Posets Arising as 1-skeleta of Simple Polytopes, the Nonrevisiting Path Conjecture and Poset Topology

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Linear Programming

• Given a matrix $A$ and vectors $\mathbf{b}, \mathbf{c}$ seek $\max \{ \mathbf{c}^T \mathbf{x} \mid A \mathbf{x} \leq \mathbf{b} \}$

• $\{ \mathbf{x} \mid A \mathbf{x} \leq \mathbf{b} \}$ is polytope $P$

if set is bounded

e.g. $A \mathbf{x} = \mathbf{b}$

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\leq
\begin{pmatrix}
5 \\
0 \\
2
\end{pmatrix}
\]

\[x_1 \leq 0 \]

\[x_1 + 2x_2 \leq 2 \]

\[-x_2 \leq 0\]
Solving Linear Programs via Simplex Method

- Define \( G(P,z) \): directed graph on 1-skeleton of \( P \), i.e. on vertex-edge graph of \( P \), with  
  \[ x_1 \rightarrow x_2 \iff \overrightarrow{c} \cdot \overrightarrow{x}_1 < \overrightarrow{c} \cdot \overrightarrow{x}_2 \]

- \( \max \{ \overrightarrow{c} \cdot \overrightarrow{x} | A \overrightarrow{x} \leq \overrightarrow{b} \} = \text{sink of } G(P,z) \)

Simplex Method: walk from some vertex \( v \in G(P,z) \) following arrows  
\[ v \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow s \]  
to sink \( s \)
- also may walk backwards to source of \( G(P,z) \) to minimize \( z \cdot \overrightarrow{x} \)
**Pivot Rule:** method to choose which out arrow \( \rightarrow \) to follow from \( v \) towards sinks \( s \).

**Key Questions:**

1. What is typical complexity of simplex method (path length)?

2. What is worst case? (i.e. diameter of \( G(P, E) \))
An Example: Traveling Salesman Problem

Polytope Vertices:

\[(1,0,1,1,0,1), (1,1,0,0,1,1), \allowbreak (0,1,1,1,1,1,0)\]

Cost Vector:

\[\vec{c} = (2.5, 7, 1.1, 3.4, 8, 1, 6)\]
**Hirsch Conjecture:** For d-dim' polytopes with n facets (max' l faces), diameter of 1-skeleton graph, denoted $\Delta(d,n)$, satisfies $\Delta(d,n) \leq n-d$.

**Francisco Santos:** Counterexamples!

**Nonrevisiting path conjecture:** For each $u,v$ in polytope $P$, there is path $u \to v$ not revisiting any facet it has left.

**Non-Revis. Path Conj $\implies$ Hirsch Conj,** (by giving short path) so BOTH FALSE!
Our Plan

Impose further conditions on $P$ and $\overline{P}$ that will imply a corollary of the following which we hope might also hold:

For each $u, v \in P$, each directed path from $u$ to $v$ never revisits any facet it has left.

This property would make all pivot rules efficient for $P$ and $\overline{P}$. 
Quick Background on Polytopes

- A polytope in $\mathbb{R}^d$ is convex hull of finite # vertices, or equivalently a bounded set that is an intersection of half spaces.

- A polytope is simple if for each vertex $v$ and each collection $e_1, e_2, \ldots, e_r$ of edges emanating out from $v$ there is an $r$-dim face containing all these edges.

  e.g. $v$ but not $v$
**New Def'n:** $G(P, c)$ has the Hasse diagram property if it is Hasse diagram of finite poset, i.e. $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_r$ for $r \geq 3$ directed path precludes $v_1 \rightarrow v_r$

**Important Non-Examples:** "Klee-Minty Cubes" (1st known examples s.t. simplex method not efficient!

- Path visits all vertices!

$\mathbf{n}$-dim $P = \{ (x_1, \ldots, x_n) | 0 \leq x_i \leq 1, 3 \leq x_i < 1 - 3x_i, 0 < x_i < 1 \}$ for $i > 1$
Lemma: Given $F \subseteq G$ with $\dim(G) = \dim(F) + 1$ in simple polytope $P$ with generic $\vec{c}$ s.t. $G(P, \vec{c})$ is a Hasse diagram, then there does not exist $v, w \in F$ with directed path $P_f$ from $v$ to $w$ in $F$, outward oriented edge $v$ to $G \setminus F$ and inward oriented edge $G \setminus F$ to $w$. 
Corollary: Monotonicity of out-degrees & partial outward directions.

3 violate Hasse diagram (at switch from out to in)

Corollary: For each face $F \in \mathcal{P}$ with $\hat{O}EF$ or $\hat{I}EF$, directed paths cannot revisit $F$ after departing from it.
Recall: A poset $L$ is a lattice if for each $u,v \in L$ there exists unique least upper bound ("join") for $u$ and $v$, denoted $u \vee v$, and unique greatest lower bound for $u$ and $v$ ("meet"), $u \wedge v$.

Note: for $P$ simple $\& G(P, \vec{e})$ Hasse diagram, an upper bound for $u, v$ both covering $x$ is sink of unique 2-face containing $x, u, v$. 
"Pseudo-joins" in a Polytope

Let \( P \) be simple polytope w/ generic cost vector \( \vec{c} \) such that \( G(P, \vec{c}) \) is Hasse diagram of poset \( L \) with \( x_1, x_2, \ldots, x_r \in L \) s.t. there exists \( u \in L \) with \( u < x_i \) for \( i = 1, 2, \ldots, r \). Define pseudo-join of \( x_1, x_2, \ldots, x_r \) as sink of unique \( r \)-face of \( P \) containing \( x_1, x_2, \ldots, x_r \).

Lemma: \( S \neq T \Rightarrow \text{pseudo}(S) \neq \text{pseudo}(T) \)
Note: Since pseudo-join of \(x_1, \ldots, x_r\) is an upper bound, there exists directed path from \(x_1, u \cdots u, x_r\) to \(\text{pseudo-join}(x_1, \ldots, x_r)\)

Thm: Let \(P\) be a simple polytope and \(\vec{c}\) be generic cost vector with \(G(P, \vec{c})\) Hasse diagram of finite lattice. Then
\[\text{pseudo-join}(x_1, x_2, \ldots, x_r) = x_1, u \cdots u, x_r\]

Pf: induction on \(r\) with \(r=2\), base case especially tricky part.
Recall: The order complex of poset $P$, denoted $\Delta(P)$, is an abstract simplicial complex whose $i$-faces are chains $v_0 < v_1 < \ldots < v_i$.

Thm: Let $P$ be a simple polytope with generic cost vector $\mathbf{c}$ such that $G(P, \mathbf{c})$ is the Hasse diagram of a finite lattice $L$. Then each open interval $(u, v) = \{ z \in L | u < z < v \}$ has order complex homotopy equivalent to a ball or a sphere.
Applications to Poset Topology

\[
\begin{array}{c|c|c}
\text{polytope } P & \mapsto & \text{poset on } G(P, \mathcal{C}) \\
\end{array}
\]

- permutohedra \(\mapsto\) weak order

- associahedra \(\mapsto\) Tamari lattice

- generalized associahedra \(\mapsto\) Cambrian lattices
Permutahedron is Weak Order

- cost vector $\mathbf{c}$ any strictly ascending vector such as $\mathbf{c} = (1, 2, 3, 4)$.

- Homotopy type 1st due to Edelman (type A) & Björner.
**Associahedron as Tamari Lattice**

- Use Laday's realization
- Poset of binary trees with cover relations: \( \preceq \)

\[
(a,b), c \quad (a, (b,c))
\]

- Homotopy type is due to Björner & Wechs via nonpure lexicographic shellability.
Some Further Questions

Qn 1: Other examples?

Qn 2: Does P simple + G(P, ≥) Hasse diagram of lattice => no directed path can revisit face it has departed? (If not, variations?)

Thanks!!