From the Weak Bruhat Order to Crystal Posets

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Perspective & Main Goal:

- Study crystal graphs regarded as posets via poset map to weak Bruhat order, namely via the (right) key map.

Poset Structure for Many Crystals

$u \lessdot_{crystal} v \iff u \xrightarrow[f_i]{} v$ for some i

- Transfer properties of weak Bruhat order to crystals.

Weak Bruhat order:

$u \lessdot_{weak} s_i u$ if $\ell(s_i u) > \ell(u)$

Motivations for Crystals

- Study representation theory of Kac-Moody algebras (e.g. affine Lie algebras)

- Take universal enveloping algebra, & its quantum algebra w/ parameter q

- $q \rightsquigarrow 1$ yields $U(A)$ for Kac-Moody algebra A

- $q \rightsquigarrow 0$ yields algebra with same dimensions of weight spaces, described by combinatorics of "crystal graphs"

(Type A) Crystals of Highest Weight Representations & their Kashiwara Lowering Operators

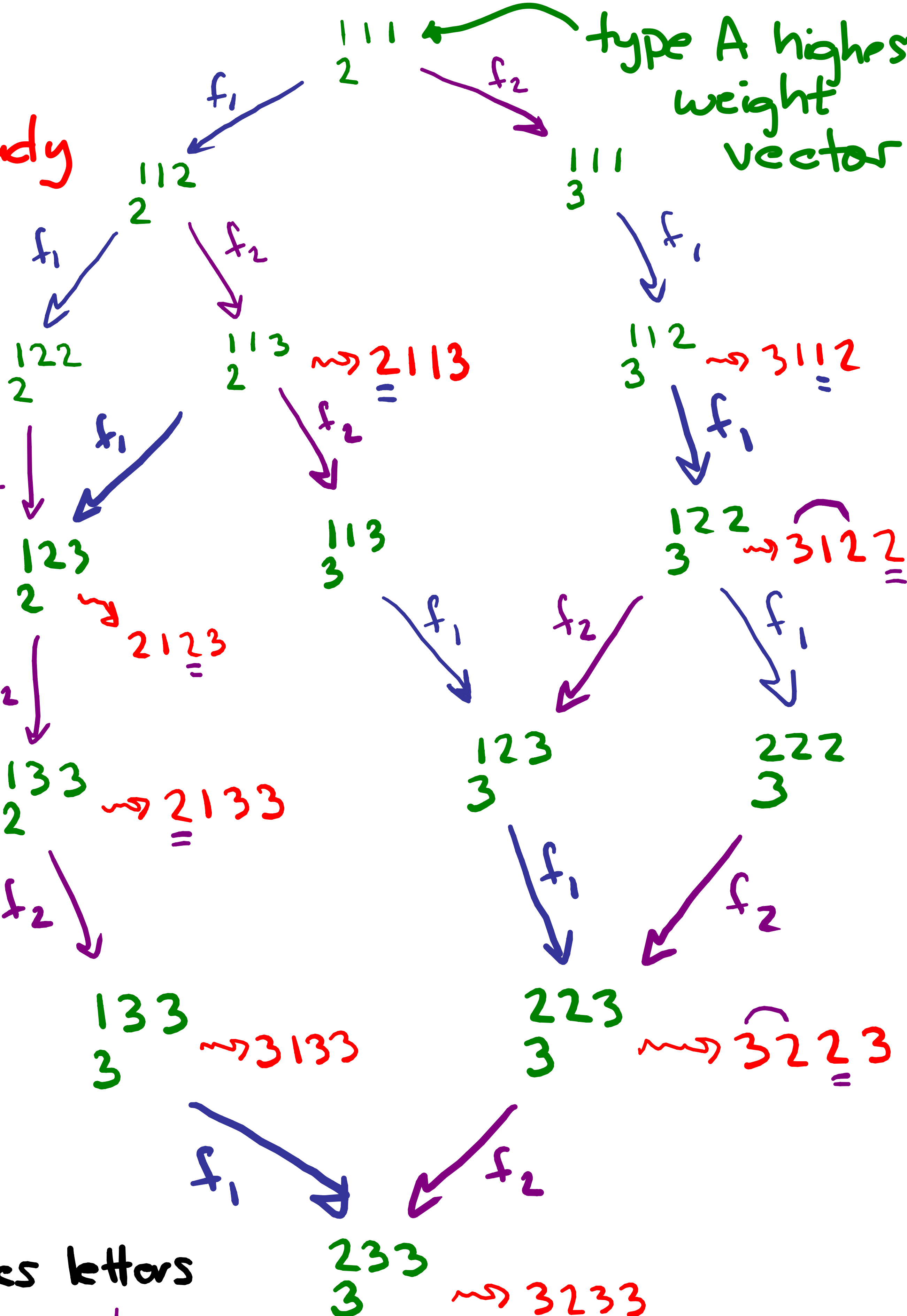
e.g. $\lambda = \ydiagram{2,1}$

$$\begin{tikzcd}[column sep=small]
& \begin{smallmatrix}1&1\\2\end{smallmatrix} \rightsquigarrow 211 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
\begin{smallmatrix}1&2\\2\end{smallmatrix} \rightsquigarrow 212 \arrow[d, "f_2"] & & \begin{smallmatrix}1&1\\3\end{smallmatrix} \rightsquigarrow 311 \arrow[d, "f_1"] \\
\begin{smallmatrix}1&3\\2\end{smallmatrix} \rightsquigarrow 213 \arrow[d, "f_2"] & & \begin{smallmatrix}1&2\\3\end{smallmatrix} \rightsquigarrow 312 \arrow[d, "f_1"] \\
\begin{smallmatrix}1&3\\3\end{smallmatrix} \rightsquigarrow 313 \arrow[dr, "f_1"'] & & \begin{smallmatrix}2&2\\3\end{smallmatrix} \rightsquigarrow 322 \arrow[dl, "f_2"] \\
& \begin{smallmatrix}2&3\\3\end{smallmatrix} \rightsquigarrow 323 &
\end{tikzcd}$$

$\lambda = $ (shape $(3,1)$)

(We study dual poset)

type A highest weight vector



f_i ignores letters other than i & $i+1$, pairs $i+1$ followed by i, then $f_i : i^r (i+1)^s \mapsto i^{r-1} (i+1)^{s+1}$

Talk Outline:

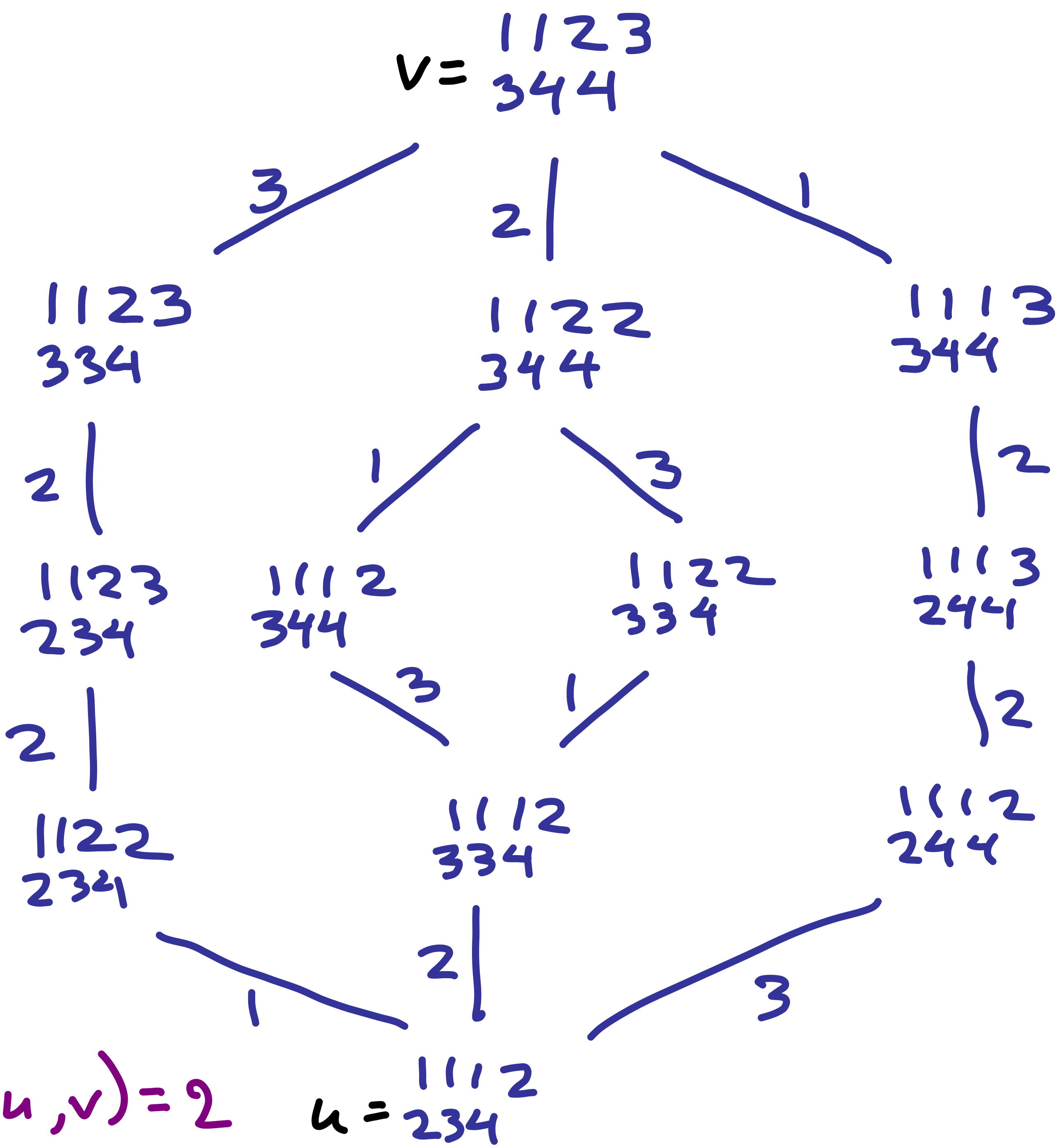
I. Background Review

II. New Algorithm to Calculate Right Key of Crystal
- does not depend on choice of model

III. Positive Results for Lower Intervals $[\hat{0}, u]$
- Möbius function & homotopy type
- Connectedness of saturated chains under "Stembridge moves" & "Sternberg moves"

IV. Negative Results for Arbitrary Type A Intervals $[u,v]$
- Arbitrarily large Möbius functions
- Arbitrarily high degree non-redundant "relations" amongst crystal operators

Base Case for Negative Examples



$M_P(u,v)=2$

& not connected by "Stembridge moves"

I. Background

Defn: The (left) weak Bruhat order on Coxeter system (W, S) is the partial order with cover relations $u \lessdot v \iff v = s_i u$ for $u, v \in W$ with $\ell(v) > \ell(u)$, for $\ell(v) = \min\{r \mid v = s_{i_1} \cdots s_{i_r} \text{ for } s_{i_1}, \ldots, s_{i_r} \in S\}$

e.g. $W = S_n$

$S = \{s_1, s_2, \ldots, s_{n-1}\}$ for $s_i = (i, i+1)$

with relations:

$$s_i^2 = e \;\&\; \underbrace{s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \;\&\; s_i s_j = s_j s_i}_{\text{"braid moves"}} \quad (\text{for } |j-i| > 1)$$

e.g. Left weak order for S_3

$$
\begin{tikzcd}
& s_1s_2s_1 = s_2s_1s_2 \arrow[dl, dash, "s_1\cdot -"'] \arrow[dr, dash, "s_2\cdot -"] & \\
s_2s_1 \arrow[d, dash, "s_2\cdot -"'] & & s_1s_2 \arrow[d, dash, "s_1\cdot -"] \\
s_1 \arrow[dr, dash, "s_1\cdot -"'] & & s_2 \arrow[dl, dash, "s_2\cdot -"] \\
& e &
\end{tikzcd}
$$

Key Fact: Saturated chains from e to w naturally labeled w/ the "reduced expressions" $s_{i_1} \cdots s_{i_{\ell(w)}}$ for w.
Likewise, saturated chains from u to v $\longleftrightarrow$ reduced expressions for vu^{-1}!

Properties of Reduced Expressions & Weak Bruhat Order

(see Björner-Brenti book for details)

Thm 3.2.1: Weak order on W is a meet-semilattice.

Corollary: Each $[u,v]$ is a lattice.

Lemma 3.2.3: For $J \subseteq S$, the join of atoms $\bigvee_{j \in J} a_j$ exists $\Leftrightarrow$ $W_J = \langle j \mid j \in J \rangle$ finite, in which case $\bigvee_{j \in J} a_j = \underbrace{w_0}_{\text{longest element}}(\underbrace{W_J}_{\text{parabolic subgroup gen'd by } J})$

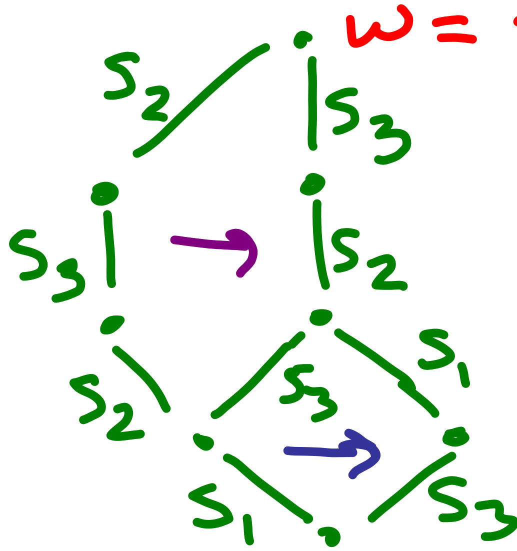
Useful Related Fact: $a_j \leq u \Leftrightarrow \exists$ red. exp. for u s.t. s_j rightmost

Connectedness under Braid Moves

Thm 3.3.1 (Björner-Brenti) Let (W,S) be a Coxeter group w/ $w \in W$. Then every two reduced expressions for w are connected via braid moves.

e.g. $\underbrace{s_1 s_2 s_3 s_2}_{} \to s_1 s_3 s_2 s_3 \to s_3 s_1 s_2 s_3$

Right weak order:

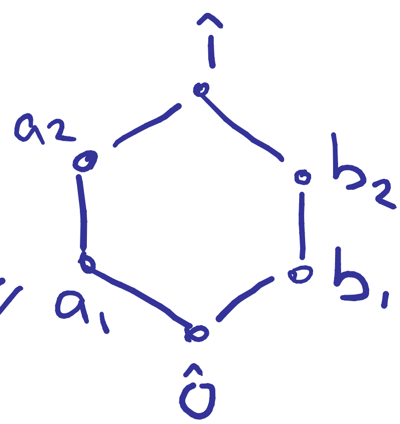


$w = s_1 s_2 s_3 s_2 = s_1 s_3 s_2 s_3$
$= s_3 s_1 s_2 s_3$

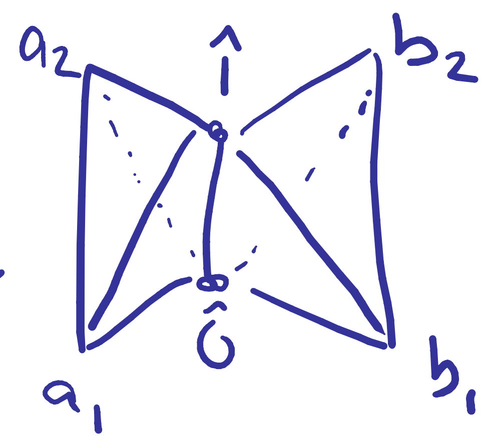
Note: Proof via lattice property for $[u,v]$

Def'n: The order complex (or nerve) of a poset P is the simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-chains $v_0 < \dots < v_i$ in P

e.g. $P \cong$



$\Delta(P) \cong$



Recall: $\mu_P(u,v) = \tilde{\chi}(\Delta(\underbrace{(u,v)}_{=\{z \in P \mid u<z<v\}}))$

($\mu_P(u,v) = 0, \pm 1$ suggests ball or sphere)

Crystals & q-Crystals

A **crystal** B of type Φ is a nonempty set B with raising & lowering operators e_i, f_i & fns

$\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$

$wt : B \to \Lambda$ = weight lattice of type Φ

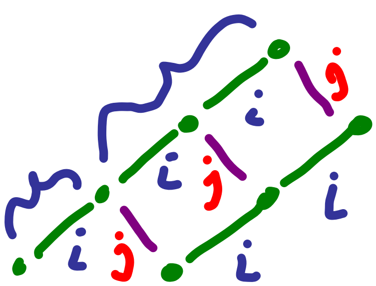
s.t.

(A1) $x, y \in B$, then $e_i(x) = y \iff x = f_i(y)$

both implying $wt(y) = wt(x) + \alpha_i$

& $\varepsilon_i(y) = \varepsilon_i(x) - 1$

$\varphi_i(y) = \varphi_i(x) + 1$



(A2) $\varphi_i(x) - \varepsilon_i(x) = \langle wt(x), \alpha_i^\vee \rangle$

Stembridge Crystals: "q-crystals"

((Crystals of highest weight rep'n's in simply laced case)

- $\chi_{B(\lambda)}(t) = \sum_{b \in B(\lambda)} t^{wt(b)}$ = character of irrep $B(\lambda)$
- if $y = e_i(x) \neq 0$ & $z = e_j(x) \neq 0$ for $i \neq j$ then either:
  - $e_i e_j(x) = e_j e_i(x) \neq 0$

    <u>OR</u>
  - $e_i e_j^2 e_i(x) = e_j e_i^2 e_j(x) \neq 0$
- likewise for f_i operators
- axioms yield this & characterize crystals of highest weight rep'n's in simply laced case

Additional Important facts

- Stembridge crystals are posets with cover reln's $u \lessdot v \iff v = f_i(u)$

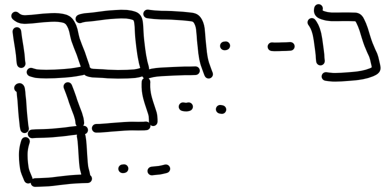
- Sternberg gave analogous "relations" for doubly laced crystals (not characterization)

A Crystal that is not a Poset

$A_3^{(1)}$

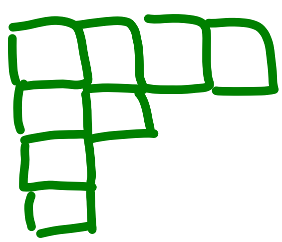
$$
\begin{tikzcd}
\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} \arrow[r, "2"] & \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array} \arrow[r, "1"] \arrow[rr, bend left, "3"] & \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array} \arrow[rr, bend left, "3"] & \begin{array}{|c|}\hline 1\\ \hline 4\\ \hline\end{array} \arrow[r, "1"] & \begin{array}{|c|}\hline 2\\ \hline 4\\ \hline\end{array} \arrow[r, "2"] \arrow[llll, bend left, "0"] & \begin{array}{|c|}\hline 3\\ \hline 4\\ \hline\end{array} \arrow[llll, bend left, "0"]
\end{tikzcd}
$$

Type A SSYT Model for Crystals

A semistandard Young tableau (SSYT) of shape λ for $\lambda \vdash n$ is a filling of a "Ferrers diagram" 

having λ_i boxes in row i for $\lambda_1 \geq \lambda_2 \geq \cdots$ with positive integers $\{a_{i,j}\}_{\substack{1 \leq i \leq \ell(\lambda) \\ 1 \leq j \leq \lambda_i}}$

s.t.
$$\begin{array}{ccccccc} a_{11} & \leq & a_{12} & \leq & a_{13} & \leq \cdots \leq & a_{1,\lambda_1} \\ \wedge & & \wedge & & \wedge & & \\ a_{21} & \leq & a_{22} & \leq & & \leq & a_{2,\lambda_2} \\ \wedge & & & & & & \\ \vdots & & & & & & \end{array}$$

e.g. $\lambda =$  $= (4,2,1,1)$

w/ SSYT
$$\begin{array}{llll} 1 & 1 & 2 & 5 \\ 3 & 4 & & \\ 4 & & & \\ 9 & & & \end{array}$$

Type A highest weight rep'n of type λ

1. $\hat{O} = \begin{matrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 & \\ 3 & 3 & \cdots & & \\ \vdots & & & & \end{matrix}$ of shape λ

2. $u \xrightarrow[i]{} v$ for v obtained from u by increment rightmost i not in "parenthesization pair" w/ $i+1$ to an $i+1$

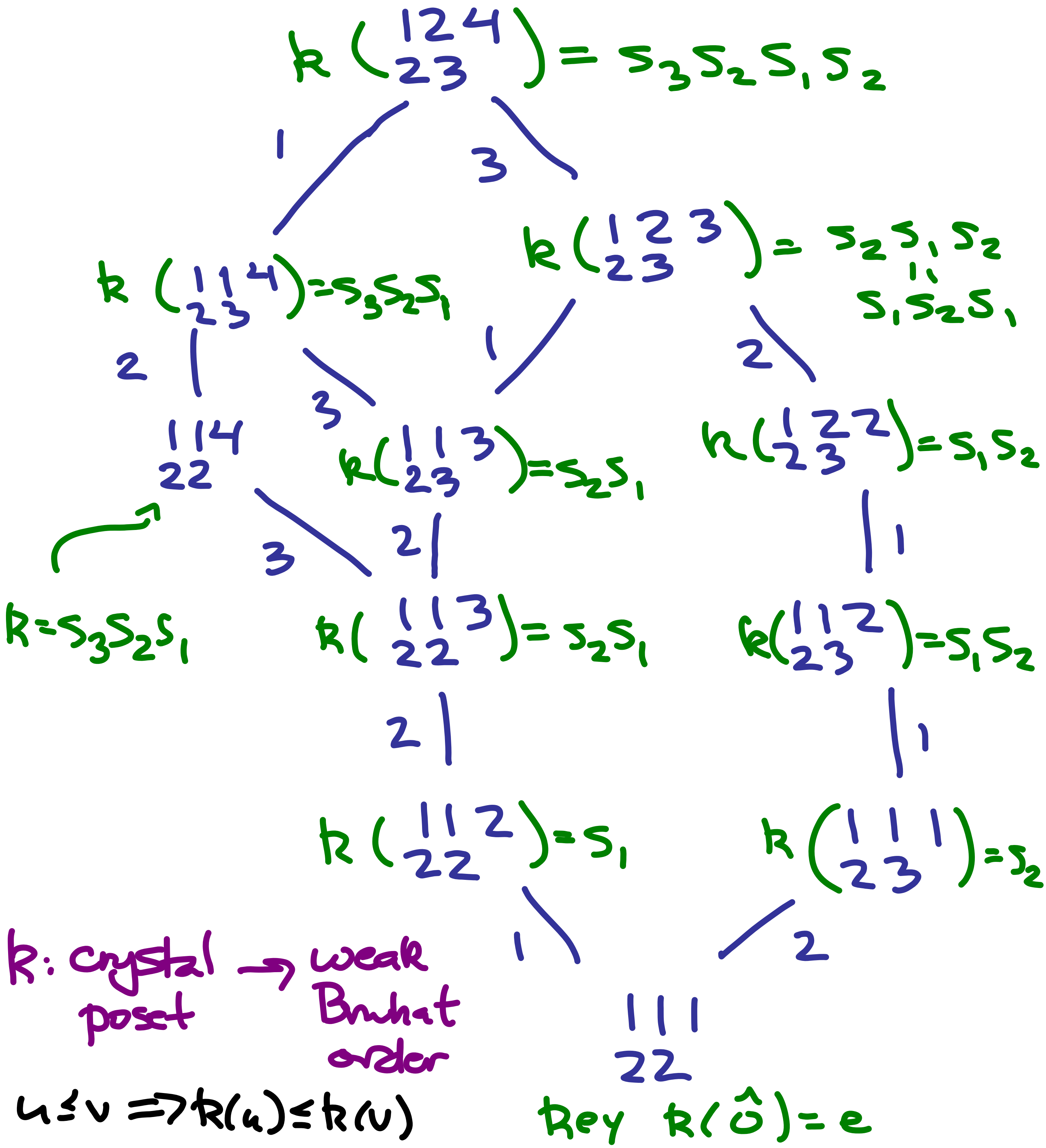
Parenthesization Pairs: Read leftmost column bottom to top, then subsequent columns L to R, ignoring all but i & $i+1$; omit consec $i+1, i$ and repeat.

e.g. $u = \begin{matrix} 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & & & & \\ \boxed{3} & 4 & 4 & & & & & \end{matrix}$ $\underset{i=3}{\rightsquigarrow}$ $\boxed{3}\,4\,4\,3\,3\,4\,4\,4$

$\uparrow e_3$

$4\,4\,4\,3\,3\,4\,4\,4 \leftrightarrow e_i(u)$

II. Right key "k" of a KM-crystal



k: crystal poset $\rightarrow$ weak Bruhat order

$u \le v \Rightarrow k(u) \le k(v)$

New Algorithm to Calculate Right Key of a KM-Crystal

(1) $key(\hat{0}) = e$

(2) if $\hat{0} \xrightarrow[i]{} a$, then $key(a) = s_i$

(i.e. $\hat{0} <\cdot a$)

(3) if v covers 2 or more elements

then $key(v) = \bigvee_{\{u \mid u \to v\}} key(u)$

(for join taken in weak order)

(4) if $u \xrightarrow[i]{} v$ and v does not cover any other elements, then:

(a) $key(v) = key(u)$ if $\exists\, u' \xrightarrow[i]{} u$

(b) $key(v) = s_i \cdot key(u)$ otherwise

Crucial Properties of Key

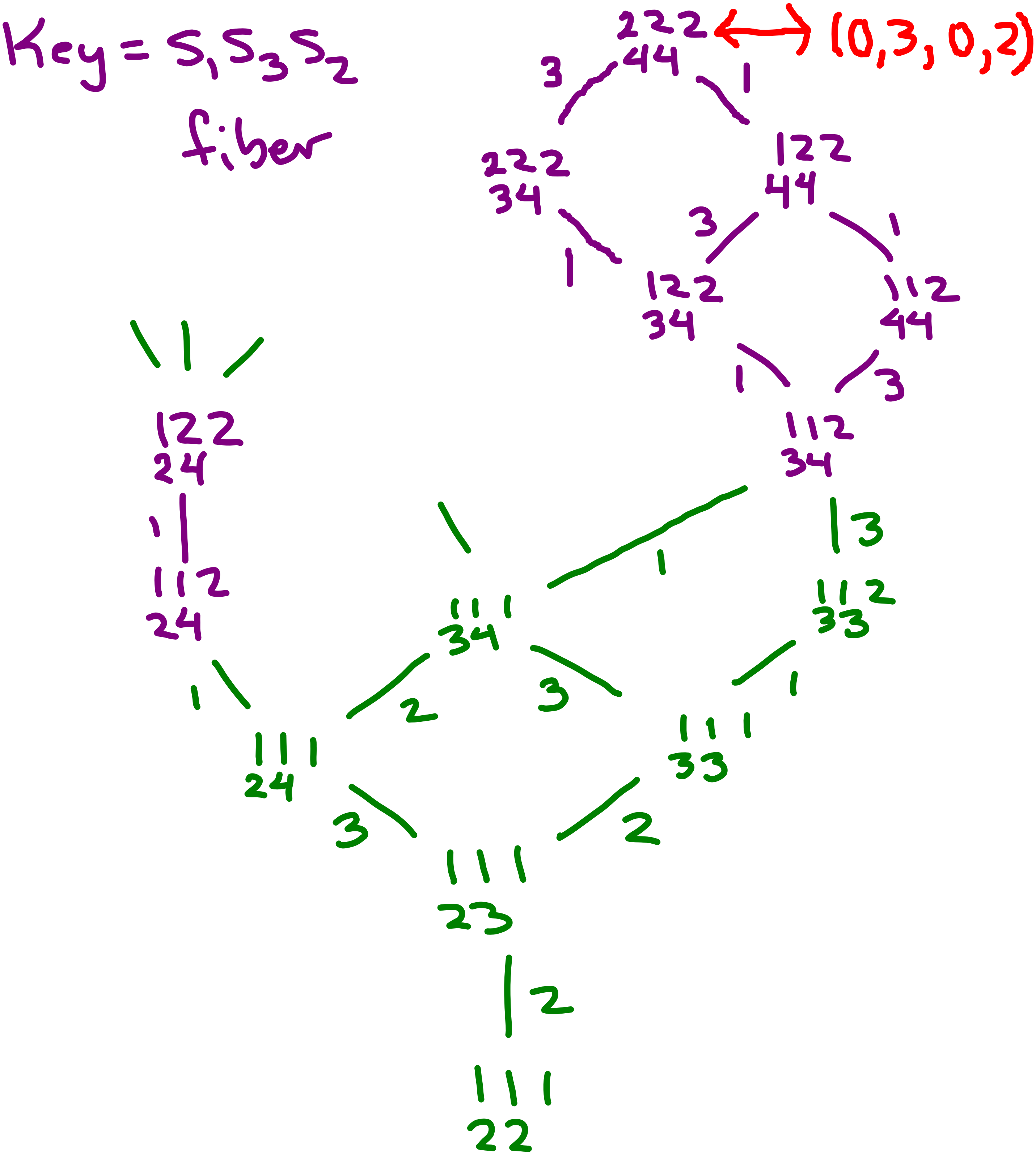
Thm (Littlemann): Given any symmetrizable Kac-Moody algebra A, the key of any crystal of type A satisfies:

$$K(f_p(F)) = \begin{cases} K(F) & \text{if } e_p(F) \neq 0 \\ s_p K(F) \text{ or } K(F) & \text{if } e_p(F) = 0 \end{cases}$$

Also, if $e_p(F) = 0$ then $s_p K(F) > K(F)$

Corollary: If $K(F) = s_{i_1} \cdots s_{i_r}$ then there exists saturated chain from F to $\hat{0}$ given by applying $e_{i_r}^{d_r} \cdots e_{i_1}^{d_1}$ to F for some $d_1, \ldots, d_r > 0$.

Fiber with Multiple Minimal Elements



Key Polynomials & right/left key

(see Lascoux-Schutzenberger & e.g.
Reiner-Shimozono)

Motivations: (1) Schubert poly. $\mathfrak{S}_w$ is positive sum of "key polynomials"

(2) Key polynomial records character for Demazure module

(3) The (closely related) right & left key maps determine smallest Demazure modules containing a given crystal element

(4) These will give us poset map from $\mathfrak{q}$-crystal to weak Bruhat order, transferring properties

Relation to Reiner-Shimozono Viewpoint on Key Polynomials

- $\partial_i = \dfrac{1-s_i}{x_i - x_{i+1}}$ & $\pi_i = \partial_i x_i$

- $K_\alpha = \pi_{i_1} \cdots \pi_{i_r} \, x^{\lambda(\alpha)}$ for α composition of n & $s_{i_1} \cdots s_{i_r}$ sorting α to $\lambda(\alpha)$

e.g. $K_{(1,0,2,1)} = \pi_2 \pi_1 \pi_3 \, x^{(2,1,1,0)}$

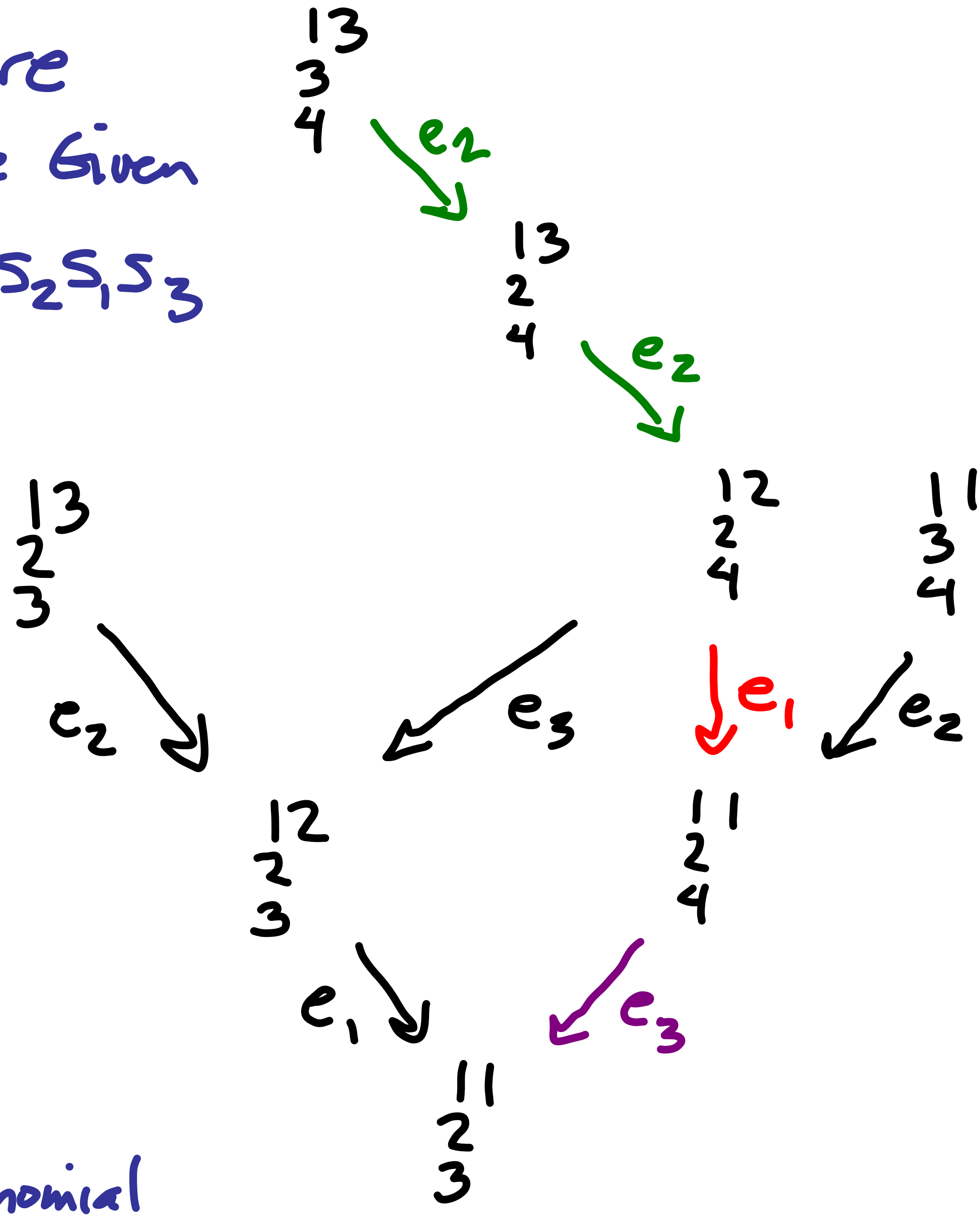
$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

$$= \pi_2 \pi_1 \left(x_1^2 x_2 (x_3 + x_4) \right)$$

$$= \pi_2 \left(x_1 x_2 x_3 (x_1 + x_2) + x_1 x_2 x_4 (x_1 + x_2) \right)$$

$$= x_1^2 x_2 x_3 + x_1 x_2 x_3 (x_2 + x_3) + x_1^2 x_4 (x_2 + x_3) + x_1 x_4 (x_2^2 + x_2 x_3 + x_3^2)$$

Demazure Module Given by $w = s_2 s_1 s_3$



Key Polynomial

$$K_{(1,0,2,1)} = \sum_{\substack{T' \leq T \\ \text{componentwise}}} x^{T'}$$

$\Updownarrow$

$K(T') \leq_{\text{Bruhat}} K(T)$ w/ no higher e_i exponents

III. Positive Results for Lower Intervals $[\hat{0}, u]$

Recall: $\mu_P(u,u) = 1$

$$\mu_P(u,v) = -\sum_{u \le z < v} \mu_P(u,z)$$

Thm 1 (H.-Lenart): Given u in a symmet. KM-crystal, then $\mu(\hat{0},u) = 0, \pm 1$. More specifically, $\mu(\hat{0},u) = 0$ unless $\mathrm{key}(u) = w_0(J)$ for some parabolic subgroup W_J with u the unique smallest element in $\mathrm{key}^{-1}(w_0(J))$, in which case $\mu(\hat{0},u) = (-1)^{|J|}$.

Proof Ingredients:

Thm 2 (H.-Lenart): Given a symmet. KM-crystal & given any parabolic W_J, then $\mathrm{key}^{-1}(w_0(W_J))$ has a unique minimal element and a unique maximal element.

(Proof via alcove path model)

Prop'n: Each $w \in W$ has unique maximal element $u \in W$ in weak Bruhat order s.t.

- $u \leq_{weak} w$
- there exists parabolic subgroup W_J s.t. $u = w_0(J)$ = longest el't

Thm 3 (H-Lenart): Each lower interval $(\hat{0}, u)$ in a symmet. KM-crystal has $\Delta(\hat{0}, u) \simeq$ ball or sphere, getting $S^{|J-2|}$ for $u = \min(k^{-1}(w_0(J)))$. Likewise for upper intervals in finite KM-crystals.

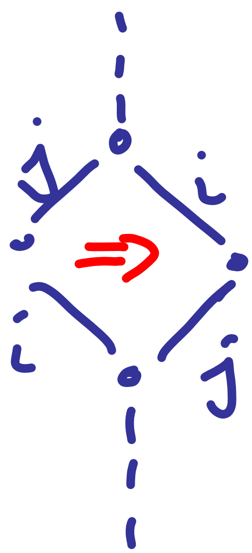
Proof Method: Quillen fiber lemma based upon:

$$f : \underset{\text{Poset}}{\text{Crystal}} \longrightarrow \underset{\{J \mid J \subseteq I\}}{\text{Boolean Algebra}}$$

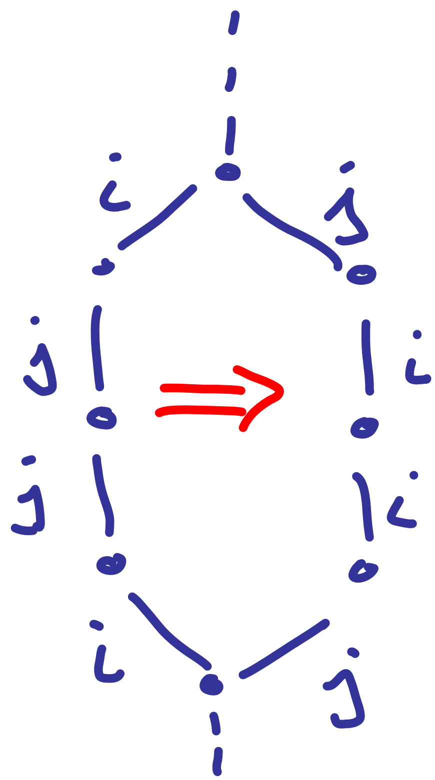
$$x \longmapsto \max\{J \mid w_0(J) \leq k(x)\}$$

Quillen Fiber Lemma: Poset map $f : P \to Q$ s.t. each $\Delta(f^{-1}_{\leq}(q))$ is contractible implies $\Delta(P) \simeq \Delta(Q)$.

Thm 4 (H.-Lenart): Given any lower interval $(\hat{0}, u)$ in a $\mathfrak{g}$-crystal, then set of saturated chains from $\hat{0}$ to u is connected by "Stembridge moves", namely moves of the form



and

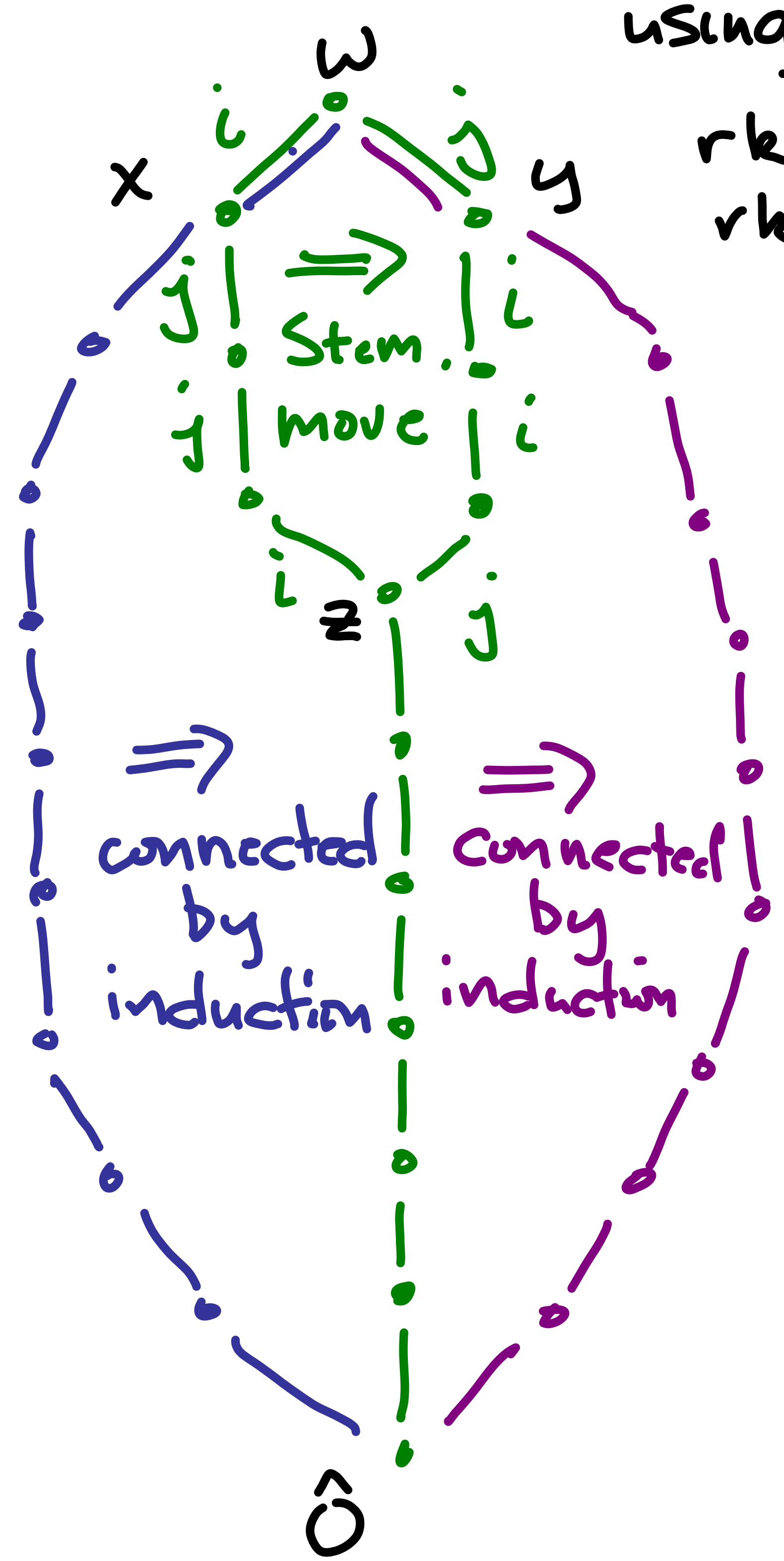


Note: Likewise in doubly-laced case via "Sternberg moves".

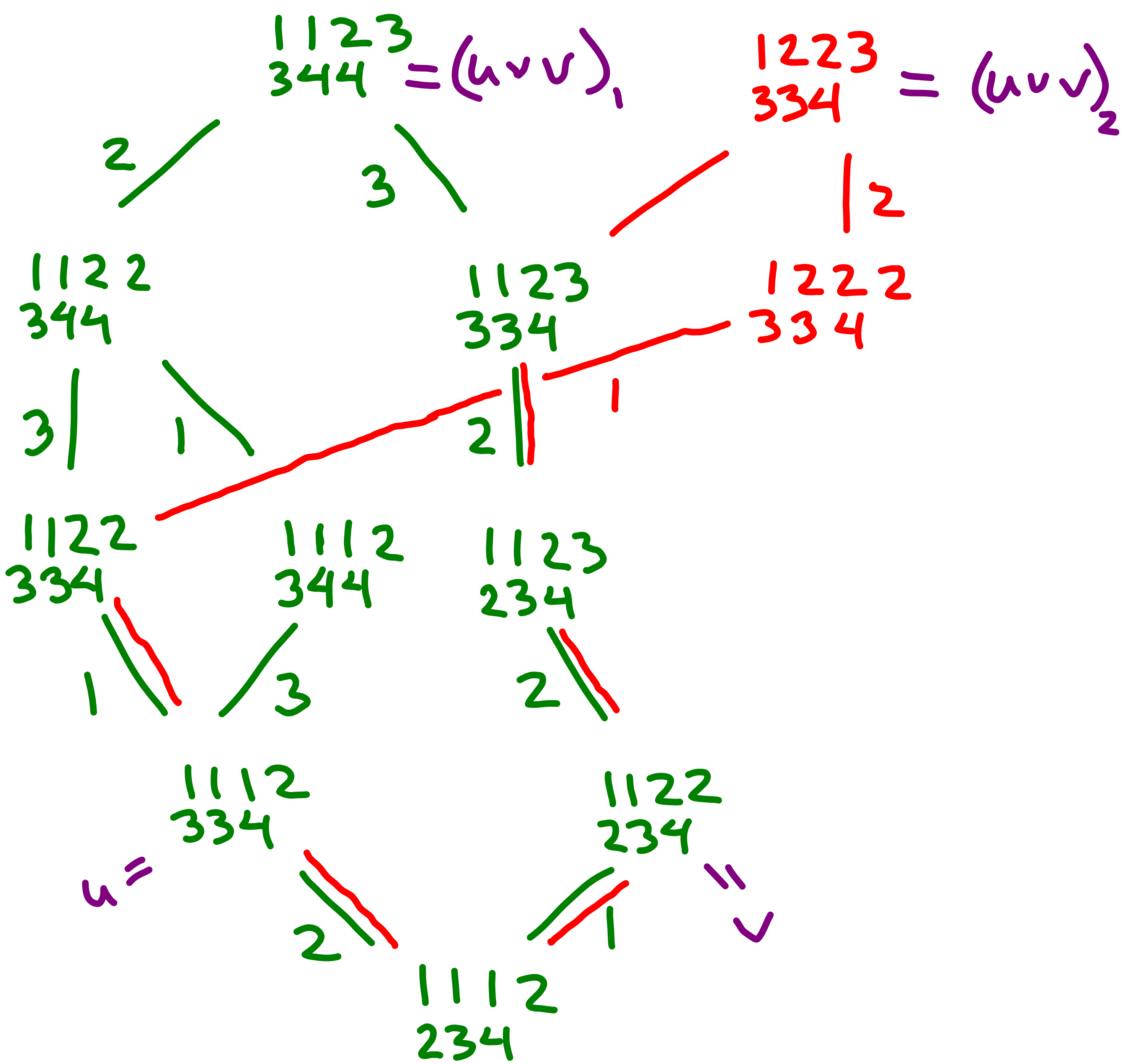
Proof Idea: Induction on rank using $z \geq \hat{0}$ &

$rk(x) < rk(w)$

$rk(y) < rk(w)$



Non-Lattice Example



(Why proof fails for arbitrary intervals)

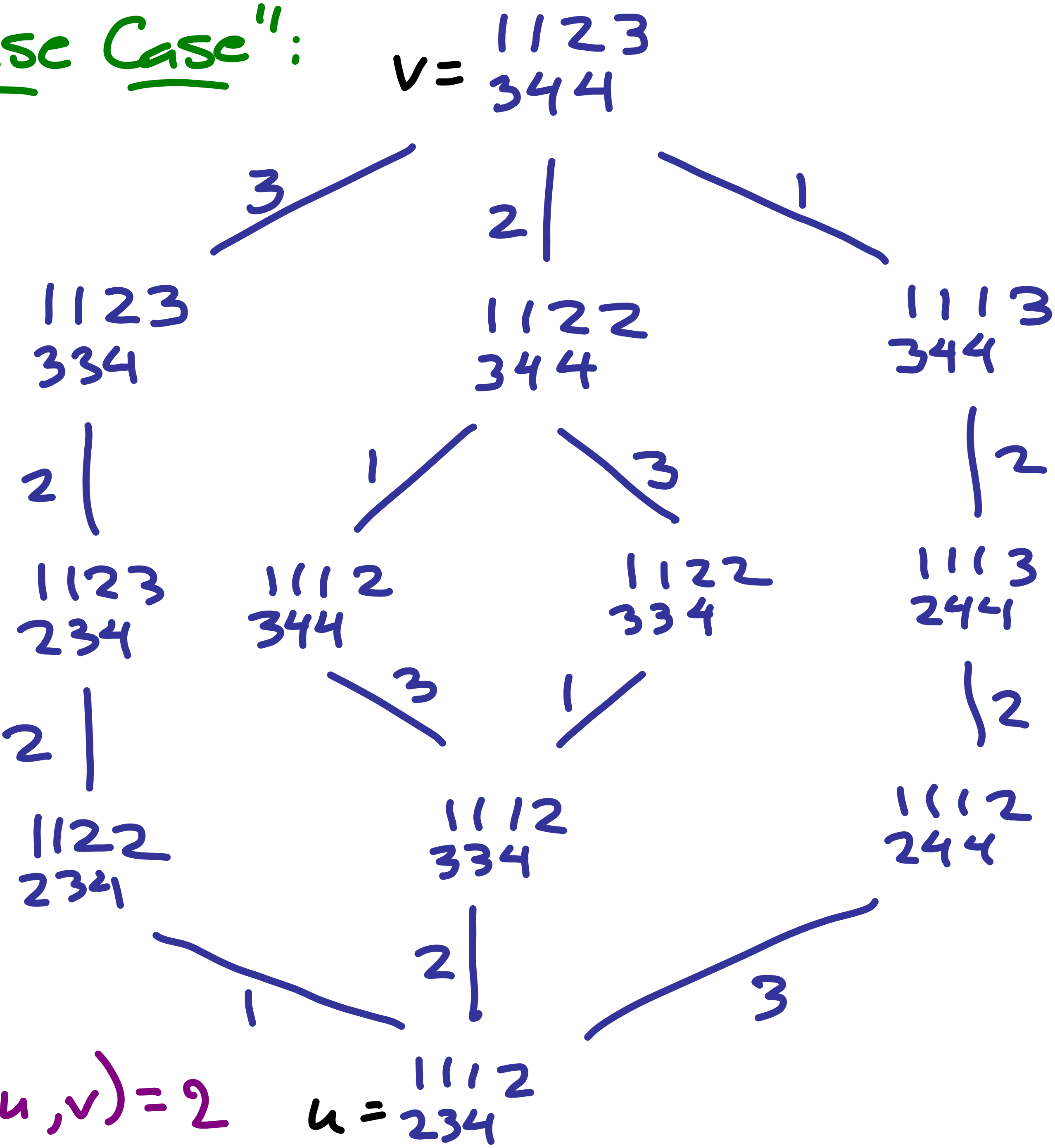
IV. Negative Results for Arbitrary (not necessarily lower) Crystal Poset Intervals (type A)

Thm 5 (H.-Lenart): There exist elements u, v in type A $\mathfrak{sl}$-crystals with $\mu(u,v) = 2^j$ for every positive integer j.

Thm 6 (H. Lenart): There exist type A intervals $[u,v]$ with $rk(v) - rk(u)$ arbitrarily large s.t. (u,v) is disconnected

Infinite Family of Examples

"Base Case":



$M_P(u,v)=2$

& not connected by "Stembridge moves"

Examples with $M(u,v)=2^j$

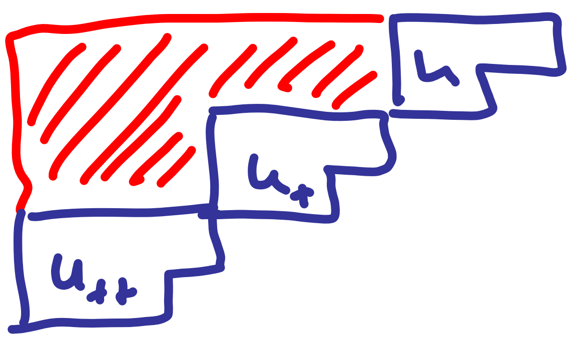
$j=1$: $u = \begin{array}{l}1112\\234\end{array}$ $\qquad$ $v = \begin{array}{l}1123\\344\end{array}$

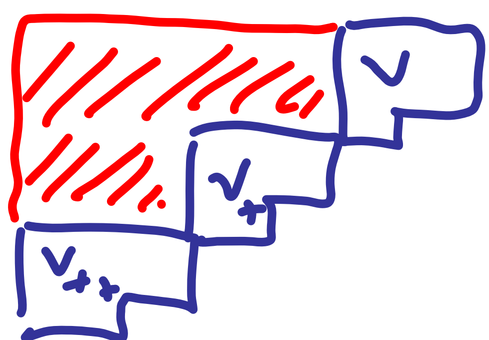
$j=2$: $u^{(2)} = \begin{array}{lll} & & \overset{u}{\shortparallel} \\ & 1111 & 1112\\ & 2222 & 234\\ 6667 & & \\ 789 & & \end{array}$ $\qquad$ $v^{(2)} = \begin{array}{lll} & & \overset{v}{\shortparallel} \\ & 1111 & 1123\\ & 2222 & 344\\ 6678 & & \\ 899 & & \end{array}$

$u_+ := u+5 = \begin{array}{l}6667\\789\end{array}$ $\qquad$ $v_+ := v+5 = \begin{array}{l}6678\\899\end{array}$

$$[u^{(2)},v^{(2)}] \cong [u,v]\times[u,v]$$

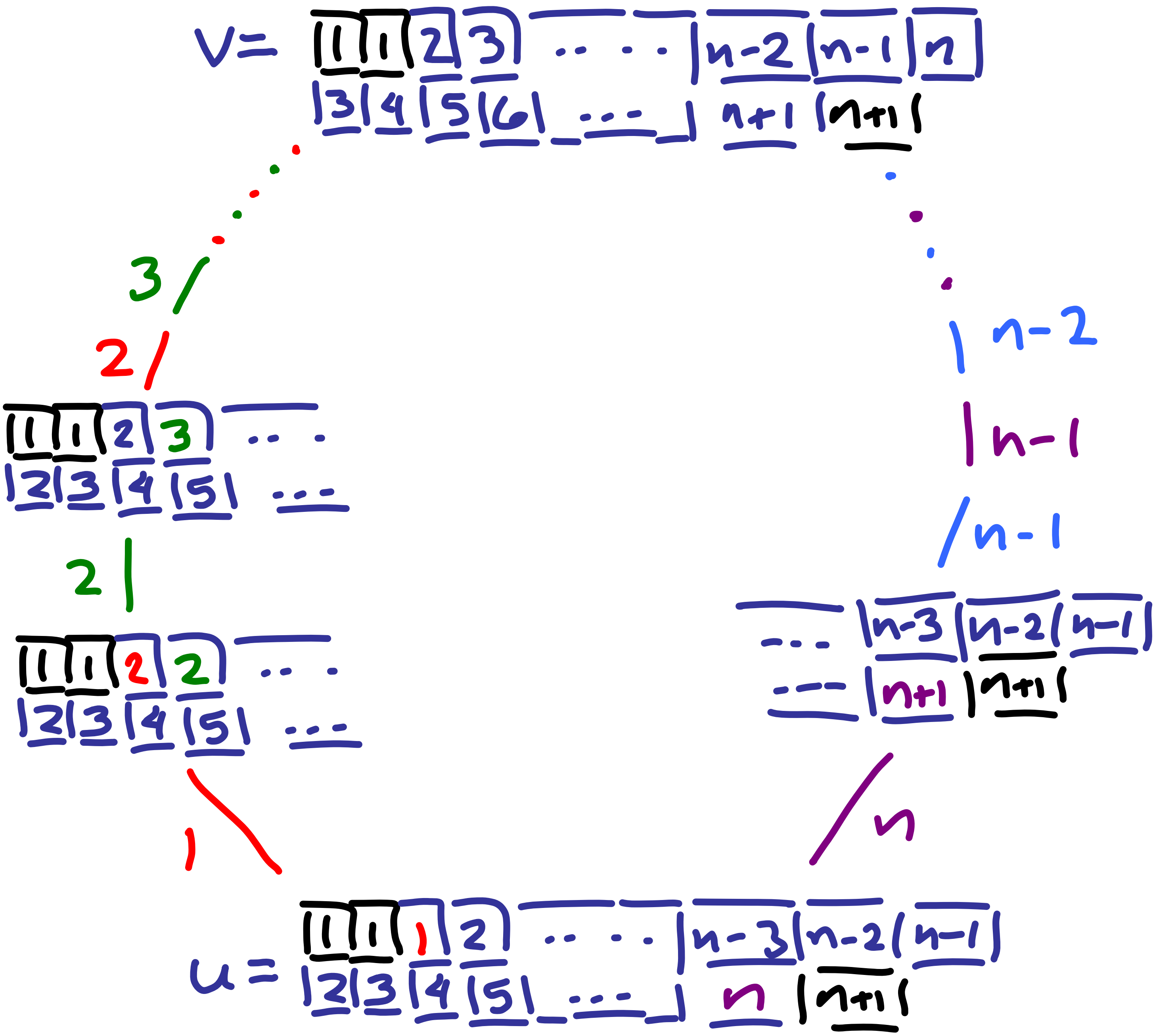
so $M(u^{(2)},v^{(2)}) = 2^2$

$u^{(3)} =$ 

$v^{(3)} =$ 

$$[u^{(k)},v^{(k)}] \cong \underbrace{[u,v]\times\cdots\times[u,v]}_{k\text{-fold}} \qquad M=2^k$$

Arbitrarily High Rank Disconnected Open Intervals



label sequences: 1, 2, 2, 3, 3, 4, 4, ..., n-1, n-1, n

& n, n-1, n-1, ..., 2, 2, 1 in distinct components

Consequence: Arbitrarily high degree relations $e_{i_1}\cdots e_{i_d}(u)=e_{j_1}\cdots e_{j_d}(u)$ amongst crystal operators applied to u not implied by any lower degree relations.

Some Further Questions:

1. Positive results in any additional generality (more general intervals)?
2. Interpretation/applications for Möbius function of a crystal?
3. Where/how exactly can the lattice property fail?