

# CW POSETS AFTER THE POINCARÉ CONJECTURE

PATRICIA HERSH

ABSTRACT. Anders Björner characterized which finite graded partially ordered sets arise as the closure relation on cells of a finite regular CW complex. His characterization of these “CW posets” required each open interval  $(\hat{0}, u)$  to have order complex homeomorphic to a sphere of dimension  $rk(u) - 2$ . Work of Danaraj and Klee showed that sufficient conditions were for the poset to be thin and shellable. The proof of the Poincaré Conjecture enables the requirement of shellability to be replaced by the homotopy Cohen-Macaulay property. This expands the range of tools that may be used to prove a poset is a CW poset.

## 1. INTRODUCTION.

Recall that the order complex  $\Delta(P)$  of a finite poset  $P$  is the simplicial complex whose  $i$ -faces are exactly the chains  $u_0 < u_1 < \dots < u_i$  in  $P$ . For any  $x < y$  in  $P$ , denote by  $(x, y)$  the open interval comprised of elements  $z \in P$  with  $x < z < y$  and let  $\Delta(x, y)$  be the order complex for the subposet  $(x, y)$  of  $P$ .

The *face poset*  $F(K)$  of a simplicial complex  $K$  is the partial order on its faces by inclusion of sets of vertices. More generally, the *closure poset*  $F(K)$  of a regular CW complex  $K$  is the partial order on its cells given by  $\sigma \leq \tau$  if and only if  $\sigma \subseteq \bar{\tau}$ . It is easy to see (and well-known) that  $\Delta(F(K) \setminus \{\hat{0}\})$  is the first barycentric subdivision of  $K$ , hence homeomorphic to  $K$ .

Thus, the closure of a cell  $\tau$  must have  $\Delta(F(\bar{\tau}) \setminus \{\hat{0}\})$  homeomorphic to a ball and therefore for each  $\sigma < \tau$  in the closure poset we must have  $\Delta(\sigma, \tau)$  homeomorphic to a sphere  $\mathbb{S}^{\dim \tau - \dim \sigma}$ .

Anders Björner introduced the notion of CW poset in [Bj]:

**Definition 1.1.** A finite graded poset  $P$  is a *CW poset* if:

- (1)  $P$  has a unique minimal element  $\hat{0}$
- (2)  $P$  has at least one additional element
- (3) For each  $y$  in  $P \setminus \{\hat{0}\}$ , the open interval  $(\hat{0}, y)$  has order complex homeomorphic to a sphere  $\mathbb{S}^{rk(y)-2}$

---

The author was supported by NSF grant DMS-1002636.

He proved the following:

**Theorem 1.2** (Björner). *A finite graded poset  $P$  is the closure poset of a regular CW complex if and only if  $P$  is a CW poset.*

Note, for instance, that all nontrivial simplicial posets (cf. [St]) with unique minimal element are CW posets. A necessary condition for a poset to be a CW poset is for each closed interval to be Eulerian:

**Definition 1.3.** A finite graded poset  $P$  with unique minimal and maximal elements  $\hat{0}$  and  $\hat{1}$  is *Eulerian* if  $\mu_P(x, y) = (-1)^{\text{rky} - \text{rkx} - 2}$  for each  $x < y$  in  $P$ .

Necessity is immediate from the fact that a sphere  $\mathbb{S}^d$  has reduced Euler characteristic  $(-1)^d$ , together with the relationship  $\mu_P(x, y) = \tilde{\chi}(\Delta(x, y))$  observed by Philip Hall. A related weaker condition than the Eulerian property is thinness:

**Definition 1.4.** A finite graded poset  $P$  is *thin* if each open interval  $(x, y)$  for  $x < y$  with  $\text{rky} - \text{rkx} = 2$  has exactly two elements.

Not only is thinness of a poset  $P$  the restriction of the Eulerian property to rank 2 intervals, but it also implies for graded posets  $P$  that  $\Delta(P)$  is a pseudomanifold.

**Definition 1.5.** A simplicial complex is *pure* if all maximal faces have the same dimension.

**Definition 1.6.** A simplicial complex is *shellable* if there is a total order on its maximal faces  $F_1, \dots, F_k$  such that  $\overline{F_j} \cap (\cup_{i < j} \overline{F_i})$  is a pure codimension one subcomplex of  $\overline{F_j}$  for each  $j \geq 2$ .

A poset is said to be shellable if its order complex is shellable. A practical set of sufficient conditions for checking that a finite graded poset  $P$  is a CW poset (cf. [DK]) is as follows:

**Theorem 1.7** (Danaraj and Klee). *If a finite graded poset  $P$  is thin and shellable, then  $P$  is a CW poset.*

For example, in [BW], Björner and Wachs proved that Bruhat order is thin and shellable, hence is a CW poset.

Denote by  $\text{lk}_K(\sigma)$  the link of a cell  $\sigma$  in a regular CW complex  $K$ . In the special case of simplicial complexes,  $\text{lk}_K(\sigma) = \{\tau \in K \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in K\}$ .

**Definition 1.8.** A pure  $d$ -dimensional simplicial complex (or regular CW complex)  $K$  is homotopy Cohen-Macaulay if for each cell  $\sigma$  in  $K$  (including the empty cell),  $\text{lk}_K(\sigma)$  is also homotopy equivalent to a wedge of  $(d - \dim \sigma - 1)$ -dimensional spheres.

It is well known that pure, shellable simplicial complexes are homotopy Cohen-Macaulay. See [Bj2] for further background.

## 2. THIN, HOMOTOPY COHEN-MACAULAY POSETS ARE CW POSETS

It seems natural to ask if the shellability requirement may be replaced by the more general hypothesis that  $P$  is homotopy Cohen-Macaulay. This would have the benefit of enabling a wider array of techniques of topological combinatorics, for instance discrete Morse theory (cf. [Fo], [BH]) and the Quillen Fiber Lemma (cf. [Qu]), to be used to prove that particular families of posets of interest are CW posets. Interestingly, this follows quite quickly from the Poincaré Conjecture, now a theorem:

**Theorem 2.1** (Poincaré Conjecture). *If  $K$  is a manifold that is a simply connected homology sphere, then  $K$  is homeomorphic to a sphere. In other words, if  $K$  is a homotopy sphere that is also a manifold, then  $K$  is homeomorphic to a sphere.*

Theorem 2.1 follows in dimension two from the classification of surfaces. It was proven in dimension at least 5 by Smale in [Sm] (see also [Ze]), in dimension 4 by Freedman in [Fr], and finally in dimension 3 by Perelman (see [Pe], [Pe2], [Pe3], with full detail in [KL]). Now to a consequence for topological combinatorics.

**Theorem 2.2.** *Let  $P$  be a finite graded poset with unique minimal element  $\hat{0}$  and at least one other element. Then  $P$  is a CW poset if  $P$  is thin and homotopy Cohen-Macaulay.*

*Proof.* First note that the theorem is true for  $P$  of rank 2, in which case  $P$  has exactly 4 elements and exactly one nontrivial open interval. Now suppose by induction the theorem holds for  $P$  of rank strictly less than  $r$ . Consider  $P$  of exactly rank  $r$ , and consider any open interval  $(x, y)$  in  $P$ . The link of any vertex in  $\Delta(x, y)$  is the order complex of a product of open intervals of strictly smaller rank. By induction, it is therefore the join of simplicial complexes each homeomorphic to a sphere, therefore is itself homeomorphic to a sphere. This implies that  $\Delta(x, y)$  is a manifold. Since  $P$  is homotopy Cohen-Macaulay, we also have that  $\Delta(x, y)$  is homotopy equivalent to a wedge of spheres; by thinness, we may assume  $\text{rky} - \text{rkx} > 2$ , implying  $\Delta(x, y)$  is connected. But the top homology group over the integers of any connected manifold is either the integers or 0, depending whether the manifold is orientable or not. Thus,  $\Delta(x, y)$  is either contractible or homotopy equivalent to a sphere. In the former case,  $\Delta(x, y)$  would be a manifold with boundary, with this boundary pure of codimension one; this contradicts thinness, which implies each codimension one face is contained in precisely two facets.

Thus,  $\Delta(x, y)$  is homotopy equivalent to a sphere. But the Poincare Conjecture yields that any manifold that is homotopy equivalent to a sphere must be homeomorphic to a sphere.  $\square$

Anders Björner has independently obtained related results [Bj3], also using the Poincare Conjecture. It seems natural now to ask whether the Poincare Conjecture has further ramifications of a similar spirit for topological combinatorics.

### 3. ACKNOWLEDGMENTS

The author thanks Anders Björner and Lenny Ng for helpful comments.

### REFERENCES

- [BH] E. Babson and P. Hersh, Discrete Morse functions from lexicographic orders, *Trans. Amer. Math. Soc.*, **357** (2005), no. 2, 509–534.
- [Bj] A. Björner, Posets, regular CW complexes and Bruhat order, *European J. Combin.* **5** (1984), no. 1, 7–16
- [Bj2] A. Björner, Topological methods. *Handbook of combinatorics*, Vol. 1, 2, 1819–1872, Elsevier, Amsterdam, 1995.
- [Bj3] A. Björner, personal communication.
- [BW] A. Björner and M. Wachs, Bruhat order of Coxeter groups and shellability, *Adv. in Math.*, **43** (1982), no. 1, 87–100.
- [DK] G. Danaraj and V. Klee, Shellings of spheres and polytopes, *Duke Math. J.*, **41** (1974), 443–451.
- [Fr] M. H. Freedman, The topology of four-dimensional manifolds, *J. Diff. Geom.* **17** (1982), 357–453.
- [Fo] R. Forman, Morse theory for cell complexes, *Adv. Math.* **134** (1998), no. 1, 90–145.
- [KL] B. Kleiner and J. Lott, Notes on Perelman’s papers, *Geom. Topol.* **12** (2008), no. 5, 2587–2855.
- [Pe] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159
- [Pe2] G. Perelman, Ricci flow with surgery on three-manifolds, arXiv:math.DG/0303109
- [Pe3] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, arXiv:math.DG/0307245
- [Qu] D. Quillen, Homotopy properties of the poset of nontrivial  $p$ -subgroups of a group, *Adv. in Math.*, **28** (1978), no. 2, 101–128.
- [Sm] S. Smale, Generalized Poincare’s conjecture in dimensions greater than four, *Ann. Math.* **117** (1961), 391–406.
- [SZ] J. Stallings, Polyhedral homotopy spheres, *Bull. Amer. Math. Soc.* **66** (1960), 485–488.
- [St] R. Stanley,  $f$ -vectors and  $h$ -vectors of simplicial posets, *J. Pure Appl. Algebra*, **71** (1991), no. 2–3, 319–331.

- [Ze] E.C. Zeeman, The Poincaré conjecture for  $n \geq 5$ , in *Topology of 3-Manifolds and Related Topics* Prentice-Hall, Englewood Cliffs, NJ, 1962, 198–204.

BOX 8205, DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC 27605

*E-mail address:* plhersh@ncsu.edu