$-chains $v_0 < \cdots < v_i$ in $P$

e.g. $P =$




$\Delta(P) =$

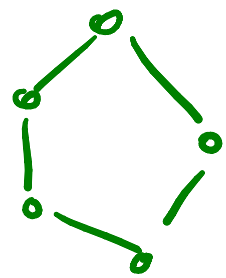


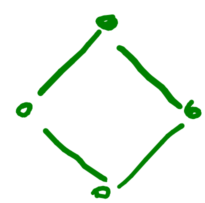

**Remark**: Studied extensively e.g. in group theory (posets of subgroups), combinatorics (Möbius fn & group actions on posets), commutative algebra (small resolutions), etc.

# Regular CW Complexes

- A CW complex is regular if the attaching map for each cell is a homeomorphism (hence injective).
  e.g. all simplicial complexes & polytopes
- $K$ regular $\Rightarrow K \cong \Delta(F(K) \setminus \{\hat{0}\}) = \mathrm{sd}K$
- seemingly encompasses several spaces of interest from combinatorial rep'n theory, real algebraic geometry, total positivity theory, electrical networks, ...

**Recall:** Poset is graded if $u \le v$ implies minimal paths $u$ to $v$ all same length

e.g.  is not graded

A graded poset is thin if each rank 2 interval has 4 elements 

**Defn (Björner):** A finite, graded poset $P$ is CW poset if

- $P$ has unique min'l elt. $\hat{0}$
- $P$ has additional element(s)
- $x \neq \hat{0} \Rightarrow \Delta(\hat{0}, x) \cong S^{\operatorname{rk} x - 2}$

Thm (Björner): $P$ is CW poset $\Leftrightarrow$ there exists regular CW complex $K$ with $P = F(K)$.

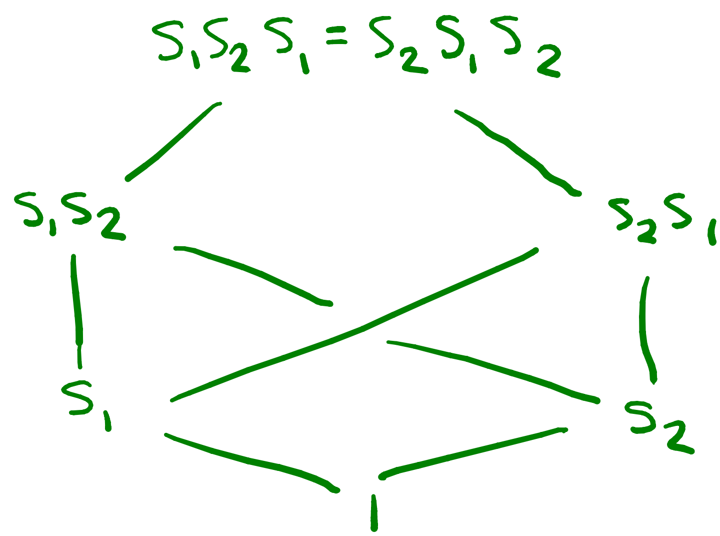
## Some Examples of CW Posets

- Shellable & thin (Danaraj-Klee)
- Bruhat order (Björner & Wachs)
- Closure poset for double Bruhat decomp. of totally nonneg. part of flag variety (Williams)
- Closure poset of triangulation of double suspension of Poincare homology 3-sphere with "big cell" glued in (due to work of R. Edwards)

The Bruhat order is partial order on Coxeter group $W$ with $u \le v \iff$ there exists reduced expressions (i.e. products of minimal # adjacent transpositions) $r(u)$ and $r(v)$ with $r(u)$ subexpression of $r(v)$.

e.g. $W = S_3$ with generators

$s_1 = (1,2)$
$s_2 = (2,3)$




- Closure poset for Schubert cell decompositions of flag varieties $G/B$ & totally nonnegative part of matrix Schubert varieties

**Question (Bernstein):** Find regular CW complexes naturally arising from rep'n theory which are homeomorphic to closed balls and have the Bruhat intervals as closure posets.

**Conjectural Solution (Sergey Fomin & Michael Shapiro, 2000):**

The Bruhat stratification of $lk(id)$ in totally nonnegative, real part of unipotent radical in semisimple, simply connected algebraic group is regular CW complex homeomorphic to closed ball.

**Theorem** (H.): Fomin-Shapiro Conjecture indeed holds.

**Special Case of Type A:**

Space of Totally nonnegative upper triangular matrices with 1's on diagonal & entries just above diagonal summing to fixed, positive constant, stratified by which minors are positive and which are 0.

# The Totally Nonnegative Part of a Space of Matrices

- $$x_i(t) = I_n + tE_{i,i+1} = \begin{array}{cc} & \text{column } i+1 \\ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & t & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} & \text{row } i \end{array}$$

  $x_i(t) = \exp(te_i)$ (general type)

  $I_n + tE_{i,i+1}$ (type A)

- $$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$$

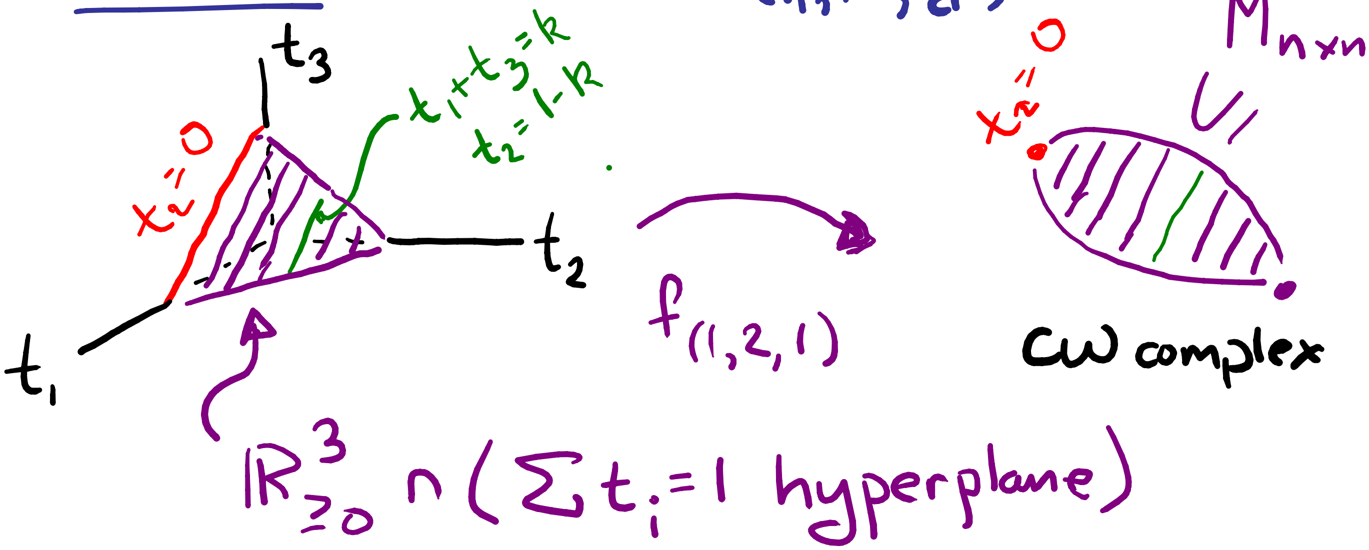
  $$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

e.g. $$f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

# "Picture" of Map $f_{(i_1,\dots,i_d)}$




$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow{\scriptstyle t_2 = 0}$$

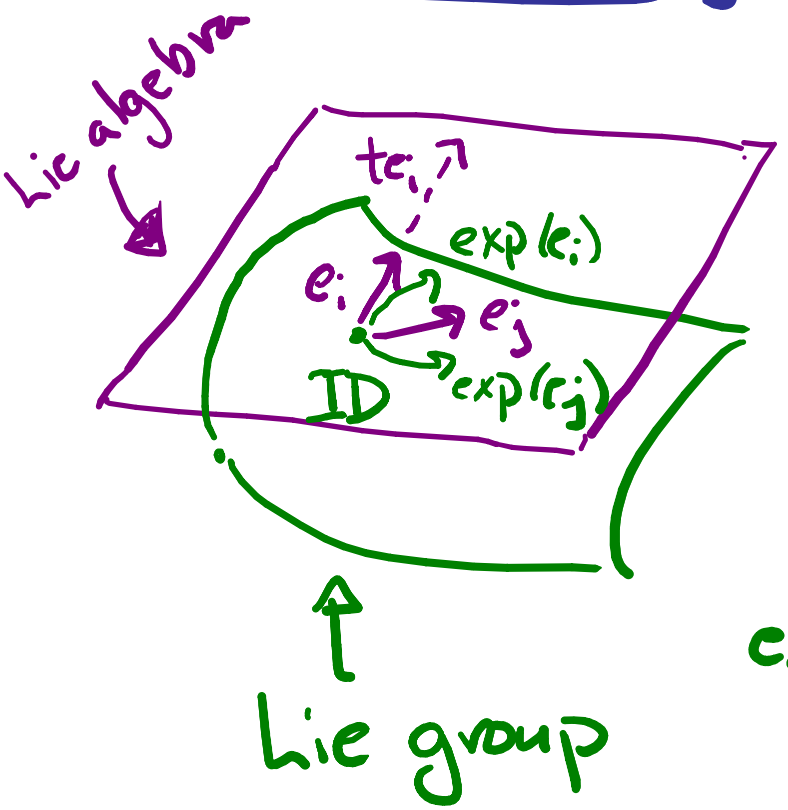
$$x_1(t_1) \circ x_1(t_3)$$

$$f_{(1,2,1)}(t_1,0,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

**Non-injectivity:** results from "nil-moves" $x_i(u)x_i(v) = x_i(u+v)$ & "long braid moves" in 0-Hecke algebra

# A Motivation: Understanding Relations Among (Exponentiated) Chevalley Generators




$$f_{(i,j)}(t_1, t_2) = \exp(t_1 e_i)\exp(t_2 e_j) = x_i(t_1)\,x_j(t_2)$$

$$\exp(te_i) = \boxed{ID + te_i} + \underbrace{\frac{t^2 e_i^2}{2}}_{0=} + \underbrace{\frac{t^3 e_i^3}{6}}_{0=} + \ldots$$

**We Prove:** Only the "obvious" relations occur

# 0-Hecke Algebra Captures which Simplex Faces have Same Image under $f_{(i_1,\ldots,i_d)}$

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$ "nil-move"

$$\Big\downarrow \text{ suppress parameters}$$

$x_i^2 = x_i$ (0-Hecke alg. rel'n, up to sign)

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1t_2}{t_1+t_3}\right)$

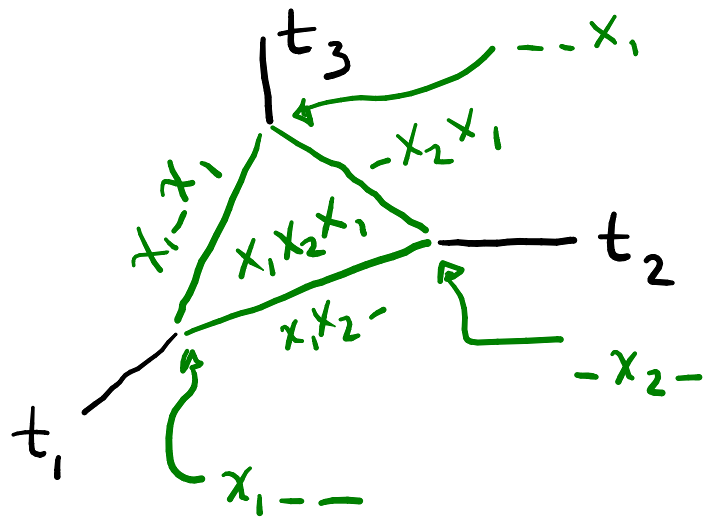
$$\Big\downarrow \text{ (type A)}$$

"long braid move" with enrichment from parameters

$x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1}$

(similar relation holds outside type A )

# Indexing Faces of Preimage by Words in 0-Hecke Algebra




**Key Observation About $f_{(i_1,\ldots,i_d)}$:**

$$\mathrm{im}(F_1)=\mathrm{im}(F_2) \iff \underbrace{x(F_1)=x(F_2)}_{\text{equal as 0-Hecke algebra elements}}$$

**Thm (Lusztig):** If $(i_1,\ldots,i_d)$ is reduced word, then $f_{(i_1,\ldots,i_d)}$ acts homeomorphically on $\mathbb{R}^d_{>0}$.

**Upshot:** $f_{(i_1,\ldots,i_d)}$ restricts to homeomorphism on each face given by reduced subword

# Properties of Change-of-Coordinates Map Given by Braid Moves

e.g. $(t_1, t_2, t_3) \mapsto \left(\frac{t_2 t_3}{t_1+t_3}, t_1+t_3, \frac{t_1 t_2}{t_1+t_3}\right)$

in type A

- Tropicalizes to change-of-basis map for Lusztig's canonical bases:

$$(a,b,c) \mapsto (b+c-\min(a,c), \min(a,c), a+b-\min(a,c))$$

- A motivation for development of cluster algebras (& mutation)

Suggested exercise: verify map is involution

# Proof Strategy for Fomin-Shapiro Conjecture (& for Images of Maps from Polytopes)

Set-up: Continuous, surjective fn

$$f: P \to Y$$

from convex polytope $P$ s.t. $f$ maps $\mathrm{int}(P)$ homeomorphically to $\mathrm{int}(Y)$.

Step 1: Perform "collapses" on $\partial P$ preserving regularity and homeomorphism type - via continuous, surjective collapsing functions $P \to P$ yielding $P/\sim$ with fewer cells s.t. $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$

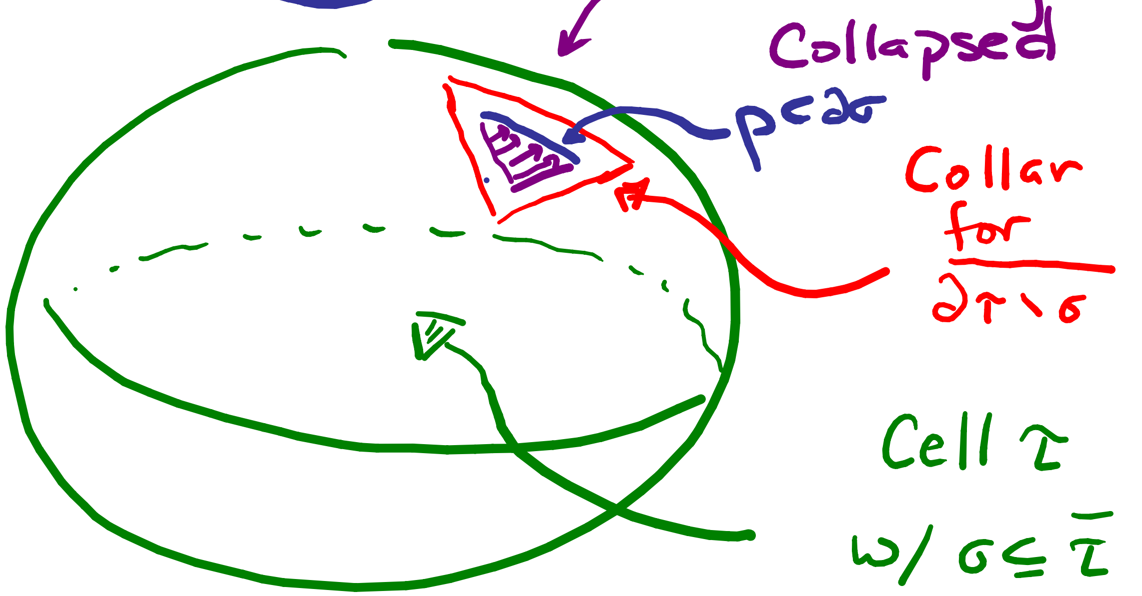
Step 2: Prove $\bar{f}: P/\sim \to Y$ is homeomorphism by new regularity criterion

# Collapsing Cell $\sigma$ onto Cell $\rho \subseteq \bar{\sigma}$ within $\partial\tau$

**Thm (M. Brown; Connelly):** Any topological manifold with boundary $\partial M$ has a collar, i.e. a nbh'd homeomorphic to $\partial M \times [0,1]$ &

**Fact:** Our collapses preserve this for:

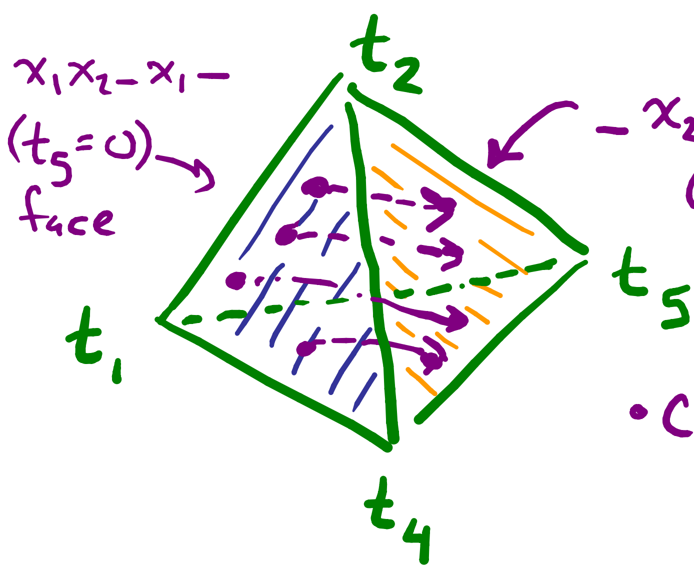
$\overline{\partial\tau \setminus \sigma} =$




**Plan:** Collapse $\bar{\sigma}$ onto $\bar{\rho} \subseteq \partial\sigma$, stretching collar for $\overline{\partial\tau \setminus \sigma}$ to cover $\bar{\sigma} \setminus \bar{\rho}$.

Collapsing "non-reduced" Face Across Curves (each in Single Fiber)

e.g. $f_{(1,2,3,1,2)}$ face with $t_3=0$




• collapse across curves identifying $t_1t_2t_4$ & $t_2t_4t_5$ faces

$$(t_1,t_2,0,t_4,t_5)\longmapsto x_1(t_1)x_2(t_2)x_1(t_4)x_2(t_5)$$

only for $t_1+t_4>0$

$$= x_2(t_1')x_1(t_2')x_2(t_4')x_2(t_5)$$

$$= x_2(t_1')x_1(t_2')x_2(t_4'+t_5)$$

for $t_1'=\dfrac{t_2t_4}{t_1+t_4}$ $\quad t_2'=t_1+t_4$ $\quad t_4'=\dfrac{t_1t_2}{t_1+t_4}$

curves: $t_1'=k_1$ & $t_2'=k_2$ & $t_4'+t_5=k_3$

# "Deletion Pairs": Transferring Coxeter Group Properties to 0-Hecke Algebra

In a non-reduced expression $s_{i_1} \cdots s_{i_d}$, let $\{s_{i_r}, s_{i_t}\}$ be a deletion pair if $s_{i_r} \cdots s_{i_{t-1}}$ and $s_{i_{r+1}} \cdots s_{i_t}$ are reduced expressions while $s_{i_r} \cdots s_{i_t}$ is nonreduced.

Key Coxeter Group Property: Any two reduced expressions for same $w \in W$ connected by series of braid moves - ensures nonreduced expressions admit modified nilmoves.

Collapsing Order: based on (1) leftmost deletion pair, (2) minimize $t-r$, (3) maximize cell dimension - amenable to induction on length & to well-defined changes-of-coord's on closed cells

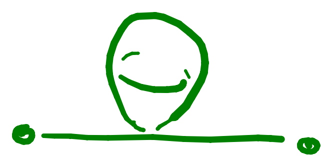
# New Regularity Criterion:

**Prop'n (H.)** Let $K$ be a finite CW complex w/ characteristic maps $\{f_\alpha\}$.

Suppose

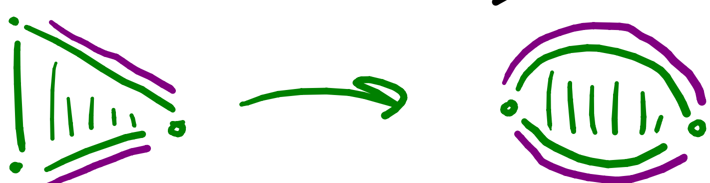
(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example:



(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example:



Then $F(K)$ is graded by cell dimension.

**Remark:** Next theorem "spreads around" injectivity requirement

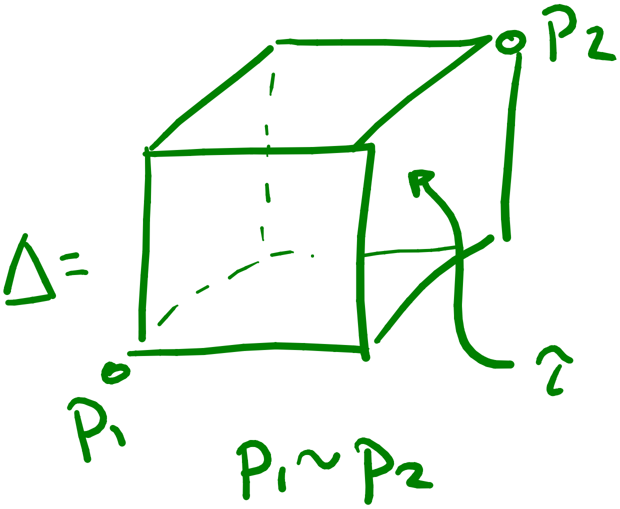
Thm (H.) Let $K$ be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then $K$ is regular w.r.t. $\{f_\alpha\}$ $\iff$

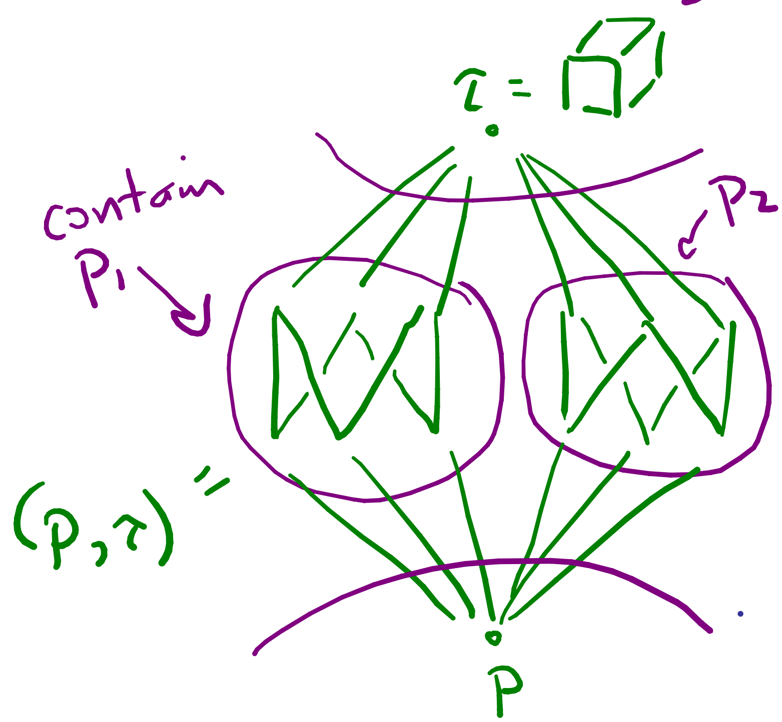
(1) $K$ meets requirements of prop'n for $F(K)$ to be graded by cell dim.

(2) $F(K)$ is thin and each open interval $(u,v)$ for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

## Non-Example



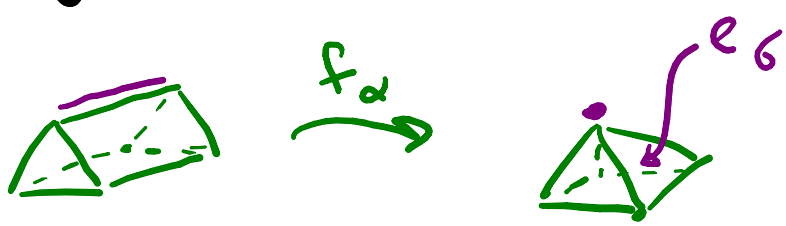
(3) For each $\alpha$, the restriction of $f_\alpha$ to preimages of codim. one cells in $\overline{e_\alpha}$ is injective.

(topological condition)

Non-Example:

(4) $\forall\, e_\sigma \leq \overline{e_\alpha}$, $f_\sigma$ factors as continuous inclusion $i: B^{\dim \sigma} \to B^{\dim \alpha}$ followed by $f_\alpha$.

Non-Example: (due to David Speyer)




Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one

# Example Where Injectivity is (Much) Easier in Codimension One

By exchange axiom for Coxeter groups

$$s_1 s_2 s_1 s_3$$

$$\swarrow \quad \swarrow \quad \searrow \quad \searrow$$

| | | | |
|---|---|---|---|
| $s_1 s_2 s_1 -$ | $s_1 s_2 - s_3$ | $\cancel{s_1 - s_1 s_3}$ | $- s_2 s_1 s_3$ |
| $\shortparallel$ | $\shortparallel$ | $\cancel{\shortparallel}$ | $\shortparallel$ |
| 3214 | 2341 | ~~1243~~ | 3142 |

various ways to delete a letter obtaining reduced expression gives <u>distinct</u> Coxeter group elements

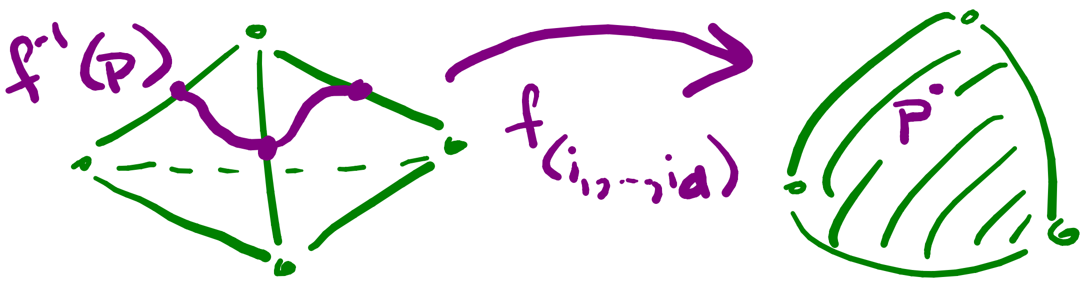
Higher Codim:
e.g.

$$s_1 s_2 s_1 s_3$$

$$\swarrow \qquad \searrow$$

$$s_1 - - - \qquad\qquad - - s_1 -$$

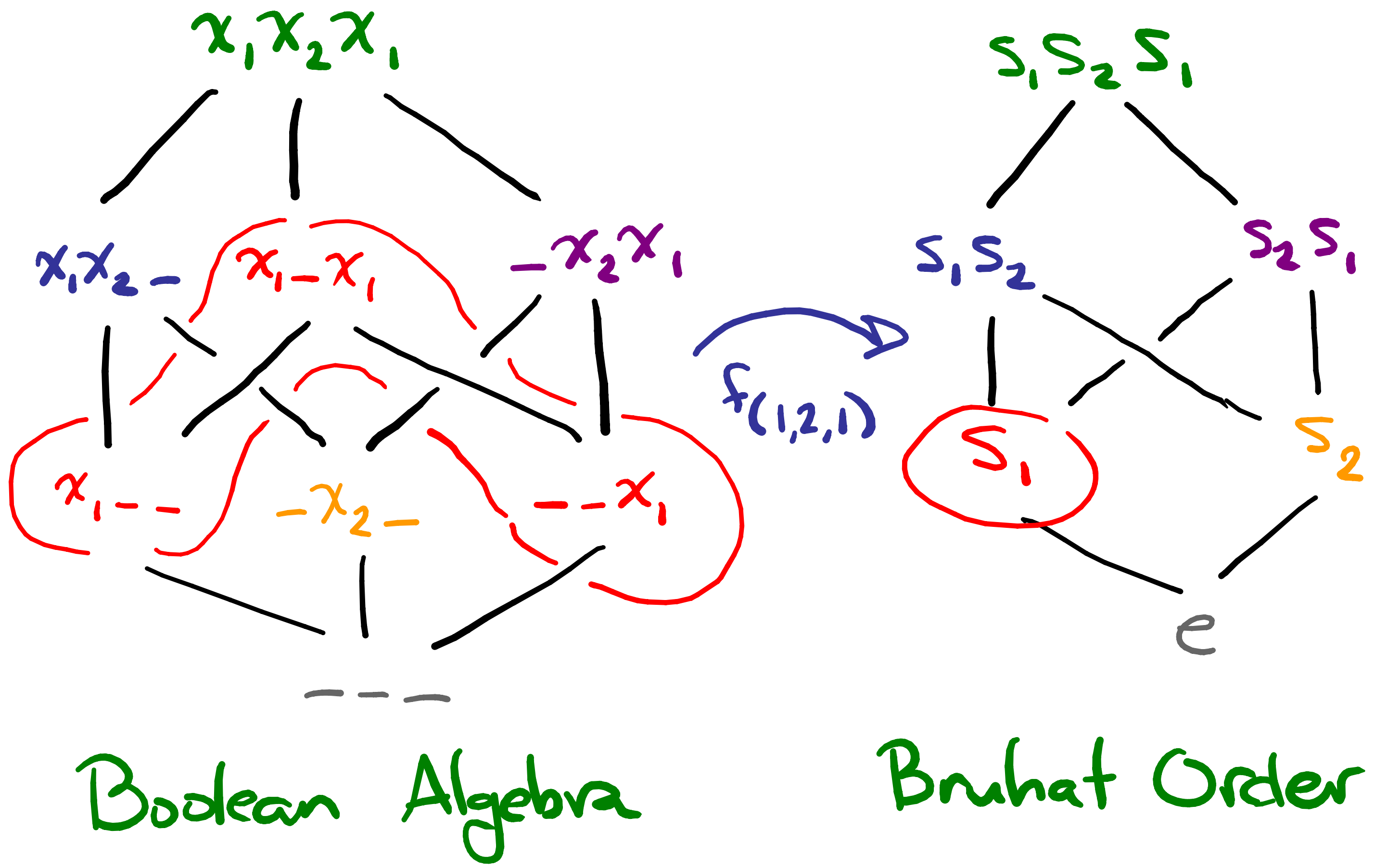
**Conjecture** (Davis-H.-Miller): $f^{-1}_{(i_1,\ldots,i_d)}(p)$ for each $p \in Y^\circ_w$ is a regular CW complex homeomorphic to a ball with closure poset dual to face poset for interior of subword complex $\Delta((i_1,\ldots,i_d),w)$.




**Remark:** Subword complexes, discussed next, first arose in work of Knutson & Miller on matrix Schubert varieties as Stanley-Reisner complexes of initial ideals of coordinate rings

**Qn:** Is there a reason / unifying picture why they arise in both settings?

# A Poset Map (on Face Posets) induced by $f_{(i_1,\ldots,i_d)}$ and an implicit definition of subword complexes




- Apply braid moves & $x_i^2 \to x_i$ to get reduced expression; replace $x_i$'s by $s_i$'s
- Fibers $f_Q^{-1}(u)$ are dual to face posets of subword complexes

"Topologist Approach" to Fomin-Shapiro Conjecture (joint work in progress with Jim Davis & Ezra Miller)

Combining Top'l Results: Let $g: B \to Z$ be continuous surjection from ball $B$ to Hausdorff space $Z$ whose restriction to $\mathrm{int}(B)$ is an embedding. Suppose also:

(1) $g(\partial B) \cong \partial B = S^n$;

(2) $g(\partial B) \cap g(\mathrm{int}(B)) = \emptyset$;

(3) $g^{-1}(p)$ is contractible $\forall p \in g(\partial B)$.

Then $Z \cong B$.

## Ingredients in this Relationship Between Fibers & Image of "Nice" Map:

- CE-approx. theorem: $g:\partial B \to \partial B$ as above may be approximated by homeomorphisms
  - Kirby-Siebenmann: dim $\geq 5$
  - Quinn: dim 4
  - Armentrout + Poincaré Conjecture: dim 3

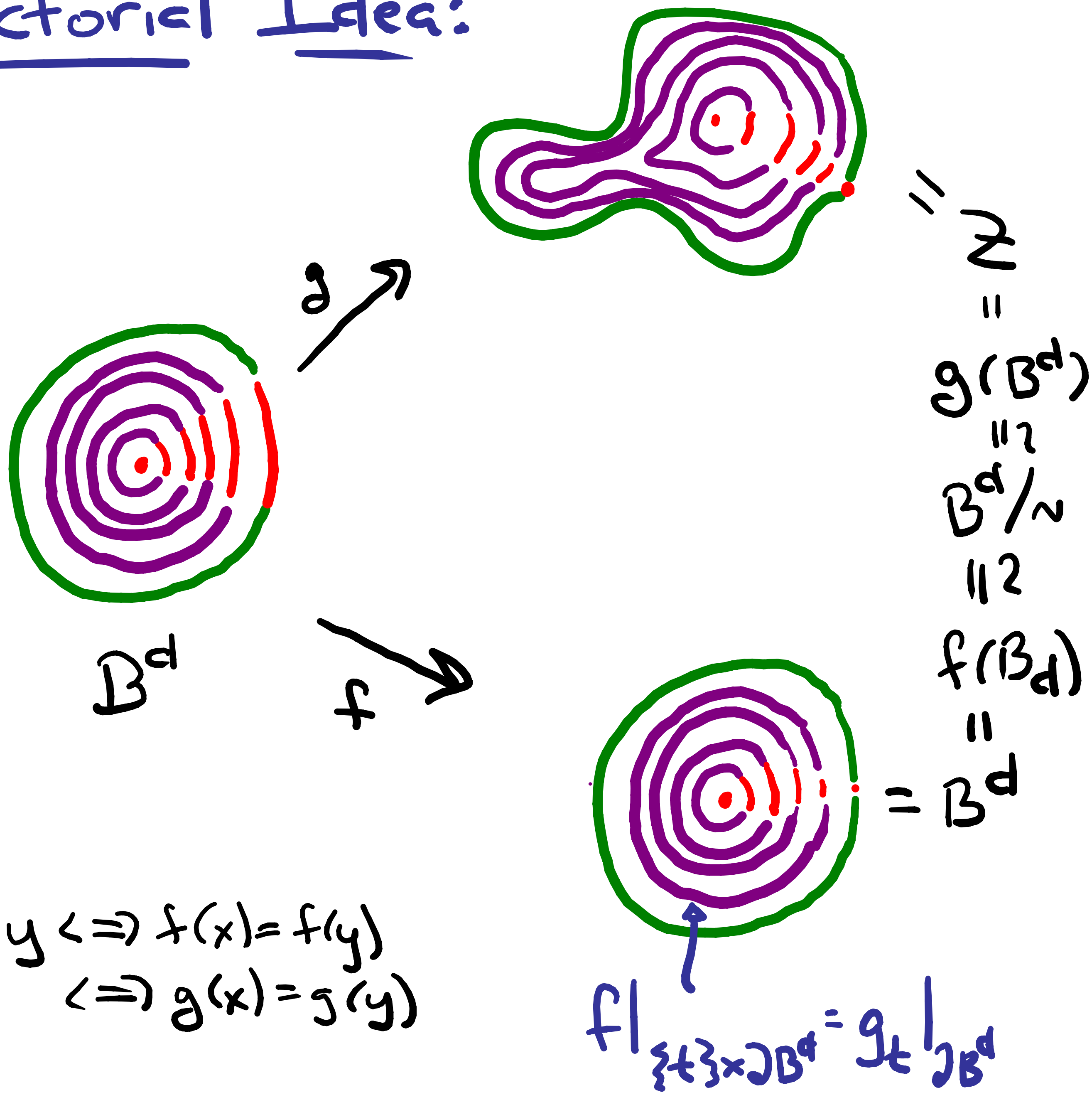
- Local Contractibility of Homeo$(S^n, S^n)$: two homeomorphisms "close enough" to each other may be connected by path of homeomorphisms

Idea: $B \cong$ metric ball $= \{\vec{0}\} \cup (0,1] \times \partial B$.

Use path of homeomorphisms converging to $g|_{\partial B}$ to construct $f: B \to B$ with $f^{-1}(p) = g^{-1}(p)\ \forall p \in B$ and $f|_{\partial B} = g|_{\partial B}$,

so $g(B) = B/\sim = f(B) \cong B$.

# Pictorial Idea:




$$x \sim y \iff f(x) = f(y)$$
$$\iff g(x) = g(y)$$

**Remark:** Completed proof of F-S Conjecture factored $f_{(i_1,\dots,i_d)}$ as series of such collapses where we construct paths of homeomorphisms & may check hypotheses combinatorially.

# Checking Sphericity for $f_{(i_1,\ldots,i_d)}(\partial \Delta^{d-1})$

1. Stratification has Bruhat intervals as closure posets.

2. By induction on dimension, cell closures in $f_{(i_1,\ldots,i_d)}(\partial B)$ are balls.

3. Hence $f_{(i_1,\ldots,i_d)}(\partial B)$ is regular CW complex, which we denote $K$.

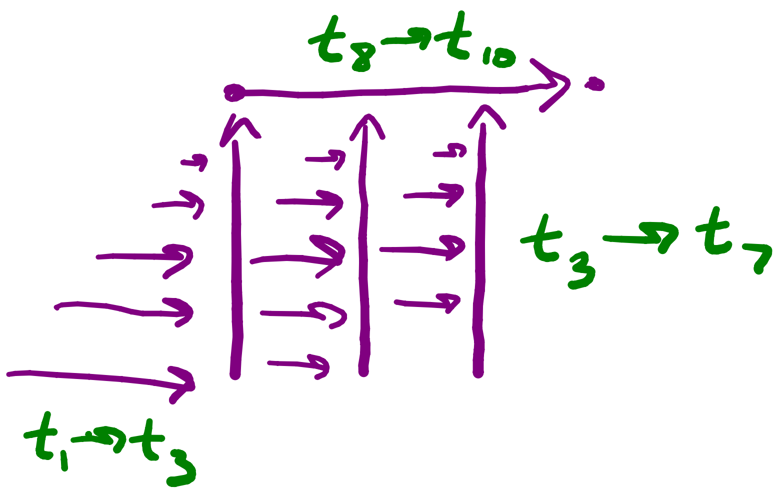
4. Hence, $K \cong \Delta(F(K)^{\setminus\{\hat{0},\hat{1}\}}) \cong$ sphere since $F(K)$ is Bruhat order, hence thin and shellable, thus a sphere.

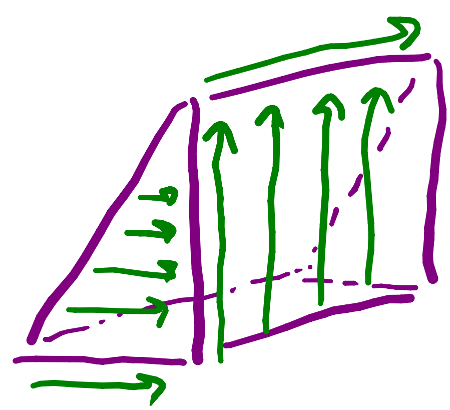
Upshot: Davis-H-Miller Conjecture would also imply Fomin-Shapiro Conjecture.

# "Flow" on a Fiber From Collapsing Process to Base Point of Fiber

e.g. $x_1 - x_1\, x_2\, x_1 - x_2\, x_3 - x_3$

$$\underbrace{\underbrace{x_1(t_1)\,x_1(t_3)}_{x_1(t_1+t_3)}\,x_2(t_4)\,x_1(t_5)\,x_2(t_7)}_{x_2(t_1')\,x_1(t_2')\,x_2(t_3'+t_7)}\,\underbrace{x_3(t_8)\,x_3(t_{10})}_{x_3(t_8+t_{10})}$$



## Further Questions

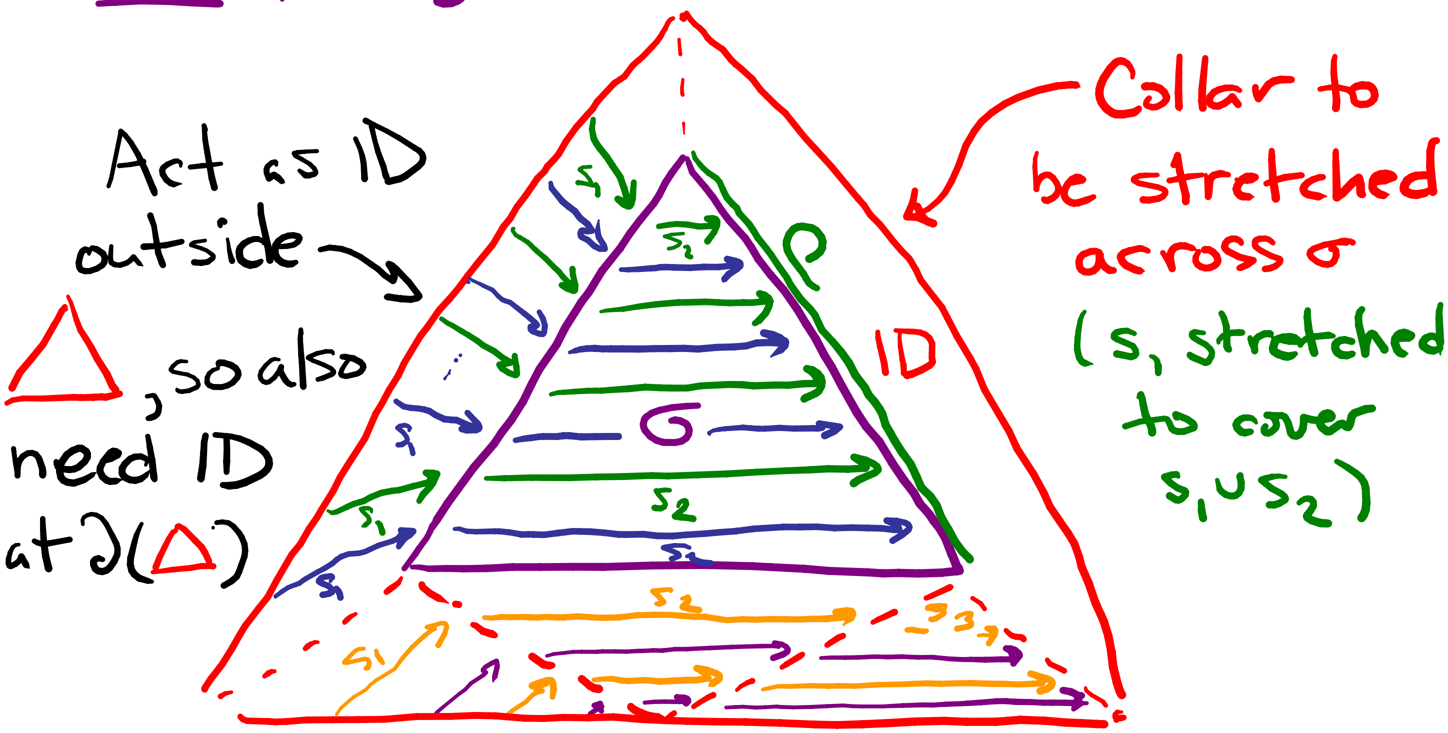
1. Analogous map, theory of "reduced expressions" & topol. results for totally nonnegative part of: Grassmannian? loop group? flag variety?
   (partial results of Postnikov, Rietsch, Williams, Speyer, Marsh, ...)
2. Explanation why subword complexes arising in distinct, but related settings? More general notion of subword complexes?

Thank you!

# Connection to Schubert Varieties & Bruhat Decompositions

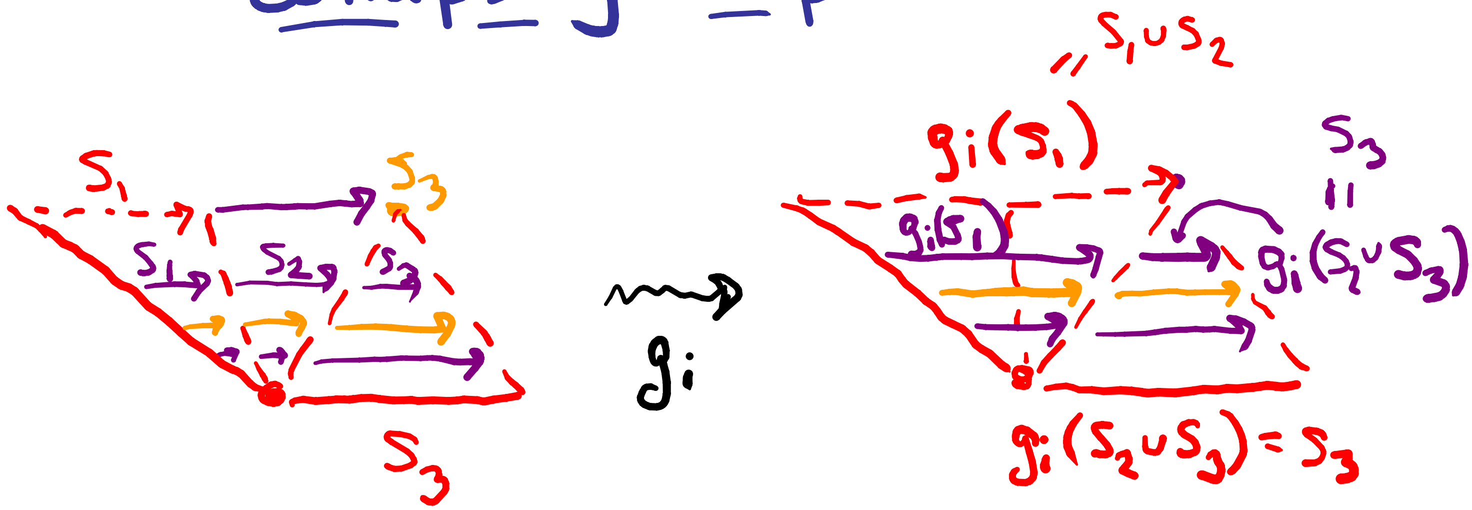
- $Y_w^\circ$ = image of $f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{>0} \to M_{n\times n}$
- $Y_w = \overline{Y_w^\circ}$ = image from $\mathbb{R}^d_{\geq 0}$
  = totally nonnegative part of $\overline{B^- w B^-} \cap \begin{pmatrix}\text{unipotent} \\ \text{radical of } B\end{pmatrix}$
- $Y_{w_0}$ = totally nonnegative part of space of upper triangular matrices w/ 1's on diagonal
  (old result of Whitney-type A)

# Collapsing a Cell $\bar{\sigma}$ onto a Cell $\bar{\rho}$




- Map segments $s_2$ in $\bar{\sigma}$ onto endpoint in $\bar{\rho}$, stretch extension $s_1 \subseteq$ collar to cover $s_1 \cup s_2$, act as ID on $\bar{\rho} \times [0,1] \subseteq$ collar.
- For $c \subseteq \partial\sigma$, collapsing map on $c \times [0,1]$ will stretch $s_1$ to cover $s_1 \cup s_2$ & shorten $s_2 \cup s_3$ to cover $s_3$, as depicted next.

# "Close-up" of bottom part of collapsing map



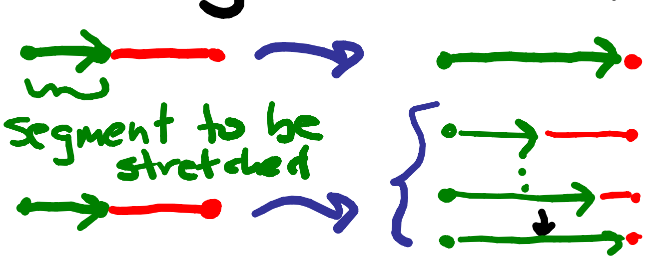

## Key Observations:

(1) This type of collapse makes sense more generally, relying on existence of continuous fn $\ln : \bar{\sigma} \to \mathbb{R}$ sending point to "length" of segment in $\bar{\sigma}$ containing it.

(2) These collapses are explicitly approximable by homeomorphisms:

Approximate

by



# (Mainly Combinatorial) Requirements Enabling Collapses Across Curves

There is a series of earlier face collapses

$= \Delta$ (as $g_1$ is surjection onto $\Delta$)

$$\begin{tikzcd} K_0 \arrow[r, "g_1"'] \arrow[d, phantom, "\shortparallel" rotate=90] & K_1 \arrow[r, "g_2"'] \arrow[d, phantom, "\cong" rotate=90] & K_2 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \cdots \arrow[r] & K_i \\ \Delta & \Delta/\sim_1 & \Delta/\sim_2 & & \end{tikzcd}$$

(new cell structure)

with closed cell of $K_i$ covered by images of parallel line segments in $K_0$ with family $\mathcal{C}_i$ of "parallel-like" curves satisfying:

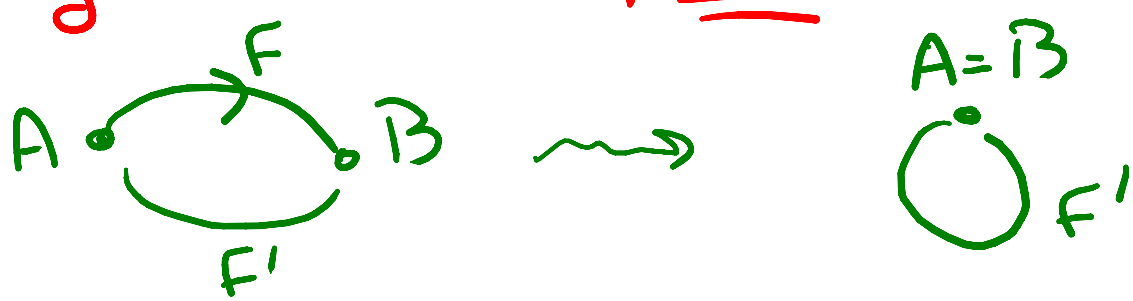
- **Distinct endpoints condition (DE):** the two ends of a nontrivial curve live in cells not yet identified

- **Distinct initial points condition (DIP):** distinct curves have distinct starting points (so collapse well-defined)

Condition to ensure collapses preserve regularity (suggested by David Speyer)

**Least Upper Bd Condition:**
If cells A & B are IDed via face collapse of F, then all least upper bounds for A & B just prior to collapse must also be collapsed in this step.

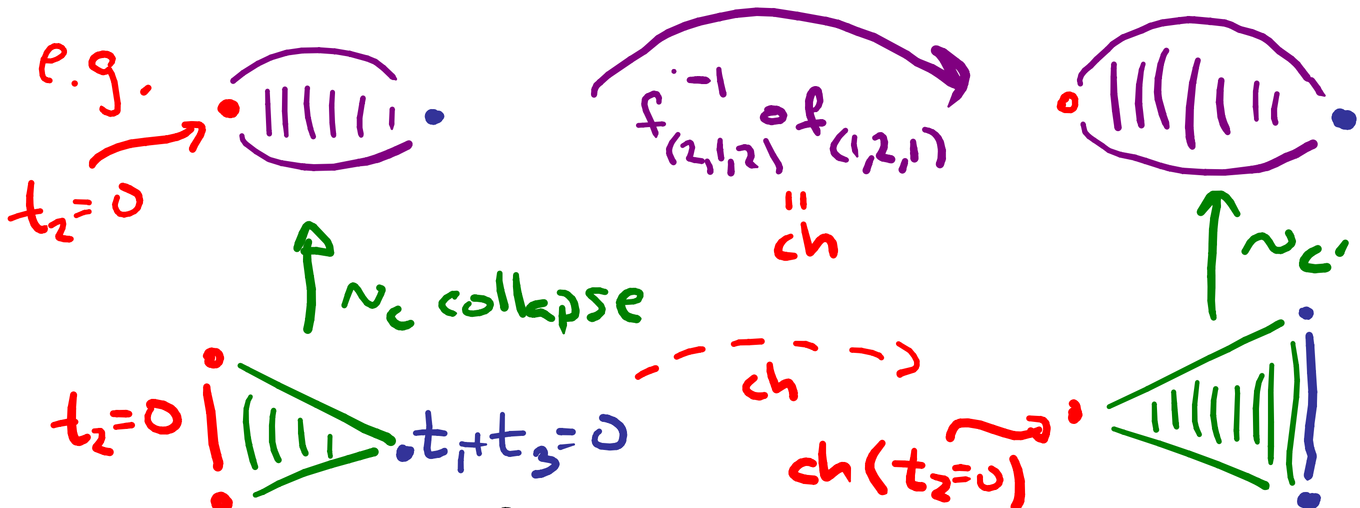
e.g. Want to prevent




Note: Conditions checkable with combinatorics of reduced/nonreduced words of 0-Hecke algebra!

# Long Braid Move as Change of Coord's

## Homeomorphism on Closed Cell to Be Collapsed




**Key Lemma:** Consider reduced expressions $s_is_js_i\ldots$ and $s_js_is_j\ldots$ of length $m(i,j)$ and equivalence relations $\sim_c$ and $\sim_{c'}$ on $\Delta^{m(i,j)-1}_{s_is_j\ldots}$ and $\Delta^{m(i,j)-1}_{s_js_i\ldots}$ given by identifications based on commutation and "slide moves". Then $\Delta^{m(i,j)-1}_{s_is_j\ldots}/\sim_c \cong \Delta^{m(i,j)-1}_{s_js_i\ldots}$ via the homeomorphism $f^{-1}_{(j,i,\ldots)} \circ f_{(i,j,\ldots)}$.

**Idea:** Subwords of $(i,j,\ldots)$ and $(j,i,\ldots)$ do not admit any long braid moves, so:

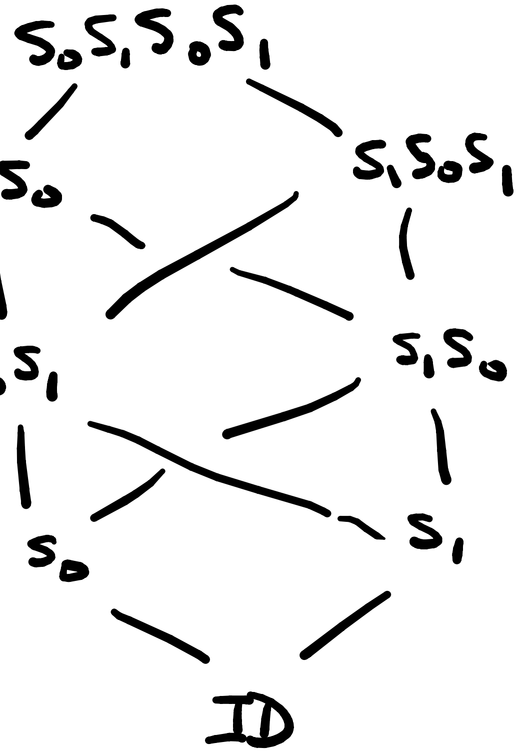
$$
\begin{tikzcd}
\Delta^{m(i,j)-1}_{s_is_j\ldots}/\sim_c \arrow[rr, "\bar{f}_{(i,j,\ldots)} = \text{homeom.}"] \arrow[rr, bend right] \arrow[d, dashed, "f^{-1}_{(j,i,\ldots)} \circ f_{(i,j,\ldots)}"'] & & Y_{s_0s_1\ldots} \arrow[dll, bend right, "\bar{f}^{-1}"'] \\
\Delta^{m(i,j)-1}_{s_js_i\ldots}/\sim_{c'} \arrow[urr, "\bar{f}_{(j,i,\ldots)} = \text{homeom.}"'] & &
\end{tikzcd}
$$

**Type B Example:**

$s_0s_1s_0\ldots = s_0s_1s_0$

$\ldots s_0s_1 \sim_c s_0 \ldots s_1 \sim_c s_0s_1 \ldots = s_0s_1$

"slide moves"



# Verifying DE (Distinct Endpoints Condition) with Combinatorics (Gives Flavor of Many Lemmas)

Suppose collapse of $F$ uses curves starting in $G_1$ and ending in $G_2$

Deletion pair

$$
\begin{array}{ccccccc}
\cdots & x_{i_r} & \cdots & x_{i_t} & \cdots & x_{i_s} & \cdots \\
 & \Uparrow & & \Uparrow & & \Uparrow & \\
 & G_1 & \longrightarrow & G' & \longrightarrow & G_2 &
\end{array}
$$

If $G_1$ were already identified earlier with $G_2$ then there exists $G'$ with earlier steps identifying $G_1$ with $G'$ and $G'$ with $G_2$. But the former would have also identified $G_1 \cup \{x_{i_s}\} = F$ with the cell $G' \cup \{x_{i_s}\} = F'$ which was already collapsed in step identifying $G'$ with $G_2$ $\Rightarrow\Leftarrow$

# Subword Complexes (introduced by Knutson & Miller)

$Q :=$ (not necessarily reduced) expression

$w :=$ Coxeter group element

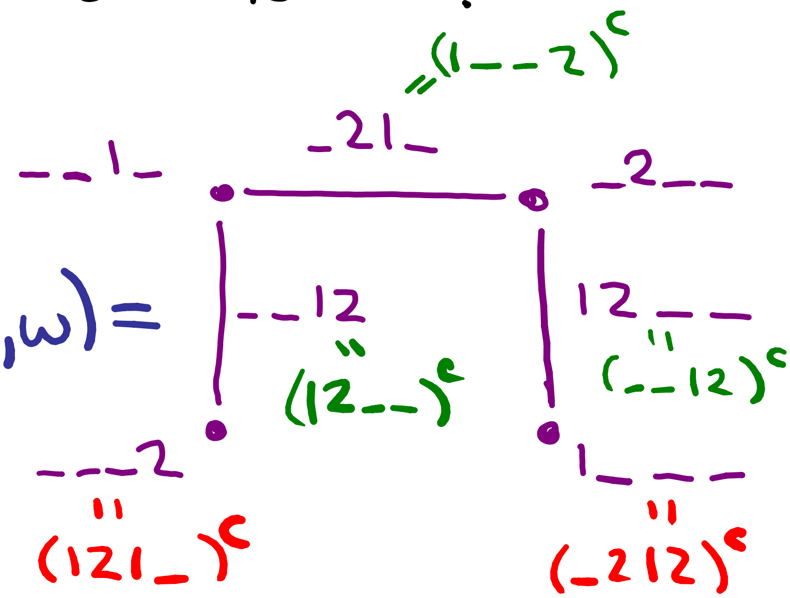
Facets of $\Delta(Q,w)$ are the subwords of $Q$ whose complements are reduced words for $w$.

e.g.

$Q = (1,2,1,2)$

$w = s_1 s_2$

$\Delta(Q,w) =$




**Thm** (Knutson-Miller): $\Delta(Q,w)$ is vertex decomposable (hence shellable) ball or sphere.

**More Generally?** "Fibers" of Parametrization Maps for Nonneg flag variety, loop groups, etc.?

# Homotopy Type of Bruhat Intervals: New Proof by Quillen Fibre Lemma

**Thm** (Armstrong-H.): The poset map $f_{(i_1 \ldots i_d)}$ yields short proof of:

$$\Delta_{Bruhat}(u,v) \simeq S^{rk\,v - rk\,u - 2} \quad \text{for all } u \leq v$$

**Idea:** • fibers $f_{\geq}^{-1}(u) = \{x \in B_n \mid f(x) \geq u\}$ are dual to face posets of subword complexes – proven to be balls by Allen Knutson & Ezra Miller.

Subword complexes previously arose as:

Stanley-Reisner complex for Gröbner degeneration of matrix Schubert variety ideal (Knutson and Miller)