

# Combinatorics & Topology of Totally Positive Spaces

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- Slides available at:

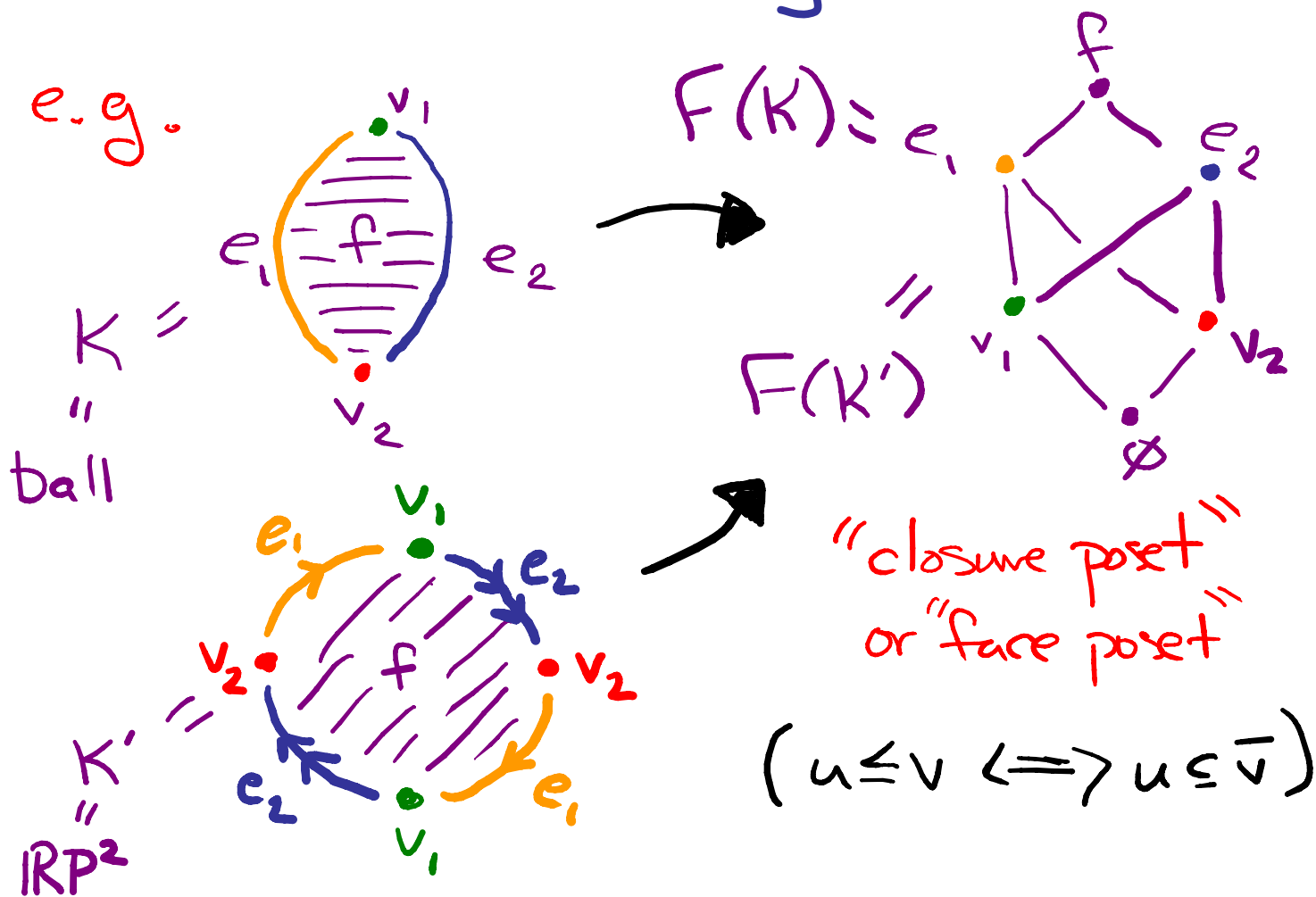
<http://www4.ncsu.edu/~p1herhsh>

- see "Regular cell complexes in total positivity", *Inventiones Mathematicae*, 197 (2014), 57-114 for details.

# Topological Aspects of Total Positivity

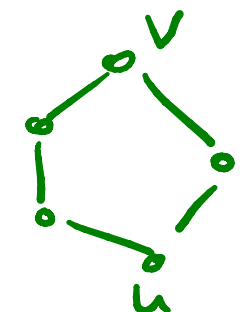
- ◆ Lusztig initiated study of **Totally nonnegative, real part of (matrix) Schubert varieties, flag varieties, ...**  
(i.e. part with minors all nonnegative in spaces of matrices or of flags)
- ◆ Conjecturally/provably homeomorphic to closed balls (after deconing)
- ◆ Proving this:
  - puts restrictions on relations among (exponentiated) Chevalley generators.
  - reveals structure in canonical bases of quantized univ. envelop. algebras of KM-algebras; a motivation for cluster algebras.
- ◆ Main Result of Talk: Proof of Fomin-Shapiro Conjecture via new tools exploiting interplay of combinatorial data & topological data.

# CW Complexes $\neq$ their Face Posets (i.e. Partially Ordered Sets)



Recall: A **CW complex**: cells  $e_\alpha \cong \mathbb{R}^{d(\alpha)}$ ,  
 characteristic maps  $f_\alpha: B^{\dim(e_\alpha)} \rightarrow \bigcup_{e_\beta \subseteq e_\alpha} e_\beta$   
 $\neq$  attaching maps  $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

• A poset is **graded** if  $u \leq v$  in  $P$  implies all minimal paths upward from  $u$  to  $v$  have same length (i.e. #edges)

e.g.  is not graded

• A graded poset is **thin** if each rank 2 interval has exactly 4 elements.

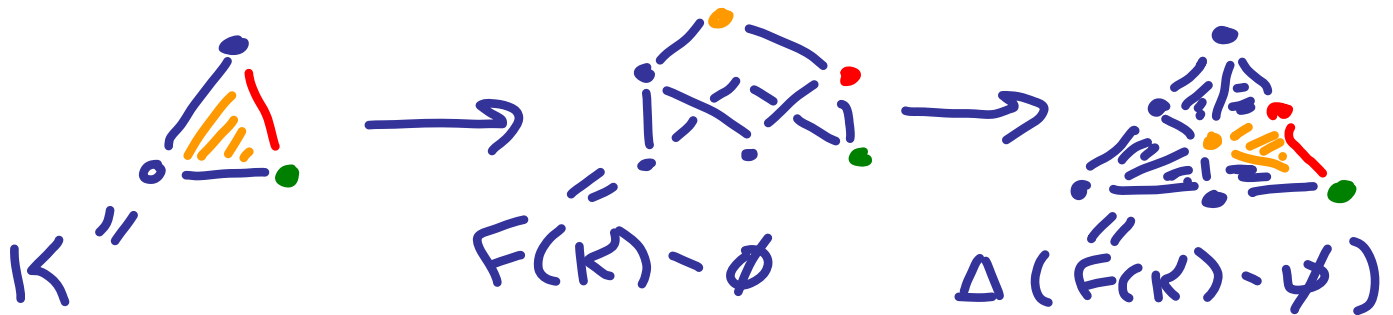


Recall: A CW complex is **regular**

if the attaching map  $f_\alpha$  for each cell  $e_\alpha$  is a homeomorphism, i.e. cell closures are closed balls.

Recall:  homeomorphic

- $K$  regular  $\Rightarrow K \cong \Delta(F(K) - \emptyset) = \text{sd}K$   
 $\underbrace{\hspace{1cm}}$   
 nerve or order complex,  
 i.e. simplicial complex whose  
 faces are chains  $u_1 < \dots < u_i$ .



Defn (Björner): A finite, graded poset  $P$  is CW poset if

- $P$  has unique min'l elt.  $\hat{0}$
- $P$  has additional element(s)
- $x \neq \hat{0} \Rightarrow \Delta(\hat{0}, x) \cong S^{\text{rank}(x)-2}$

Thm (Björner):  $P$  is CW poset  $\Leftrightarrow$   
 there exists regular CW complex  
 $K$  with  $P = F(K)$ .

## Some Examples of CW Posets

- Bruhat order (Björner & Wachs) of finite Coxeter group
- Closure poset for double Bruhat decomp. of totally nonneg. part of flag variety (Williams)
- thin, "shellable" posets (Danaraj & Klee)
- Closure poset of double suspension of triangulation of homology sphere with "big cell" glued in (due to work of J. Cannon & R. Edwards)

A Goal of Mine: Use combinatorics of  $F(K)$  + limited topological info (codim. one cell incidences) to understand  $K$ .

# Finite Coxeter Groups $\hat{=}$ finite

## Reflection Groups: Quick Review

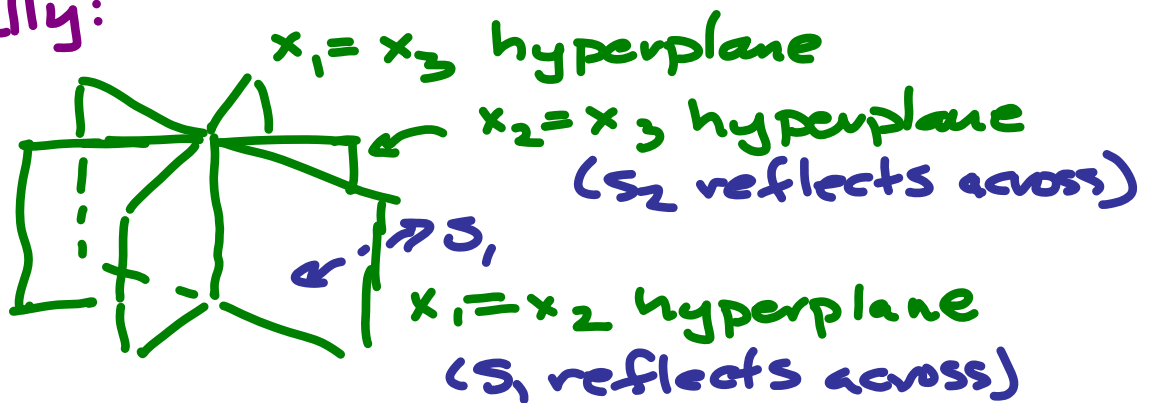
Coxeter group: generators  $\{s_i \mid i \in S\}$

$$\hat{=} \text{rels } s_i^2 = e \hat{=} (s_i s_j)^{m(i,j)} = e$$

e.g.  $S_3 = \langle \underset{\substack{\text{"} \\ (1,2)}}{s_1}, \underset{\substack{\text{"} \\ (2,3)}}{s_2} \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle$   
Symmetric group ( $\hat{=} \text{ other Weyl gps}$ )

geometrically:

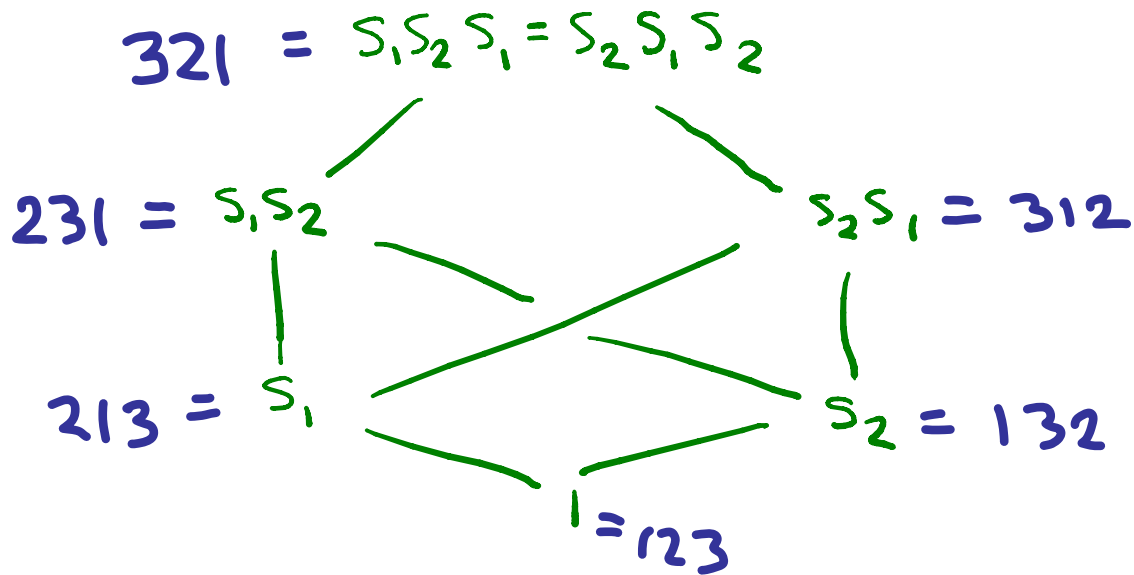
(as group of reflections)



A **reduced expression** for  $w \in W$  is an expression of minimal "length" e.g.  $s_1 s_2$  but not  $s_1 s_2 s_1 s_2$  (which equals  $s_2 s_1$ )

The **Bruhat order** is partial order on elements of Coxeter group  $W$  with  $u \leq v$   $\Leftrightarrow$  there exist **reduced expressions**  $r(u)$  and  $r(v)$  for  $u \neq v$ , respectively, with  $r(u)$  subexpression of  $r(v)$ .

e.g.  $W = S_3$  with generators  $s_1 = (1,2)$   
 $s_2 = (2,3)$




- **reduced word**  $(i_1, \dots, i_d)$  for  $s_{i_1} s_{i_2} \dots s_{i_d}$
- Closure poset for Schubert cell decompositions of flag varieties  $G/B$  (all cells have even real dim)



Qn (Bernstein & Björner): Find regular CW complexes naturally arising from rep'n theory which are homeomorphic to closed balls and have the (lower) Bruhat intervals as closure posets.

Conjectural Solution (Fomin & Shapiro):

The Bruhat stratification of  $\text{lk}(\text{id})$  in totally nonneg. real part of unipotent radical of Borel in semisimple, simply connected algebraic group defined and split over  $\mathbb{R}$ .

i.e. a "slice"  within:

$$U_{i\omega} = \left[ \overline{B^- \omega B^-} \cap (\text{unipotent subsp of } B) \right]_{\geq 0}$$

lower triangular opposite Borel  $B^-$   $\nearrow$   
 permutation  $\omega$   $\nearrow$   
 upper triang. w/ 1's on diagonal  $\nearrow$  totally nonneg. part

Theorem (H.): Fomin-Shapiro

Conjecture indeed holds.

Special Case (Running Example for Talk):

Space of totally nonnegative upper triangular matrices with 1's on diagonal & entries just above diagonal summing to fixed, positive constant, stratified by which minors are positive and which are 0.

Concrete Realization: products of certain elementary matrices, by results of Whitney & Lusztig.

# The Totally Nonnegative Part of a Space of Matrices

$\bullet \chi_i(t) = I_n + t E_{i,i+1} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1+t \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$

(general finite type, exponential Chevalley generator)

Annotations:
 

- $\exp(te_i)$  (with  $i$  above) points to  $I_n$
- (type A) points to  $E_{i,i+1}$
- column  $i+1$  (with arrow) points to the  $t$  entry in the matrix
- row  $i$  (with arrow) points to the  $t$  entry in the matrix

$\bullet f_{(i_1, \dots, i_d)}: \mathbb{R}_{\geq 0}^d \longrightarrow M_{n \times n} \subseteq \mathbb{R}^{n^2}$

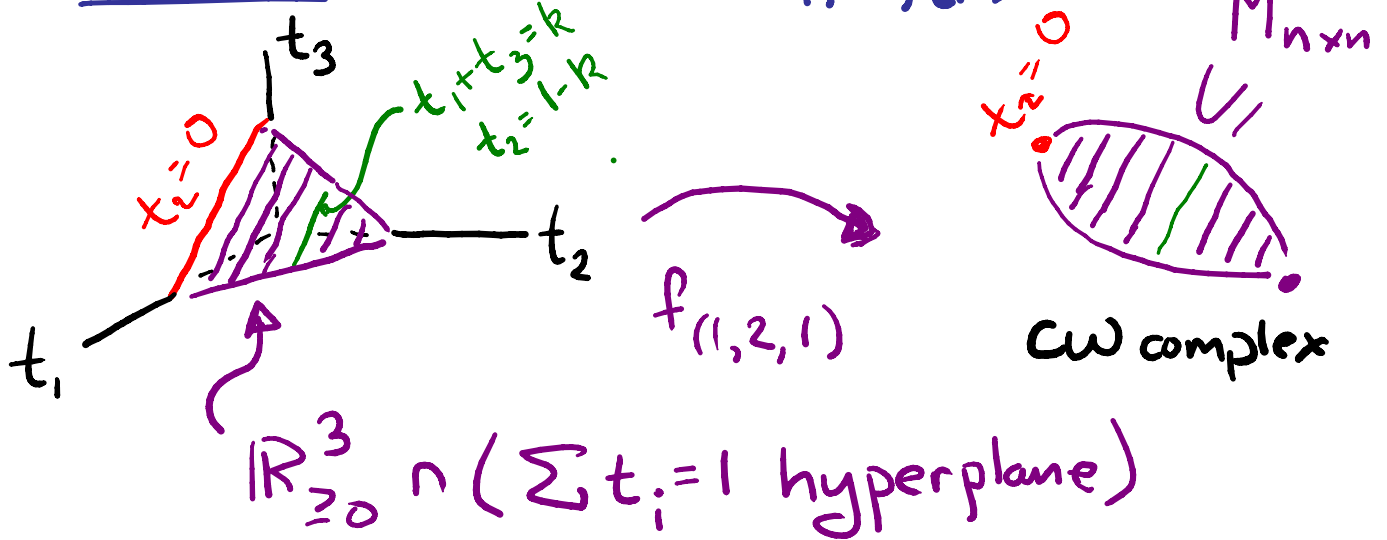
$(t_1, \dots, t_d) \longmapsto \chi_{i_1}(t_1) \cdots \chi_{i_d}(t_d)$

e.g.  $f_{(1,2,1)}(t_1, t_2, t_3) = \chi_1(t_1) \chi_2(t_2) \chi_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1+t_2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1 t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

# "Picture" of Map $f_{(1,2,1)}$



$$f_{(1,2,1)}(t_1, t_2, t_3) = \begin{pmatrix} 1 & t_1 \\ & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & t_3 \\ & 1 \\ & & 1 \end{pmatrix}$$

$t_2 = 0$

$$x_i(t_1) = x_i(t_3)$$

$$\begin{aligned} f_{(1,2,1)}(t_1, 0, t_3) &= \begin{pmatrix} 1 & t_1 \\ & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & t_3 \\ & 1 \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t_1 + t_3 \\ & 1 \\ & & 1 \end{pmatrix} = x_i(t_1 + t_3) \end{aligned}$$

Non-injectivity: results from "modified nil-moves"  $x_i(u)x_i(v) \rightarrow x_i(u+v)$  directly & after "long braid moves" in OHecke algebra.

# 1st Key Idea to Proof of F-S Conjecture

0-Hecke Algebra Captures which Simplex Faces have Same Image under  $f_{(i_1, \dots, i_d)}$

$$(1) x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$$

"nil-move"

↓ suppress parameters

$$x_i^2 = x_i \quad (\text{0-Hecke alg. rel'n, up to sign})$$

$$(2) x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2+t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1+t_2}{t_1+t_3}\right)$$

↓ (type A)

assuming  $t_1+t_3 > 0$

$$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$$

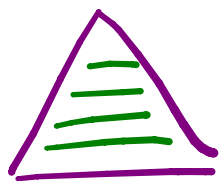
(similar relation holds outside type A)

"long braid move"  
with enrichment  
from parameters

Fibers as Curves:

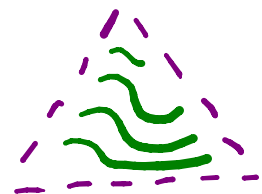
$$x_i^2 \rightarrow x_i$$

$$t_1+t_2 = k$$

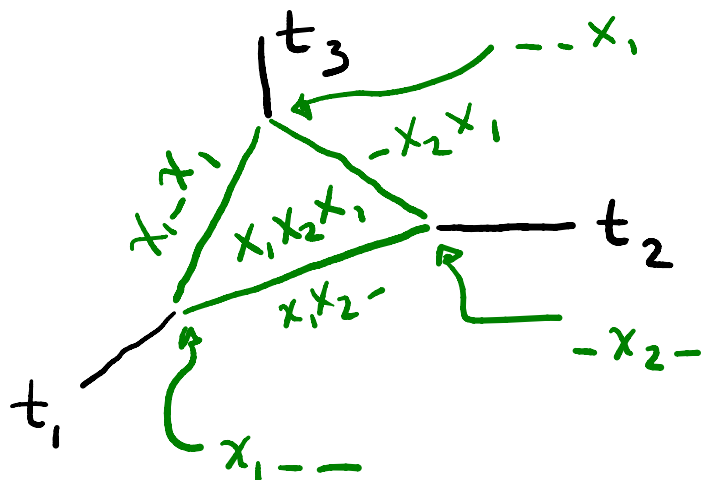


! !

OR after a braid move:



# Indexing Faces of Preimage by Words in 0-Hecke Algebra



Key Observation About  $f_{(i_1, \dots, i_d)}$ :

$$\text{im}(F_1) = \text{im}(F_2) \Leftrightarrow \underbrace{x(F_1) = x(F_2)}_{\text{equal as 0-Hecke algebra elements}}$$

equal as  
0-Hecke algebra elements

Thm (Lusztig): If  $(i_1, \dots, i_d)$  is reduced, then  $f_{(i_1, \dots, i_d)}$  is homeomorphism on  $\mathbb{R}_{>0}^d$

Observation: "non-reduced" subwords give redundant faces covered by curves, each in single fiber of  $f_{(i_1, \dots, i_d)}$

# Properties of Change-of-Coordinates Map Given by Braid Moves

e.g.  $(t_1, t_2, t_3) \mapsto \left( \frac{t_2 t_3}{t_1 + t_3}, t_1 + t_3, \frac{t_1 t_2}{t_1 + t_3} \right)$

in type A

- Tropicalizes to change-of-basis map for Lusztig's canonical bases:

$$(a, b, c) \mapsto (b + c - \min(a, c), \min(a, c), a + b - \min(a, c))$$

- A motivation for development of cluster algebras ( $\pm$  mutation)

Exercise: check this is an involution.

- Note: birational map which induces homeomorphism between quotient spaces

# Proof Strategy (Phrased for Possible Future Applic's too)

Set-up: Continuous, surjective fn

$$f: P \rightarrow Y$$

from convex polytope  $P$  (e.g.  $\Delta_n$ ) s.t.  $f$  maps  
 $\text{int}(P)$  homeomorphically to  $\text{int}(Y)$ .

Step 1: Perform "collapses" on  $\partial P$ , each  
preserving regularity and homeomorphism type,  
via continuous, surjective collapsing functions  
 $P \rightarrow P$  yielding  $P/\sim$  with fewer cells  
s.t.  $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$

(to accomplish IDs we know are needed  
by collapsing all non-reduced faces)

Step 2: Prove  $\tilde{f}: P/\sim \rightarrow Y$  is

homeomorphism by **new regularity criterion**

(to prove no further IDs are needed)

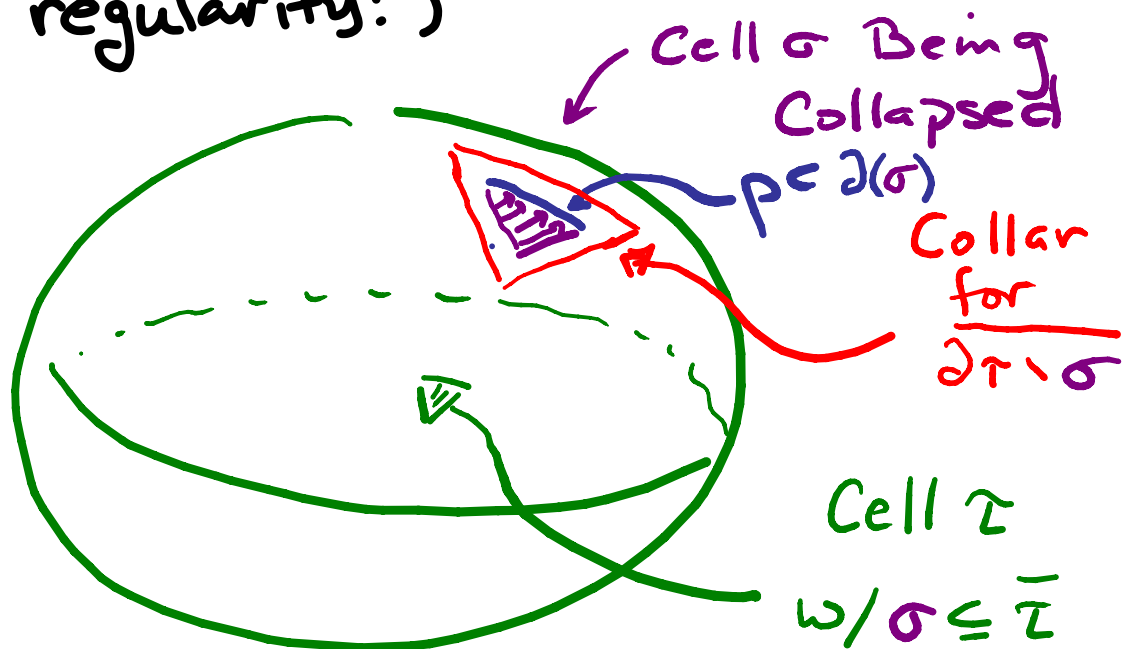


# Step 1 Collapsing Cell $\sigma$ onto Cell $\rho \subseteq \bar{\sigma}$ within $\partial \tau$

Thm (M. Brown; Cannonally): Any topological manifold with boundary  $\partial M$  has a collar (i.e. a nbhd homeomorphic to  $\partial M \times [0, 1]$ ).

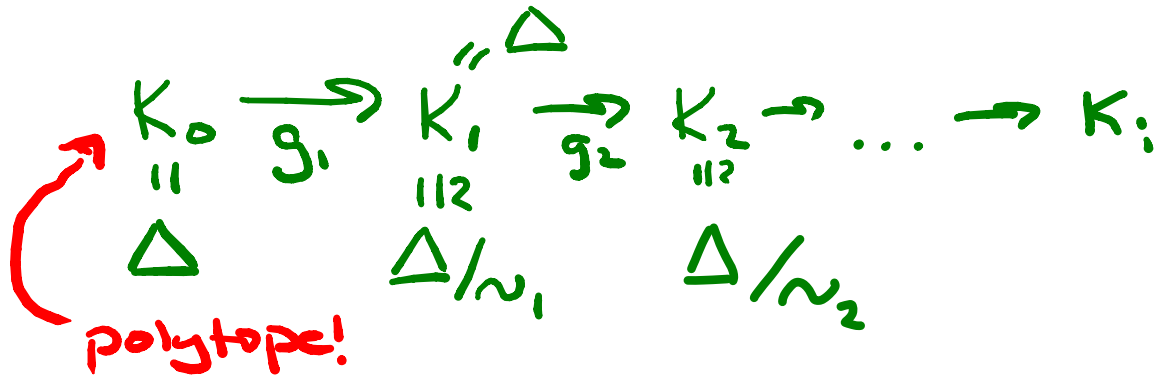
e.g.  $\overline{\partial \tau \setminus \sigma} =$

Plan: Collapse  $\bar{\sigma}$  onto  $\bar{\rho} \subseteq \partial \sigma$ , stretching collar for  $\overline{\partial \tau \setminus \sigma}$  to cover  $\bar{\sigma} \setminus \bar{\rho}$ , preserving top'l manifold structure ( $\ddagger$  homeom. type  $\ddagger$  regularity!)



$$\ddagger \dim \tau = \dim \sigma + 1$$

# (Mainly Combinatorial) Conditions Allowing Such Face Collapses Across Curves



• collapse face in  $K_i$  across images of parallel line segments in  $K_0$  satisfying:

• Distinct endpoints condition (DE):



• Distinct initial points condition (DIP):



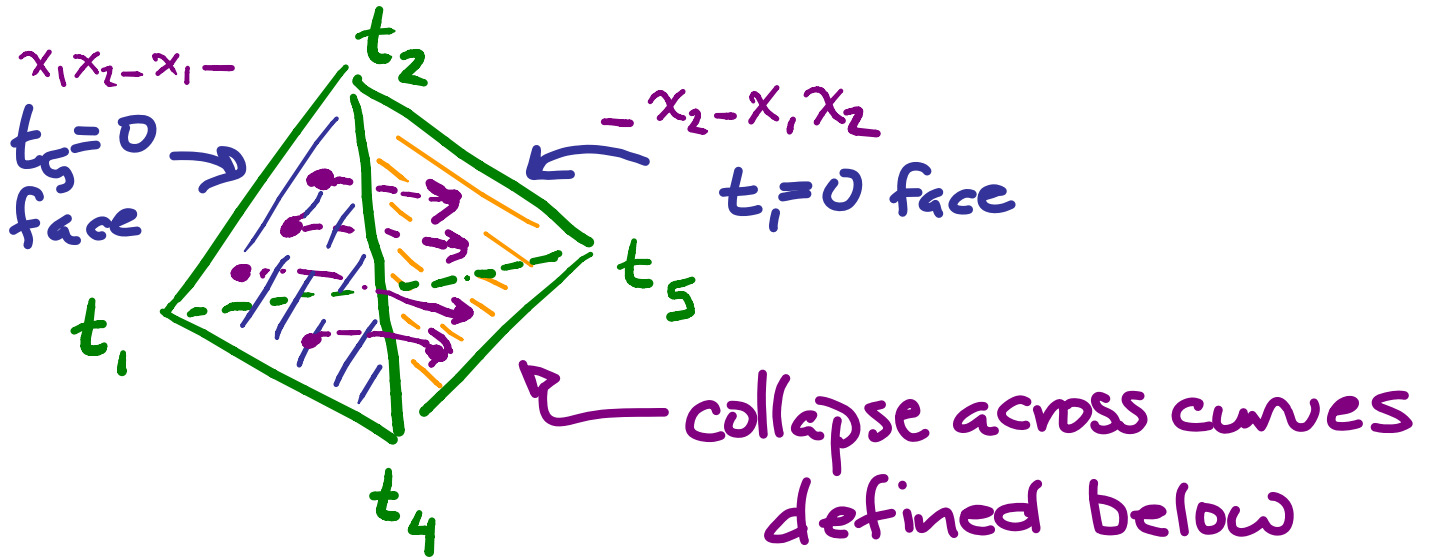
• Least upper bound condition (LUB)



(conditions checkable via O-Hecke algebra)

# Example of a Face Collapse

e.g.  $f(1,2,3,1,2)$  face with  $t_3=0$



$$(t_1, t_2, 0, t_4, t_5) \mapsto x_1(t_1) x_2(t_2) x_1(t_4) x_2(t_5)$$

for  $t_1 + t_4 > 0$

$$\begin{aligned} & \parallel \\ & x_2(t'_1) x_1(t'_2) x_2(t'_4) x_2(t_5) \\ & \parallel \\ & x_2(t'_1) x_1(t'_2) x_2(t'_4 + t_5) \end{aligned}$$

for  $t'_1 = \frac{t_2 t_4}{t_1 + t_4}$      $t'_2 = t_1 + t_4$      $t'_4 = \frac{t_1 t_2}{t_1 + t_4}$

**Curves:**  $t'_1 = k_1 \neq t'_2 = k_2 \neq t'_4 + t_5 = k_3$

Step 2: Proving that induced

map  $\bar{f}_{(1,1,\dots,1)}: \Delta_n / \sim \rightarrow \left\{ \begin{array}{l} \text{space} \\ \text{of} \\ \text{matrices} \end{array} \right\}$

on quotient

space

is a homeomorphism.

↑ identifications  
from  
collapsing  
non-reduced  
faces

# New Regularity Criterion for finite CW complexes

Preparatory Lemma (H.): Let  $K$  be a finite CW complex w/ characteristic maps  $\{f_\alpha\}$ . Suppose:

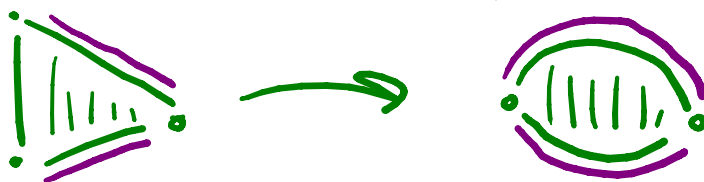
(1)  $\forall \alpha, f_\alpha(\partial B^{\dim \alpha})$  is a union of open cells (surjectivity)

Non-Example:



(2)  $\forall f_\alpha$ , the preimages of the open cells of codim. one in  $\bar{e}_\alpha$  are dense in  $\partial(B^{\dim \alpha})$

Non-Example:



Then  $F(K)$  is graded by cell dimension.

Insightful feedback (Quinn): Next theorem "spreads around" injectivity requirement.

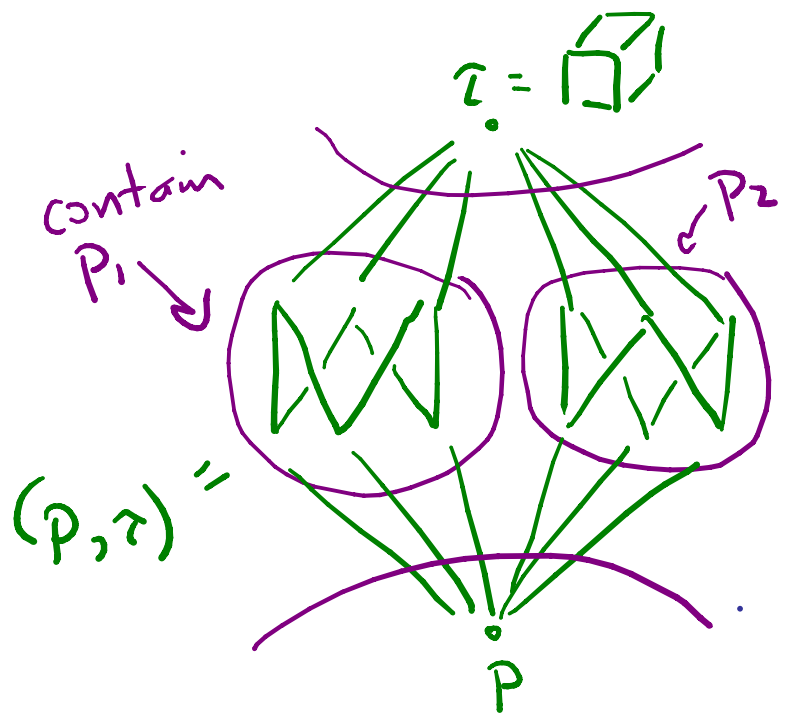
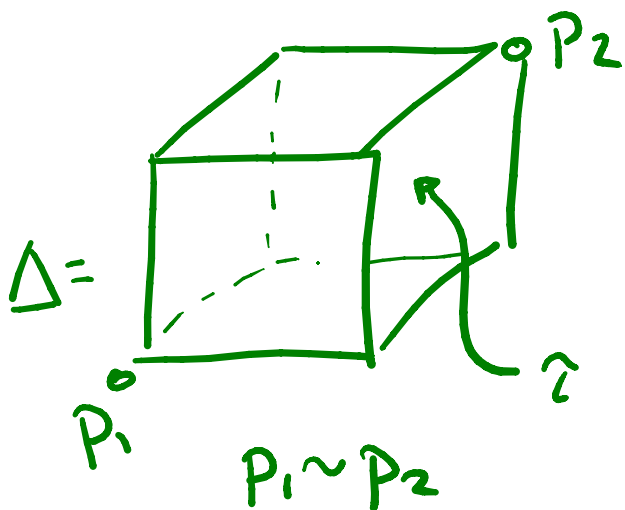
Thm (H.) Let  $K$  be finite CW complex w.r.t. characteristic maps  $\{f_\alpha\}$ . Then  $K$  is regular w.r.t.  $\{f_\alpha\} \iff$

(1)  $K$  meets requirements of prop'n for  $F(K)$  to be graded by cell dim.

(2)  $F(K)$  is thin and each open interval  $(u, v)$  for  $\dim(v) - \dim(u) > 2$  is connected (as graph)

(combinatorial condition)

### Non-Example



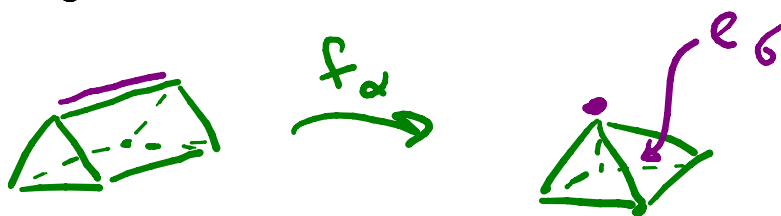
(3) For each  $\alpha$ , the restriction of  $f_\alpha$  to preimages of codim. one cells in  $\bar{e}_\alpha$  is injective.  
 (topological condition)

Non-Example:



(4)  $\forall e_\sigma \subseteq \bar{e}_\alpha$ ,  $f_\sigma$  factors as continuous inclusion  $i: B^{\dim \sigma} \rightarrow B^{\dim \alpha}$  followed by  $f_\alpha$ .

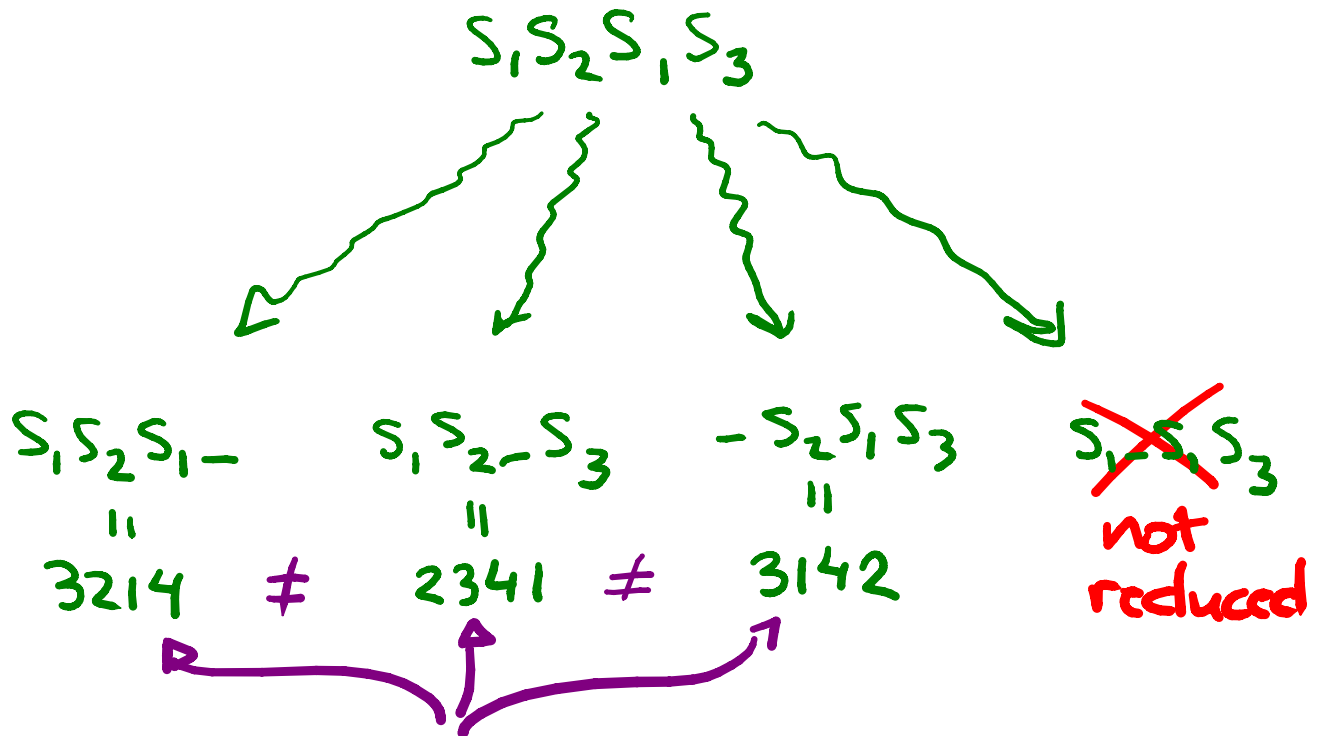
Non-Example:



Notably Absent: Injectivity requirement for  $\{f_\alpha\}$  beyond codim. one.

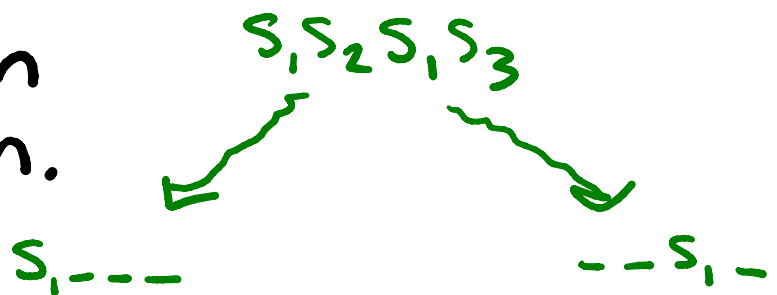
Proof: Induction on difference in dim.

# Injectivity of Attaching Maps in Codimension One via Coxeter group strong exchange axiom



reduced subexpressions of reduced expression obtained by deleting one letter give **distinct** Coxeter group elements.

In contrast: fails in higher codimension.





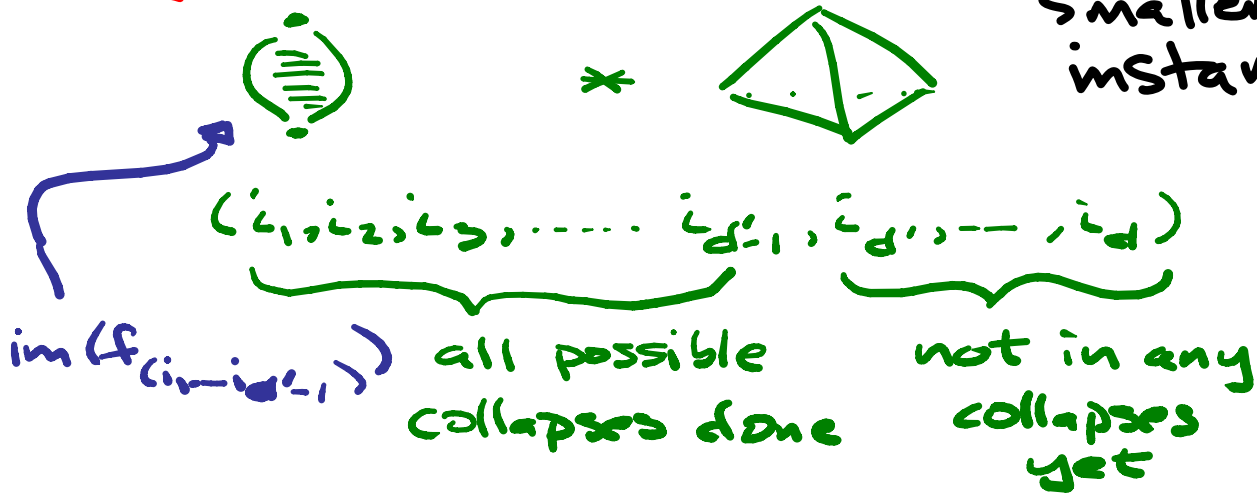
# Additional Key Challenges

1. 0-Hecke algebra lacks inverses!  
lacks cancellation law.

Key Idea: find ways to transfer properties from Coxeter gp

2. Need change-of-coords for braid moves as homeom's on closed cells

Key Idea: induction by embedding smaller instance



3. Need maps to extend to full complex

4. Tricky combinatorics to verify LUB at each collapsing step.

# Other Stratified Spaces with Seemingly Similar Features

1. Totally nonnegative part of  
Grassmannian

Postnikov: polytope of plabic graphs  
w/ "measurement map" to  $Gr_{\geq 0}$   
+ elaborate theory of plabic graphs

Postnikov-Speyer-Williams:  $Gr_{\geq 0}$   
is CW complex (via attaching  
maps that are not homeomorphisms)

2. Closed cells in totally  
nonnegative part of loop group

Lam-Plyevskiy: developed theory  
of these spaces

3. Totally nonnegative part of flag variety

Rietsch: poset of closure rel's

Marsh-Rietsch: parametrization

Williams: poset is CW poset

Rietsch-Williams: CW complex w/ attaching maps via canonical bases.

Note: our spaces arise as links of cells

4. Stratified spaces of electrical networks

Kenyon-Propp-Wilson, Lam,

Curtis-Ingerman-Morrow, Kenyon,...

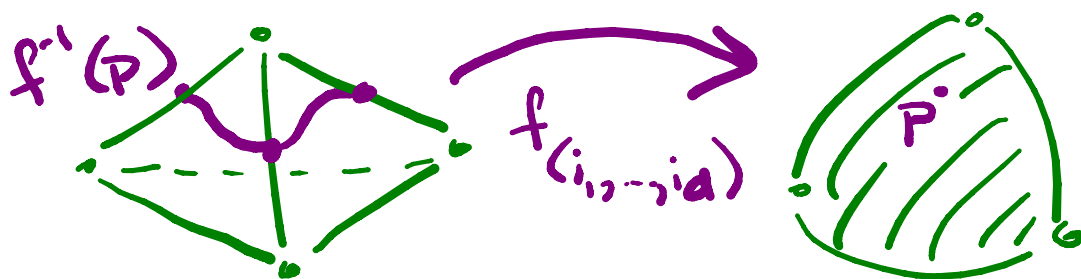
Open Qn: homeomorphism type & other topol. structure for these spaces?

# A Follow-up Project:

(with Jim Davis & Ezra Miller)

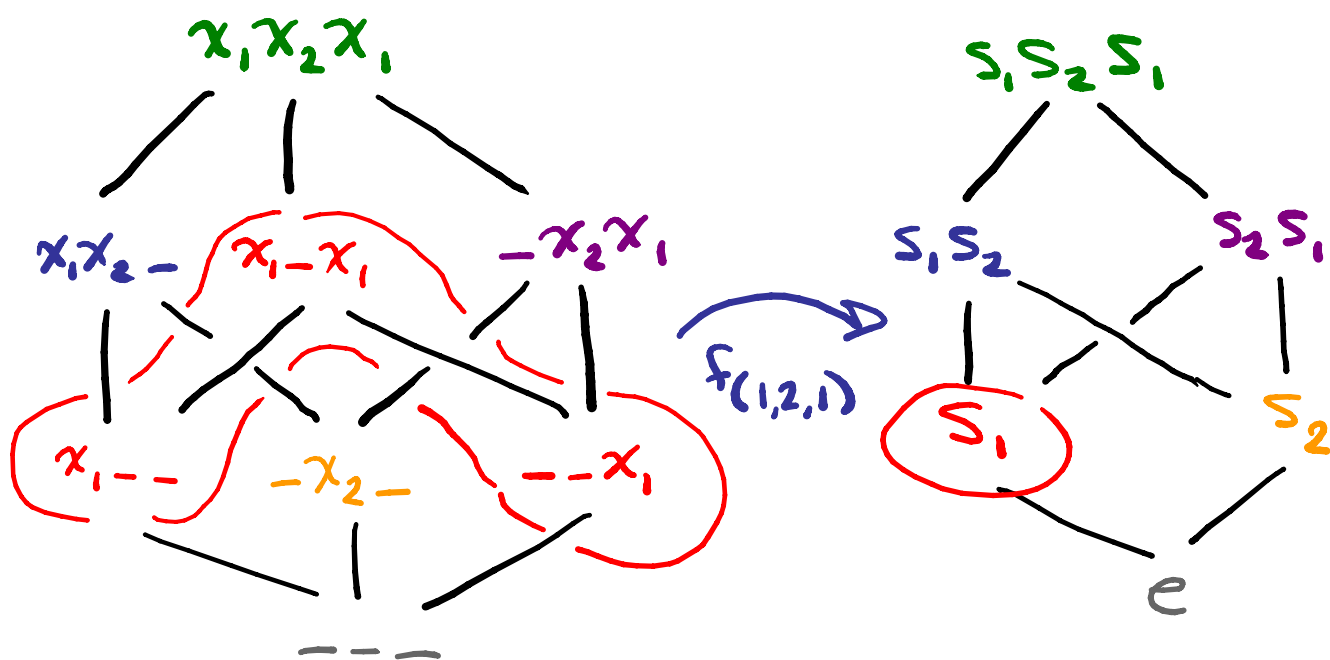
Conjecture (Davis-H. Miller):  $f_{(i_1, \dots, i_d)}^{-1}(p)$

for each  $p \in Y_\omega^\circ$  is a regular CW complex homeomorphic to a ball with closure poset dual to face poset for interior of "subword complex"  $\Delta((i_1, \dots, i_d), \omega)$ .



Remark: Subword complexes introduced by Knutson & E. Miller as Stanley-Reisner complexes of initial ideals of coordinate rings associated to matrix Schubert varieties.

A Poset Map (on Face Posets)  
induced by  $f(i, \dots, -id)$  ( $\dagger$  implicit  
Def'n of Subword Complexes)



Bodean Algebra  $B_n$

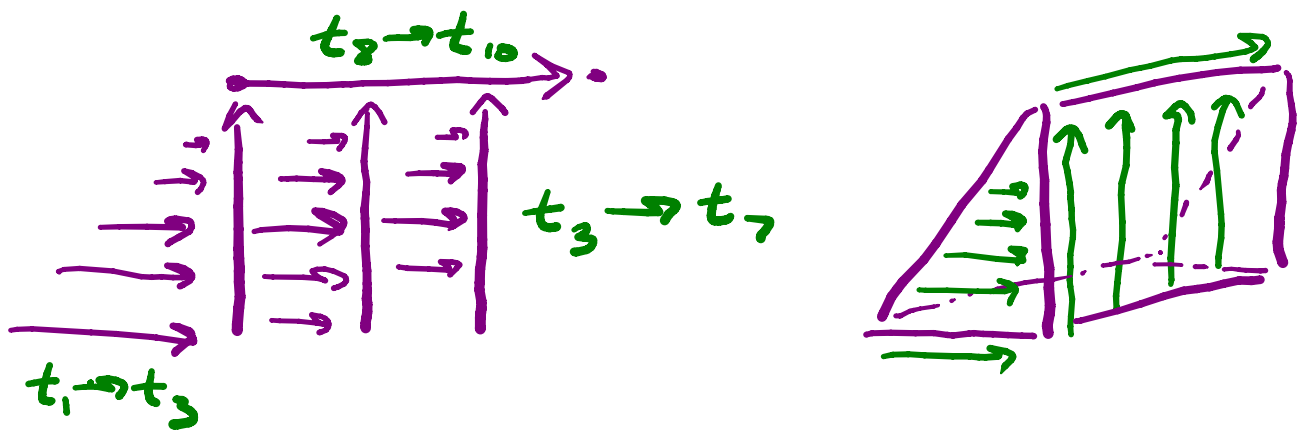
Bruhat Order

- Apply braid moves  $\dagger x_i^2 \rightarrow x_i$  to get reduced expression; replace  $x_i$ 's by  $s_i$ 's
- Fibers  $f_{\geq}^{-1}(u) = \{x \in B_n \mid f(x) \geq u\}$  are dual to face posets of subword complexes (fibers as in Quillen's Lemma A)

# Description of fibers via Flow (Based on Collapses) to Base Point

e.g.

$$\begin{array}{c}
 \xrightarrow{1} \quad \xrightarrow{2} \quad \xrightarrow{3} \\
 x_1 - x_1, x_2 \quad x_1 - x_2 \quad x_3 - x_3 \\
 \\
 \underbrace{x_1(t_1) \quad x_1(t_3)}_{x_1(t_1+t_3)} \quad \underbrace{x_2(t_4) \quad x_1(t_5) \quad x_2(t_7) \quad x_3(t_8) \quad x_3(t_{10})}_{x_3(t_8+t_{10})} \\
 \\
 \underbrace{x_2(t'_1) \quad x_1(t'_2) \quad x_2(t'_3+t_7)}
 \end{array}$$



- Remarks:
- DTM Conjecture  $\Rightarrow$  FS Conjecture (via CE-Approx. Thm)
  - Our proof factors  $f(i, \dots, t)$  as product of "nice" maps



2. Combinatorics of **generalized subword complexes** as combinatorial model for fibers

- Grassmannian?

- loop group?

Vertex decomposability?

Gallery connectedness?

3. Combinatorics of **generalized Bruhat order** as combinatorial model for images

- loop group?

- electrical networks?

Lam: proved thin

In particular: are these CW posets?



4. Explicit polytope & map for flag variety (perhaps by developing combinatorics of suitable reduced & non-reduced objects for canonical bases explicitly)
5. Explanation for subword complexes arising in seemingly disparate settings?
6. More general setting explaining very strong analogies between reduced words, reduced planar graphs, etc.

Thank you!