Posets Arising as 1-skeleta of Simple Polytopes, the Nonrevisiting Path Conjecture and Poset Topology

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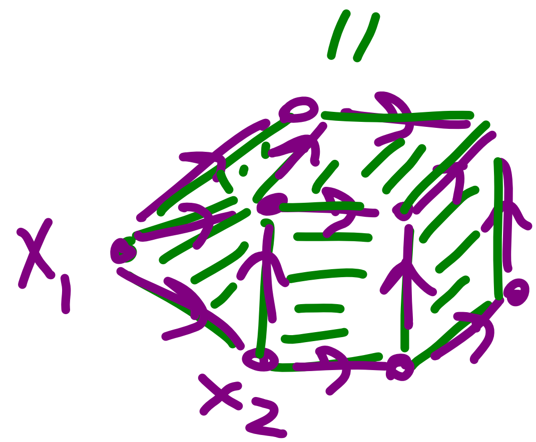
- with thanks to Karola Mészáros for fruitful discussions early in project

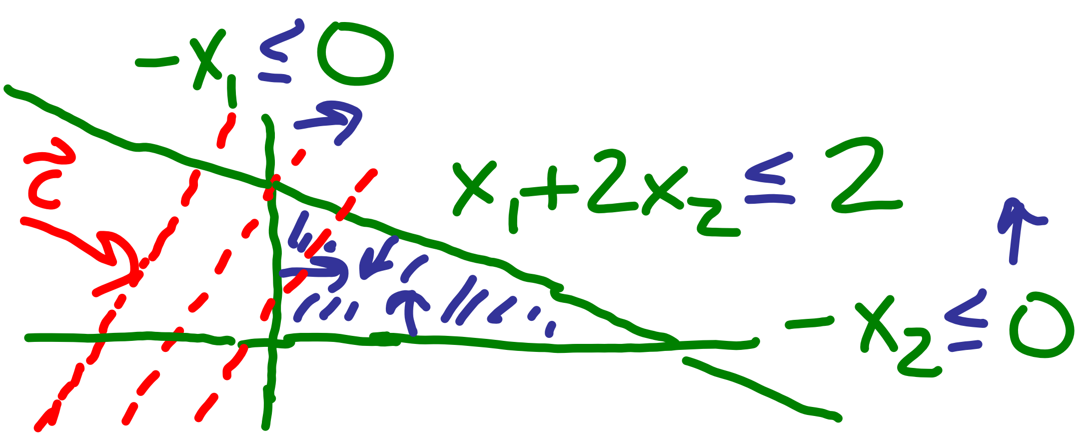
Linear Programming

- Given a matrix A & vectors $\vec{b}, \vec{c}$ seek $\max\{\vec{c}\cdot\vec{x} \mid A\vec{x} \le \vec{b}\}$
- $\{\vec{x} \mid A\vec{x} \le \vec{b}\}$ is polytope P if set is bounded



e.g.

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}} \le \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}_{\vec{b}}$$



Solving Linear Programs via Simplex Method

- Define $G(P,\vec{c}) :=$ directed graph on 1-skeleton of P, i.e. on vertex-edge graph of P, with

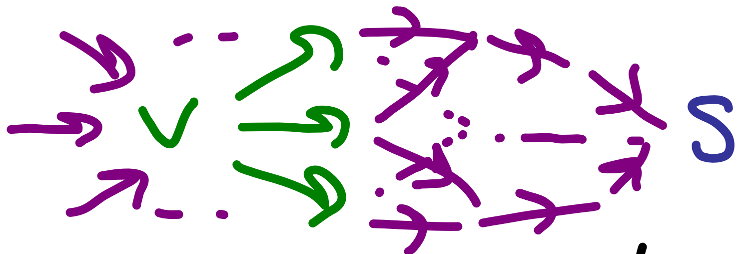
$$x_1 \to x_2 \iff \vec{c}\cdot\vec{x}_1 < \vec{c}\cdot\vec{x}_2$$

- $\max\{\vec{c}\cdot\vec{x} \mid A\vec{x} \le \vec{b}\} =$ sink of $G(P,\vec{c})$

Simplex Method: walk from some vertex $v \in G(P,\vec{c})$ following arrows $v \to v_1 \to v_2 \to \cdots \to s$ to sink s

- also may walk backwards to source of $G(P,\vec{c})$ to minimize $\vec{c}\cdot\vec{x}$

Pivot Rule: method to choose which out arrow



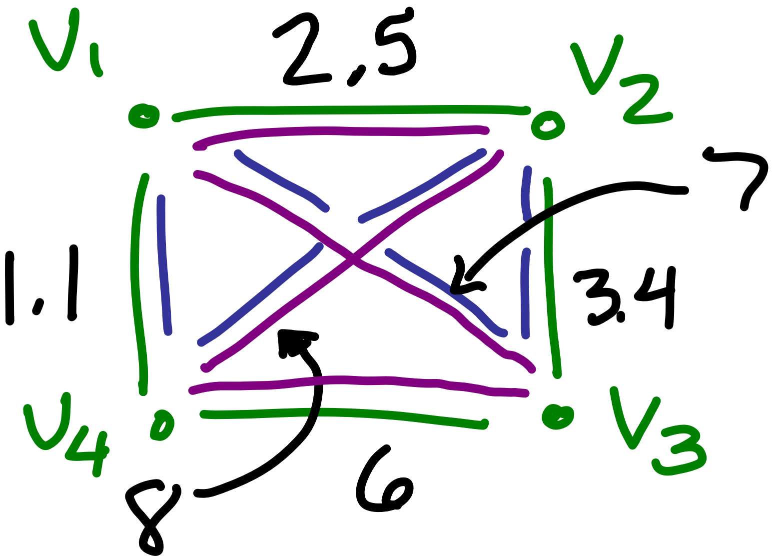
to follow from v towards sink s.

Key Questions:

1. What is typical complexity of simplex method (path length)?

2. What is worst case? (i.e. diameter of $G(P, \vec{c})$)

An Example: Traveling Salesman Problem

$G =$ 

Polytope Vertices:

$(1,0,1,1,0,1)$, $(1,1,0,0,1,1)$,

e_{12} e_{14} e_{23} e_{34}

$(0,1,1,1,1,0)$

Cost Vector:

$$\vec{c} = (2.5, 7, 1.1, 3.4, 8, 6)$$

Hirsch Conjecture: For d-dim'l polytopes with n facets (max'l faces), diameter of 1-skeleton graph, denoted $\Delta(d,n)$, satisfies

$$\Delta(d,n) \leq n-d.$$

Francisco Santos: Counterexamples!

Nonrevisiting path conjecture:

For each u,v in polytope P, there is path u to v not revisiting any facet it has left.

Non-Revis. Path Conj $\Rightarrow$ Hirsch Conj. (by giving short path) so BOTH FALSE!

Our Plan

Impose further conditions on P and $\vec{c}$ that will imply a corollary of the following which we hope might also hold:

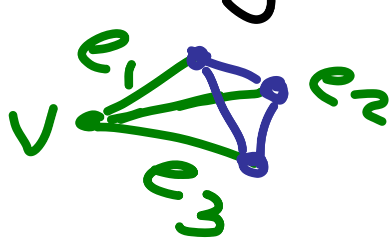
> For each $u, v \in P$, each directed path from u to v never revisits any facet it has left.

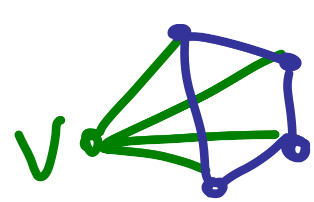
This property would make all pivot rules efficient for P and $\vec{c}$.

Quick Background on Polytopes

- A polytope in $\mathbb{R}^d$ is convex hull of finite # vertices, or equiv'ly a bounded set that is an intersection of half spaces.

- A polytope is simple if for each vertex v and each collection $e_1, e_2, .., e_r$ of edges emanating out from v there is an r-dim'l face containing all these edges

e.g. 

but not 

New Def'n: $G(P,\vec{c})$ has the **Hasse diagram property** if it is Hasse diagram of finite poset, i.e. $v_1 \to v_2 \to \cdots \to v_r$ for $r \geq 3$ directed path precludes $v_1 \to v_r$

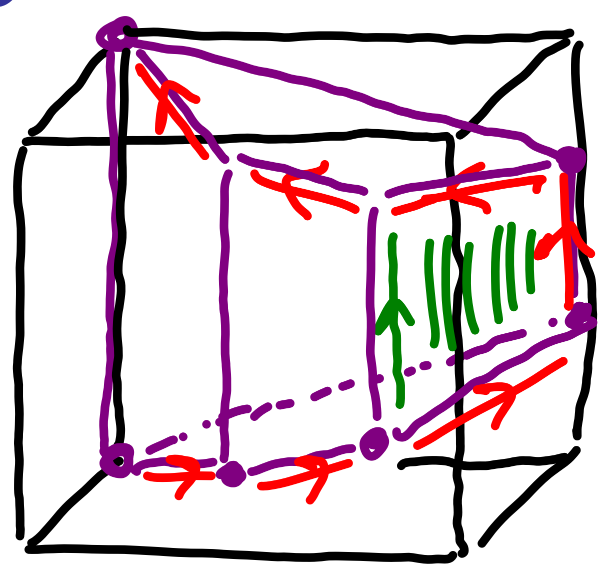
Important Non-Examples:

"Klee-Minty Cubes" (1st known examples s.t. simplex method not efficient!)

e.g. $n=3$

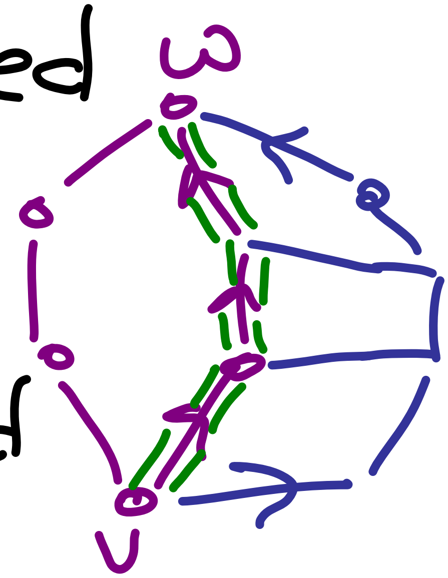
$\vec{c} = (0,0,1)$

- path visits all vertices!

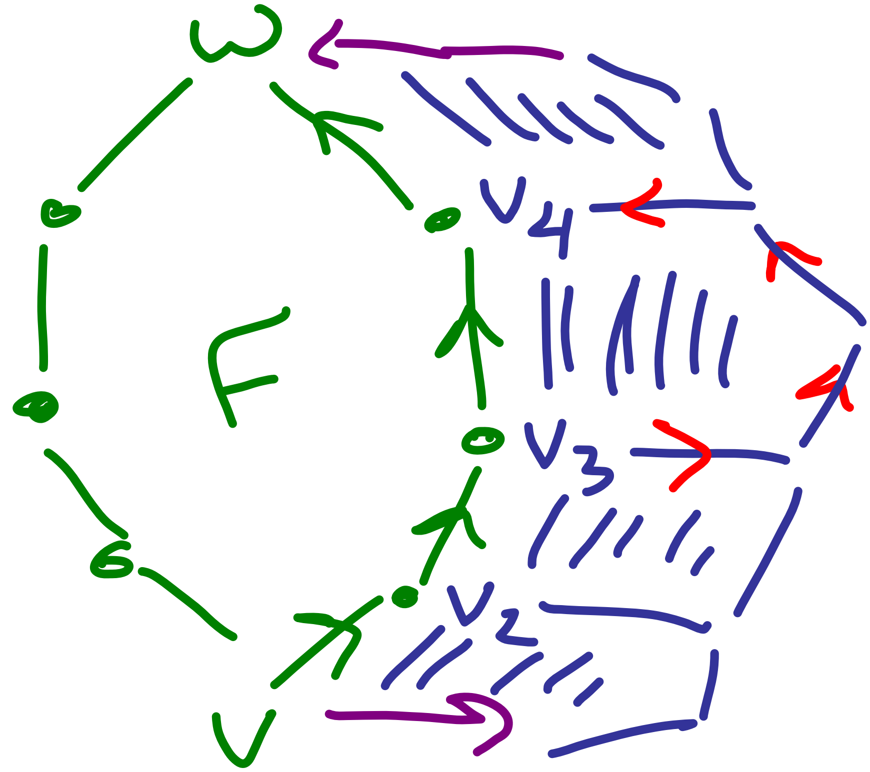


$$\text{n-dim'l KM-cube} = \left\{ (x_1, \ldots, x_n) \,\middle|\, 0 \leq x_1 \leq 1;\ \begin{array}{l} \varepsilon x_{i-1} < x_i < 1 - \varepsilon x_{i-1} \\ \text{for } i > 1 \end{array};\ 0 < \varepsilon < \tfrac{1}{2} \right\}$$

Lemma: Given $F \subseteq G$ with $\dim(G) = \dim(F) + 1$ in simple polytope P w/ generic $\vec{c}$ s.t. $G(P, \vec{c})$ is a Hasse diagram, then there does not exist $v, w \in F$ with directed path P_F from v to w in F, outward oriented edge v to $G \setminus F$ and inward oriented edge $G \setminus F$ to w.



Corollary: Monotonicity of out-degrees & partic. outward directions.

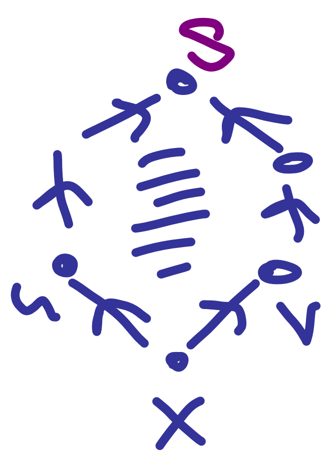


violate Hasse diagram (at switch from out to in)

Corollary: For each face $F \subseteq P$ with $\hat{0} \in F$ or $\hat{1} \in F$, directed paths cannot revisit F after departing from it.

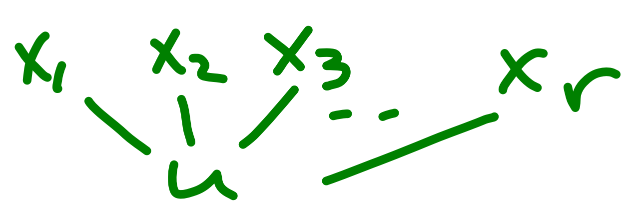
Recall: A poset L is a **lattice** if for each $u, v \in L$ there exists unique least upper bound ("join") for u and v, denoted $u \vee v$, and unique greatest lower bound for u and v ("meet"), $u \wedge v$.

Note: for P simple & $G(P, \vec{c})$ Hasse diagram, an upper bound for u, v both covering x is sink s of unique 2-face containing x, u, v.



"Pseudo-joins" in a Polytope

Let P be simple polytope w/ generic cost vector $\vec{c}$ such that $G(P,\vec{c})$ is Hasse diagram of poset L with $x_1, x_2, \ldots, x_r \in L$ s.t. there exists $u \in L$ with $u \lessdot x_i$ for $i = 1, 2, \ldots, r$. Define pseudo-join of $x_1, x_2, \ldots, x_r$ as sink of unique r-face of P containing



<u>Lemma</u>: $S \neq T \Rightarrow \text{pseudo}(S) \neq \text{pseudo}(T)$.

Note: Since pseudo-join of $x_1, \dots, x_r$ is an upper bound, there exists directed path from $x_1 \vee \dots \vee x_r$ to $\text{pseudo-join}(x_1, \dots, x_r)$

Thm: Let P be a simple polytope and $\vec{c}$ be generic cost vector with $G(P, \vec{c})$ Hasse diagram of finite lattice. Then

$\text{pseudo-join}(x_1, x_2, \dots, x_r) = x_1 \vee \dots \vee x_r$

Pf: induction on r with $r=2$ base case especially tricky part.

Recall: The order complex of poset P, denoted $\Delta(P)$, is abstract simplicial complex whose i-faces are chains $v_0 < v_1 < \cdots < v_i$

Thm: Let P be a simple polytope with generic cost vector $\vec{c}$ such that $G(P, \vec{c})$ is the Hasse diagram of a finite lattice L. Then each open interval $(u,v) = \{z \in L \mid u < z < v\}$ has order complex homotopy equivalent to a ball or a sphere

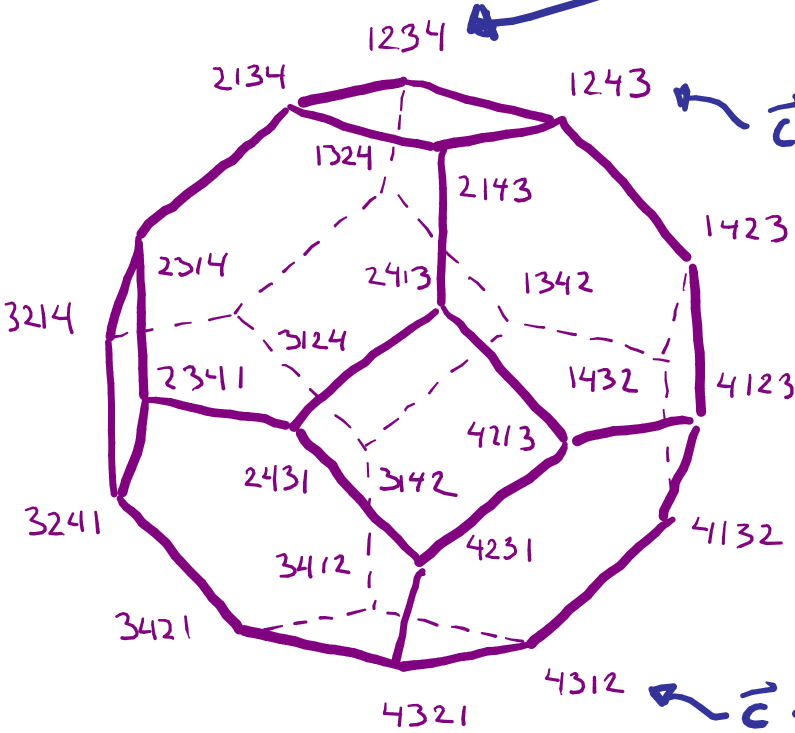
Applications to Poset Topology

polytope P $\rightsquigarrow$ poset on $G(P, \vec{c})$

- permutahedra $\rightsquigarrow$ weak order
- associahedra $\rightsquigarrow$ Tamari lattice
- generalized associahedra $\rightsquigarrow$ Cambrian lattices

Permutahedron as Weak Order

- cost vector $\vec{c}$ any strictly ascending vector such as $\vec{c}=(1,2,3,4)$.



$\vec{c}\cdot(1,2,3,4)=30$

$\vec{c}\cdot(1,2,4,3)=29$

$\vec{c}\cdot(1,4,2,3)=27$

$\vec{c}\cdot(4,1,2,3)=24$

$\vec{c}\cdot(4,1,3,2)=23$

$\vec{c}\cdot(4,3,1,2)=21$

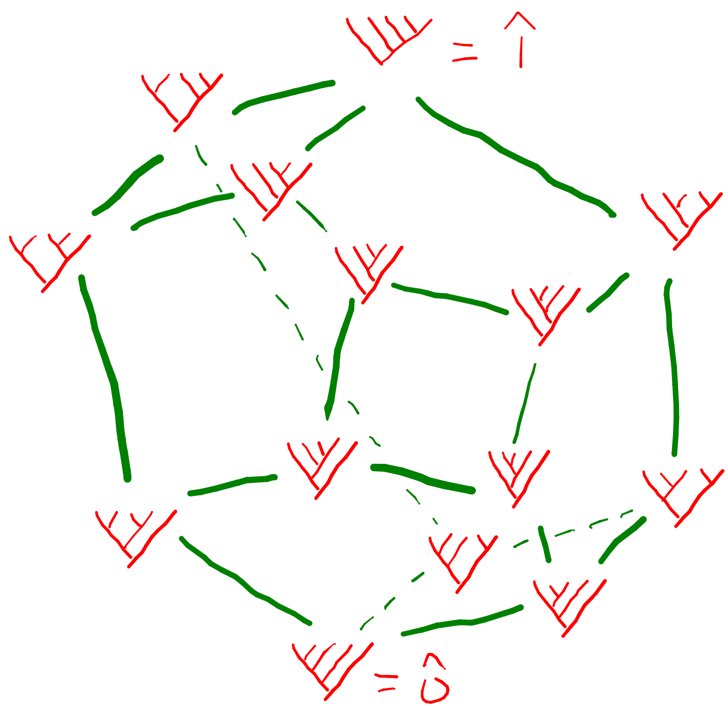
$\vec{c}\cdot(4,3,2,1)=20$

- Homotopy type 1st due to Edelman (type A) & Björner

Associahedron as Tamari Lattice

- Use Loday's realization
- Poset of binary trees with cover relations: $\lessdot$

$$((a,b),c) \lessdot (a,(b,c))$$



- Homotopy type 1st due to Björner & Wachs via nonpure lexicographic shellability

Some Further Questions

Qn 1: Other examples?

Qn 2: Does P simple + $G(P, \gtrless)$ Hasse diagram of lattice $\Rightarrow$ no directed path can revisit face it has departed? (If not, variations?)

Thanks!!