FIBERS OF MAPS TO TOTALLY NONNEGATIVE SPACES

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Abstract. This paper undertakes a study of the structure of the fibers of a family of maps $f_{(i_1,\dots,i_d)}$ arising from representation theory, motivated both by connections to Lusztig's theory of canonical bases and also by the fact that these fibers encode the nonnegative real relations amongst exponentiated Chevalley generators. In particular, we prove that the fibers of these maps $f_{(i_1,\dots,i_d)}$ (restricted to the standard simplex Δ_{d-1} in a way that still captures the full structure) admit cell decompositions induced by the decomposition of Δ_{d-1} into open simplices of various dimensions. We also prove that these cell decompositions have the same face posets as interior dual block complexes of subword complexes and that these interior dual block complexes are contractible.

We conjecture that each such fiber is a regular CW complex homeomorphic to the interior dual block complex of a subword complex. We show how this conjecture would yield as a corollary a new proof of the Fomin–Shapiro Conjecture by way of general topological results regarding approximating maps by homeomorphisms.

1. Introduction

Let U be the unipotent radical of a Borel subgroup in a semisimple, simply connected algebraic group defined and split over $\mathbb{R}$. The totally nonnegative part of the link of the identity in U is stratified into Bruhat cells. Sergey Fomin and Michael Shapiro [FS00] conjectured that this stratification is a regular CW decomposition of a topological closed ball. They proved that this stratified space has Bruhat order as the partial order of closure relations on its cells and obtained homological results (especially in type A) supporting their conjecture. This conjecture from [FS00] was proven by Hersh in [Her14], proving this in a way that heavily involved a realization of these spaces as images of maps $f_{(i_1,\dots,i_d)}$ that are quite interesting and fundamental in their own right. A main goal of the present paper, which may be regarded as a sequel to [Her14], is to better understand the overall structure of the fibers of these maps $f_{(i_1,\dots,i_d)}$. We refer readers to Section 2.3 for a review of notation used in the discussion below (and throughout the paper), along with related background material.

Much of the interest in these maps $f_{(i_1,\dots,i_d)}$ (and in some closely related families of maps) comes from a desire to understand their fibers (see e.g. [BZ01], [BFZ96], [BFZ05], [KW] and [PSW]). One motivation for interest in the case of $f_{(i_1,\dots,i_d)}$ is the

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fact that these fibers describe the nonnegative real relations amongst exponentiated Chevalley generators. One consequence of the results in [Her14] is that all nonnegative real relations amongst exponentiated Chevalley generators are direct consequences of what might be regarded as the "obvious" relations amongst them. Our work will give a much more full understanding of the combinatorial and topological structure of the fibers in their entirety, namely of stratified spaces which may be regarded as topological spaces of nonnegative real relations amongst Chevalley generators. An important aspect of this work will be the clarification it will give to the role of subword complexes in this story.

An important special case is when the algebraic group is of type A, which is precisely the case of $SL_n(\mathbb{R})$, with $(i_1, \ldots, i_d)$ a reduced word for the longest element in S_n. In this case, the image of the map $f_{(i_1,\ldots,i_d)}$ applied to the standard simplex $\sum t_i = K$ for $t_1, \ldots, t_d \geq 0$ and $K > 0$ is exactly the set of totally nonnegative, real $n \times n$ matrices that are upper triangular with 1's on the diagonal and entries just above the diagonal summing to a fixed positive constant K, or in other words the link of the identity in the unipotent radical of the standard Borel subgroup. The stratification (cell decomposition) of the image of $f_{(i_1,\ldots,i_d)}$ is based on which matrix minors are strictly positive and which are 0. It is proven in [Her14] that this is a regular CW decomposition, and that the space itself is a closed ball. In particular, this shows that each cell closure is a closed ball, including ones indexed by every other element w of the symmetric group, with these given by maps $f_{(i_{j_1},\ldots,i_{j_s})}$ given by subwords of $(i_1, \ldots, i_d)$ where $(i_{j_1}, \ldots, i_{j_s})$ is a reduced word for w.

Now to the precise definition of these maps $f_{(i_1,\ldots,i_d)}$, both in type A and in more general finite type. Let $x_i(t) = I_n + tE_{i,i+1}$ in type A, and more generally let $x_i(t) = \exp(tu_i)$, namely let it be an exponentiated Chevalley generator. In type A, $x_i(t)$ is obtained from the $n \times n$ identity matrix by putting a t in row i immediately to the right of the diagonal. Now for any reduced word $(i_1, \ldots, i_d)$, let

$$f_{(i_1,\ldots,i_d)}(t_1, \ldots, t_d) = x_{i_1}(t_1) \cdots x_{i_d}(t_d)$$

for each $(t_1, \ldots, t_d) \in \mathbb{R}^d_{\geq 0}$. We study the restriction of $f_{(i_1,\ldots,i_d)}$ to the simplex in which each parameter t_i is nonnegative and these parameters sum to a fixed positive constant (which we typically choose to be 1). It suffices to understand such a restriction of the map $f_{(i_1,\ldots,i_d)}$, since it is easy to see that varying the fixed, positive constant simply dilates the structure, with the overall structure being that of a cone. It is natural to think of this restriction as being the link of the identity, namely of the cone point.

Let Y_w for $w \in W$ be the closure of any cell Y_w^o in the Bruhat decomposition of the link of the identity in the totally nonnegative, real part of the unipotent radical of a Borel subgroup in a semisimple, simply connected algebraic group over $\mathbb{C}$ defined and split over $\mathbb{R}$. Given any reduced word $(i_1, \ldots, i_d)$ for w, the image of $f_{(i_1,\ldots,i_d)}$ applied to the set of points $(t_1, \ldots, t_d) \in \mathbb{R}^d_{>0}$ satisfying $\sum_{i=1}^d t_i = 1$ will be exactly Y_w^o.

One of the main results in [Her14] was a determination of the homeomorphism type of each cell closure in the natural stratification of the image of this map (restricted to domain the simplex). For instance in type A the (open) cells are determined by which minors of the image matrix are 0 and which are strictly positive. This result of [Her14] proved the Fomin-Shapiro Conjecture, following up on work of Fomin and Shapiro proving that the poset of closure relations for this stratified space is Bruhat order, determining homological structure (in type A) and providing properties of an intriguing and quite useful projection map.

Equivalently in type A (and also holding more generally), the aforementioned cell decomposition for the image of $f_{(i_1,\dots,i_d)}$ is based on which of the parameters $t_1,\dots,t_d$ are strictly positive and which are 0; one takes the subword of $(i_1,\dots,i_d)$ given by those positions with strictly positive parameters, calculates the Demazure product of this subword (or equivalently the unsigned 0-Hecke algebra product) and assigns the point $(t_1,\dots,t_d)$ to a cell in the image based on which Coxeter group we get as this Demazure product.

In type A, a different proof of the homeomorphism type of the closure of the big cell for the image of $f_{(i_1,\dots,i_d)}$ for the special case of $(i_1,\dots,i_d)$ a reduced word for the longest element was recently given in [GKL17], doing so in a way that relies heavily on special properties of the longest element. Further related results of Galashin, Karp and Lam which are even more recent and again focus on the big cell (but in related stratified spaces) appear in [GKL18]. Subsequent to the completion of our paper, Galashin, Karp and Lam posted to the math arXiv the paper [GKL19] which appears to be a major advance, proving all cell closures are homeomorphic to closed balls for the totally nonnegative real part of any partial flag variety (including the case of the Grassmannian which was conjectured by Postnikov). In each of these papers and preprints, the focus has been on understanding the images of maps. This work still leaves completely open the question of understanding the fibers of these same maps, which is the topic of our paper. Specifically, we focus on the fibers of the maps $f_{(i_1,\dots,i_d)}$ whose images were studied in [Her14].

The main goal of the present paper is to carry out a comprehensive study of the fibers $f^{-1}_{(i_1,\dots,i_d)}(p)$ of the map $f_{(i_1,\dots,i_d)}$ restricted to the simplex Δ_{d-1}, a domain having the benefit of being compact but nonetheless capturing the full structure of the fibers for the map $f_{(i_1,\dots,i_d)}$ on all of $\mathbb{R}^d_{\geq 0}$. There is a remarkably rich and intriguing structure to the fibers, as we hope our results (and conjectures) enumerated below will help to illuminate.

A main accomplishment of this paper, proven in Theorem 4.20, is as follows:

Theorem 1.1. *The inverse image $f^{-1}_{(i_1,\dots,i_d)}(p)$ of each point p admits a cell stratification induced by the natural cell stratification of the simplex by intersecting each cell of the simplex with $f^{-1}_{(i_1,\dots,i_d)}(p)$.*

Theorem 1.1 is obtained as an immediate corollary of a stronger, more technical result, Theorem 4.19, that gives a homeomorphism for each strata σ from a space $\tilde{\sigma}$ satisfying $\sigma \subsetneq \tilde{\sigma} \subsetneq \overline{\sigma}$ to $[0,1)^{\dim \sigma}$, doing so in such a way that this homeomorphism restricts to a homeomorphism from σ to $(0,1)^{\dim \sigma}$. While an assortment of different properties of fibers are developed throughout this paper, nearly all of the results of Section 4 come into play as ingredients in the proof of Theorem 4.19, and thereby for the proof of Theorem 1.1. Thus, Theorem 1.1 pulls together nearly all of the topological results regarding fibers as well as many of the combinatorial results that are proven in this paper.

In Proposition 3.4, we determine the combinatorial structure of each fiber $f^{-1}_{(i_1,\ldots,i_d)}(p)$:

Theorem 1.2. *The face poset for the cell decomposition for $f^{-1}_{(i_1,\ldots,i_d)}(p)$ restricted to Δ_{d-1} that is induced by the natural cell decomposition of the simplex Δ_{d-1} is isomorphic to the face poset of the interior dual block complex of the subword complex $\Delta(Q,w)$ for $p \in Y_w^o$ and $Q = (i_1,\ldots,i_d)$.*

We also prove contractibility of interior dual block complexes of subword complexes in Section 3, specifically in Proposition 3.5:

Theorem 1.3. *The interior dual block complex of any nonempty subword complex $\Delta(Q,w)$ is contractible.*

Taken together, Theorems 1.1, 1.2 and 1.3 give strong evidence that the fibers $f^{-1}_{(i_1,\ldots,i_d)}(p)$ themselves should be contractible (a property that seems to be essentially proven in [Her14] in a very complicated way within the body of other proofs there). We conjecture contractibility and more in regards to the structure of the fibers. Specifically, we make the following pair of conjectures:

Conjecture 1.4. *Given any fiber $f^{-1}_{(i_1,\ldots,i_d)}(p)$ for $p \in Y_w^o$, the stratification of this fiber that is induced by the standard cell decomposition of the simplex is a regular CW decomposition of the fiber.*

This would imply the following conjecture, by virtue of Theorems 1.2 and 1.3.

Conjecture 1.5. *There is a cell structure preserving homeomorphism from the cell decomposition for $f^{-1}_{(i_1,\ldots,i_d)}(p)$ for $p \in Y_w^o$ that is induced by that of a simplex to the interior dual block complex for the subword complex $\Delta((i_1,\ldots,i_d),w)$. In particular, $f^{-1}_{(i_1,\ldots,i_d)}(p)$ is contractible.*

One reason for interest in knowing that the fibers are contractible is that this can be used to give a new proof of the Fomin-Shapiro Conjecture, as discussed in Section 5.

Remark 1.6. One might hope for each fiber $f^{-1}_{(i_1,\ldots,i_d)}(p)$ to be a closed ball, or at least to be "pure", namely for all of its maximal cells to have the same dimension as each other. However, there are counterexamples to both of these statements.

For example, consider $(i_1,\dots,i_d) = (1,3,2,1,3,2)$ for any choice of $p \in Y_w^o$ for $w = s_1s_3s_2$. Theorems 1.1 and 1.2 together imply in this case that maximal cells are not all of the same dimension as each other.

In Section 2, we review background, doing so in a way that aims to make this paper accessible to readers coming from an assortment of fields including combinatorics, topology and representation theory. In Section 3, we determine the combinatorial structure of fibers in terms of interior dual block complexes of subword complexes; we also prove in Section 3 that the unique regular CW complexes having the same face posets as our fibers are contractible. Section 4 proves that the standard regular CW decomposition of a simplex restricted to any fiber $f^{-1}_{(i_1,\dots,i_d)}(p)$ of $f_{(i_1,\dots,i_d)}$ gives a cell decomposition of $f^{-1}_{(i_1,\dots,i_d)}(p)$. Finally, we show in Section 5 how contractibility of fibers would combine with other results in this paper together with results in the literature regarding approximating maps by homeomorphisms to yield a new proof of the Fomin-Shapiro Conjecture.

2. Background

2.1. **Cell decompositions and their closure posets.** A *decomposition* of a topological space X is a collection $\{X_\alpha\}_{\alpha\in I}$ of disjoint subsets whose union is X. A *stratification* of X is a decomposition in which $X_\alpha \cap \overline{X_\beta} \neq \emptyset$ implies $X_\alpha \subseteq \overline{X_\beta}$. A *$d$-cell* is a topological space homeomorphic to the interior of the d-ball. A *cell* is a d-cell for some d. A *cell decomposition (respectively, cell stratification)* is a decomposition (respectively, stratification) where each X_α is a cell.

A finite *CW complex* is a decomposition of a Hausdorff space into a finite number of cells so that (i) a set is closed if and and only if its intersection with the closure of each cell is closed, (ii) the topological boundary of every d-cell is contained in a finite union of cells of dimension strictly less than d, and (iii) for every d-cell there is a continuous surjective map from an d-ball to the closure of the cell which restricts to a homeomorphism from the interior of the d-ball to the cell. A map satisfying (iii) above is called a *characteristic map* and the restriction of a characteristic map to the sphere S^{d-1} is called an *attaching map.* A map $f : A \to B$ is an *embedding* if it is a homeomorphism onto its image. A finite CW complex is a cell stratification. A *regular CW complex* is a CW complex so that for each cell there exists an attaching map which is an embedding.

A stratification induces a partial order on the index set by defining $\alpha \le \beta$ if and only if $X_\alpha \subset \overline{X_\beta}$. That is, the *closure poset* (or *face poset*) of a stratified space or CW complex, when this poset is well-defined, is the partial order on cells given by $\sigma \le \tau$ iff $\sigma \subseteq \overline{\tau}$. We denote its unique minimal element, corresponding to the empty face, by $\hat{0}$. A map $f : P \to Q$ from a poset P to a poset Q is a *poset map* if $u \le v$ in P implies $f(u) \le f(v)$ in Q. Recall that a poset is *graded* if for each $u \le v$, all paths from u to v have the same length. A graded poset is *thin* if each rank 2 interval $[u, w]$ includes exactly 2 elements v_1, v_2 satisfying $u < v_i < w$, namely the open interview (u, w) consists of exactly these 2 elements. For regular CW complexes, the closure poset will be a poset graded by cell dimension with each open interval $(\hat{0}, v) = \{z \mid \hat{0} < z < v\}$ having order complex that is homeomorphic to a sphere $S^{\mathrm{rk}\, v-2}$. Björner proved in [Bjö84] that this together with having a unique minimal element and at least one other poset element is enough to ensure that a finite, graded poset is the closure poset of a regular CW complex; finite, graded posets with these properties are therefore called *CW posets*. Results of Danaraj and Klee from [DK] imply that finite, graded posets with unique minimal element and at least one additional element will be CW posets if they are thin and shellable; this was used to prove that Bruhat order is a CW poset, a fact we will use in Section 5.

2.2. **Coxeter groups and the associated 0-Hecke algebras.** We will make use of numerous well known properties of finite Coxeter systems as well as versions of these properties that transfer to associated 0-Hecke algebras. The unsigned 0-Hecke algebra will emerge out of a need to use a non-standard product on a Coxeter group called the Demazure product. We now review these notions and the properties we will need.

A *Coxeter system* (W, Σ) consists of a finite group W and a finite set of generators Σ so that W has a presentation of the form

$$W = \langle s_i \in \Sigma \mid (s_i s_j)^{m(s_i, s_j)} = e \rangle$$

where the $m(s_i, s_j)$ are positive integers with $m(s_i, s_i) = 1$ and with $m(s_i, s_j) = m(s_j, s_i) \geq 2$ for $i \neq j$. The set Σ is a minimal generating set whose elements are called *simple reflections*. Every element of Σ has order 2. The elements s_i and s_j commute if and only if $m(s_i, s_j) = 2$. Sometimes we refer to commutation relations $s_i s_j = s_j s_i$ as *short braid relations*. More generally, the relation $(s_i s_j)^{m(s_i, s_j)} = e$ is equivalent to the *braid relation* $s_i s_j s_i \cdots = s_j s_i s_j \cdots$ where each side of the equation is a product of $m(s_i, s_j)$ simple reflections alternating between s_i and s_j. We call this a *long braid relation* for $m(s_i, s_j) > 2$. See [Hum90] or [BB05] for background on Coxeter groups.

For instance, when $W = S_n$ is the symmetric group, we can take $\Sigma = \{s_1, \ldots, s_{n-1}\}$, where $s_i = (i\ i+1)$ is an adjacent transposition. The relations are $s_i^2 = e$, $(s_i s_j)^2 = e$ for $|j - i| > 1$ and $(s_i s_{i+1})^3 = e$ for $1 \leq i \leq n-1$. The corresponding long braid relation in this case is $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

An *expression* for $w \in W$ is a product $s_{i_1} \cdots s_{i_d}$ of simple reflections equalling w under the standard group-theoretic product. This is called a *reduced expression* for w if d is minimal among all possible expressions for w. A *word* of size d is an ordered sequence $Q = (i_1, \ldots, i_d)$ of subscripts each indexing an element of Σ. Since one may pass easily back and forth between an expression and the corresponding word, one often speaks in terms of words just because they encode the same data more compactly. An ordered subsequence P of a word Q is called a *subword* of Q, written

$P \subseteq Q$. The expression corresponding to such P is called a *subexpression* of the expression corresponding to Q.

Subwords of Q come with their embeddings into Q, so two subwords P and P' involving reflections at different positions in Q are treated as distinct even if the sequences of reflections in P and P' coincide. To simplify notation, often we write Q as a string without parentheses or commas, and abuse notation by saying that Q is a word in W, without explicit reference to Σ. An expression for $w \in W$ as a product $w = s_{i_1} \cdots s_{i_d}$ is *reduced* if d is as small as possible; this minimal d is the *length* of w.

The following results may be found, e.g., in [Hum90] and [BB05] where they appear in Section 1.7 and Theorem 3.3.1, respectively.

Lemma 2.1 (Exchange Condition)**.** *Let $w = s_{i_1} \cdots s_{i_r}$ (not necessarily reduced), where each s_{i_j} is a simple reflection. If $\ell(ws_i) < \ell(w)$ for a simple reflection $s_i \in \Sigma$, then there exists an index j for which $ws_i = s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_r}$. In particular, w has a reduced expression ending in a simple reflection $s_i \in \Sigma$ if and only if $\ell(ws_i) < \ell(w)$.*

Theorem 2.2. *Any two reduced expressions for the same element w of a finite Coxeter group W are connected by a series of (long and short) braid moves, where a short braid move is $s_is_j \to s_js_i$ for $m(i,j) = 2$ and a long braid move is $s_is_js_i \cdots \to s_js_is_j \cdots$ with each of these expressions alternating s_i and s_j consisting of $m(i,j) > 2$ letters. Moreover, any expression for w is connected to any reduced expression for w by a series of long and short braid moves together with nil-moves $s_i^2 \to s_i$.*

In particular, the following is an immediate consequence of the above results.

Lemma 2.3. *Fix a reduced expression $s_{i_1} \ldots s_{i_d}$ for $w \in W$ and a simple reflection $s \in \Sigma$ such that $s_{i_1} \ldots s_{i_d} s$ is non-reduced. Then there is a reduced expression $s_{j_1} \ldots s_{j_d}$ for w with $s_{j_d} = s$ and a sequence of (long and short) braid moves that transforms $s_{i_1} \ldots s_{j_d}$ into $s_{j_1} \ldots s_{j_{d-1}} s$.*

Thus any two reduced expressions for a word have the same length and multiplication by a simple reflection always changes the length.

Proposition 2.4. *There is a unique associative map $\delta : W \times W \to W$ such that*

$$\delta(w, s_i) = \begin{cases} ws_i & \text{if } \ell(ws_i) > \ell(w) \\ w & \text{if } \ell(ws_i) < \ell(w) \end{cases}$$

for $w \in W$ and $s_i \in \Sigma$.

Proof. See [KM04, Section 3]. □

Definition 2.5. The map δ in Proposition 2.4 is the *Demazure product* on W. Using associativity, extend it to a map $\delta : W^d \to W$ for all positive integers d. Given a word $Q = (i_1, \ldots, i_d)$ in the sense of Definition 2.11, define $\delta(Q) = \delta(s_{i_1}, \ldots, s_{i_d})$. The key relations in the case of the symmetric group are $\delta(s_i, s_{i+1}, s_i) = \delta(s_{i+1}, s_i, s_{i+1})$, $\delta(s_i, s_j) = \delta(s_j, s_i)$ when $|i - j| > 1$, and $\delta(s_i, s_i) = \delta(s_i)$.

The following alternative description for the Demazure product will justify the equivalence of this map δ to the standard product for the unsigned 0-Hecke algebra, defined immediately after Lemma 2.6, with this equivalence using the bijective correspondence between generators of W and its (unsigned) 0-Hecke algebra:

Lemma 2.6. *The definition above is equivalent to the following set of requirements for an associative map δ:*

1. $\delta(w, s_i) = ws_i$ *if* $l(ws_i) > l(w)$
2. $\delta(s_i, s_i) = s_i$ *for* $s_i \in \Sigma$.
3. $\delta(s_i, s_j, s_i \dots) = \delta(s_j, s_i, s_j, \dots)$ *where each side is an alternation of length* $m(i,j)$ *of the simple reflections* s_i *and* s_j.

Proof. Each of these three conditions follows easily from special cases of the conditions given in Definition 2.5. Conversely, we obtain the condition $\delta(w, s_i) = w$ for $l(ws_i) < l(w)$ from Definition 2.5 from these three conditions as follows. We use the fact that w must have a reduced expression with s_i as its rightmost letter to have $l(ws_i) < l(w)$ (see Lemma 2.1) together with the fact (recalled in Theorem 2.2) that any reduced expression for w may be obtained from any expression for w via a series of (long and short) braid moves and nil-moves (with each nil-move giving rise to a modified nil-move $(s_i, s_i) \to s_i$ when using the Demazure product). □

A finite Coxeter system (W, Σ) gives rise to the Demazure product (W, δ) which in turn gives rise to a ring, called the 0-Hecke algebra. Abstracting a bit, let G be a set, $e \in G$ an element, and $\phi : G \times G \to G$ be an associative function, so that $\phi(e, g) = g = \phi(g, e)$ for all $g \in G$. Let R be a ring. Define a ring $R[G, \phi]$ to be additively the left free R-module with basis G and give it the multiplication

$$\left(\sum r_i g_i\right)\left(\sum r'_j g'_j\right) = \sum r_i r'_j \phi(g_i, g'_j)$$

The ring $\mathbb{F}_2[W, \delta]$ is the 0-Hecke algebra. Recasting this a bit, let $\mathbb{F}_2\langle s_1, s_2, \dots, s_k\rangle$ be the free, noncommutative, associative, unital $\mathbb{F}_2$-algebra generated by the simple reflections, and let I be its 2-sided ideal generated by $s_i^2 - s_i$ and by the "braid relations" $(s_i s_j s_i \dots) - (s_j s_i s_j \dots)$ where both terms are an alternation of the letters s_i and s_j of length $m(s_i, s_j)$. Rewriting the image of s_i in the quotient ring by x_i we set

$$\mathbb{F}_2[x_1, x_2, \dots, x_k] = \mathbb{F}_2\langle s_1, s_2, \dots, s_k\rangle / I$$

where the x_i satisfy the relations $x_i^2 = x_i$ and $x_i x_j x_i \cdots = x_j x_i x_j \dots$. These relations are called a *modified nil move* and a *braid move* respectively.

Part 2 and 3 of Lemma 2.6 give a map

$$\mathbb{F}_2[x_1, x_2, \dots, x_k] \to \mathbb{F}_2[W, \delta],$$

Part 1 and the expression of a group element as a reduced word gives surjectivity of the map and the uniqueness gives injectivity. Henceforth we identify the two rings and call them the 0-Hecke algebra.

We note that this is exactly the specialization of the usual Hecke algebra over the field of two elements where the usual parameter q is set to 0; in this context, we may ignore signs.

Definition 2.7. The *Bruhat order* is the partial order on elements of a Coxeter group W with $u \leq v$ if and only if there exist reduced expressions for u and v such that the reduced expression for u is a subexpression of the reduced expression for v.

The next notion made an early appearance in [KM04, Lemma 3.5.2] and was formally defined (and named) in [Her14], where it played a key role in several proofs.

Definition 2.8. The letters s_{i_j} and s_{i_k} in an expression $s_{i_1} \dots s_{i_d}$ constitute a *deletion pair* if $j < k$ with $s_{i_j} \dots s_{i_{k-1}}$ and $s_{i_{j+1}} \dots s_{i_k}$ both reduced expressions for the same Coxeter group element while $s_{i_j} \dots s_{i_k}$ is a non-reduced expression.

For example, in the symmetric group, $s_3s_1s_2s_1s_2$ has a deletion pair $\{s_{i_2}, s_{i_5}\}$. It is proven in [Her14] that the condition that these two reduced expression are for the same Coxeter group element actually follows from the other parts of the definition.

Proposition 2.9. *Given a deletion pair $\{s_{i_j}, s_{i_k}\}$ in an expression $s_{i_1} \dots s_{i_d}$, there is a series of braid moves that may be applied to $s_{i_j} \dots s_{i_{k-1}}$ yielding another reduced expression $s_{i'_j} \dots s_{i'_{k-1}}$ such that $i'_{k-1} = i_k$.*

Proof. This is a consequence of the exchange axiom for Coxeter groups along with Theorem 2.2 and Lemma 2.3. ☐

Lemma 2.10. *Under the conditions of Definition 2.8, the two words $s_{i_j} \dots s_{i_{k-1}}$ and $s_{i_{j+1}} \dots s_{i_k}$ are reduced expressions for the same Coxeter group element $\delta(s_{i_j}, \dots, s_{i_k})$.*

Proof. This follows from [KM04, Lemma 3.5] or from [Her14, Lemma 5.5]. ☐

The Demazure product is quite useful for understanding relationships between reduced subwords for a fixed element w inside of a given ambient word Q [KM04]. These relationships are expressed topologically using the subword complexes discussed next, complexes which were first introduced in [KM05, Definition 1.8.1] and [KM04, Definition 2.1].

Definition 2.11. A word Q *represents* $w \in W$ if the ordered product of the simple reflections in Q is a reduced decomposition for w. A word Q *contains* $w \in W$ if some subsequence of Q represents w.

The *subword complex* $\Delta(Q, w)$ for a word Q and an element $w \in W$ is the simplicial complex whose k-simplicies are given by $(k + 1)$-letter subwords R of Q so that $P = Q \setminus R$ contains w.

The facets of the subword complex $\Delta(Q, w)$ are given by those words $R = Q \setminus P$ where P is a reduced word for w.

Theorem 2.12 ([KM04, Theorems 2.5 and 3.7 and Corollary 3.8]). *The subword complex $\Delta(Q, w)$ is shellable and homeomorphic to either a ball or a sphere. It is homeomorphic to a ball if and only if $\delta(Q) \neq w$. A face $Q \smallsetminus P$ lies in the boundary of $\Delta(Q, w)$ if and only if P satisfies $\delta(P) \neq w$.*

2.3. **Total positivity.** Recall that the *minors* of a matrix are the determinants of its $i \times i$ submatrices. A matrix in $M_n(\mathbb{R})$ is *totally nonnegative* if all of its minors are greater than or equal to zero; it is *totally positive* if all of its minor are strictly positive. We are interested in the space of totally nonnegative, real matrices which are upper triangular with ones on the main diagonal, and more generally in the totally nonnegative real part of the unipotent radical of a Borel subgroup in a semisimple, simply connected algebraic group.

Given any (not necessarily reduced) word $Q = (i_1, \ldots, i_d)$, Lusztig defined a continuous map

$$f_Q : \mathbb{R}^d_{\geq 0} \to U^+_{\geq 0}$$

which in type A is the map

$$f_Q : \mathbb{R}^d_{\geq 0} \to M_n(\mathbb{R})$$

given by $f_Q(t_1, \ldots, t_d) = x_{i_1}(t_1) \ldots x_{i_d}(t_d)$, where $x_j(t) = I_n + tE_{j,j+1}(n)$ with $E_{j,j+1}(n)$ the n-by-n matrix which is all zeroes except for a 1 in row j and column $j+1$. More generally, we have the map $f_Q(t_1, \ldots, t_d) = x_{i_1}(t_1) \ldots x_{i_d}(t_d)$, with $x_i(t)$ denoting the exponentiated Chevalley generator $exp(te_i)$.

We will be especially focused on the structure of each *fiber* of the map $f_{(i_1,\ldots,i_d)}$, by which we mean a set $f^{-1}_{(i_1,\ldots,i_d)}(p)$ for some $p \in M_n(\mathbb{R})$ (in type A) or more generally some $p \in U^+_{\geq 0}$. Now we recall results of Lusztig from [Lus94] after first introducing notation; these results from [Lus94] will prove vital to our work.

Denote U^+ the unipotent radical of a Borel subgroup B in a real reductive group and $U^+_{\geq 0}$ denote the nonnegative part.

Proposition 2.13. *[Lus94, Proposition 2.6] Let $s_i, s_{i'}$ be distinct simple reflections in W. Let $m(i, i') \geq 2$ be the order of $s_i s_{i'}$ in W. Let $a_1, a_2, \ldots, a_m$ be a sequence in $\mathbb{R}_{>0}$. Then there exists a unique sequence $a'_1, a'_2, \ldots, a'_m$ in $\mathbb{R}_{>0}$ such that*

$$x_i(a_1)x_{i'}(a_2)x_i(a_3)x_{i'}(a_4) \cdots = x_{i'}(a'_1)x_i(a'_2)x_{i'}(a'_3)x_i(a_4) \cdots$$

with each side of the equation consisting of a product of $m(i, i')$ factors with alternating subscripts i and i'.

Example 2.14. In type A for $1 \leq i < n$, Proposition 2.13 specializes to the statement

$$x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2t_3}{t_1 + t_3}\right) x_i(t_1 + t_3)x_{i+1}\left(\frac{t_1t_2}{t_1 + t_3}\right)$$

for any $t_1, t_2, t_3 > 0$. One may easily confirm this identity by matrix multiplication.

Proposition 2.15. *[Lus94, Proposition 2.7] Let $w \in W$, and let $s_{i_1} s_{i_2} \cdots s_{i_n}$ be a reduced expression for w. Then the following all hold.*

(a) The map $f_{(i_1,\dots,i_d)}$ from $\mathbb{R}^n_{>0}$ to U^+ given by $(a_1, a_2, \dots, a_n) \mapsto x_{i_1}(a_1)x_{i_2}(a_2)\cdots x_{i_n}(a_n)$ is injective.

(b) The image of the map $f_{(i_1,\dots,i_n)}$ from part (a) is a subset $U^+(w)$ of $U^+_{\geq 0}$ which depends on w but not on the choice of reduced word $(i_1, i_2, \dots, i_n)$ for w.

(c) If $w' \neq w$ for $w', w \in W$, then $U^+(w) \cap U^+(w') = \varnothing$.

(d) For $a_1, \dots, a_n$ nonzero elements of $\mathbb{R}$, we have

$$x_{i_1}(a_1)\cdots x_{i_n}(a_n) \in B^- s_{i_1} B^- s_{i_2} B^- \cdots s_{i_n} B^- \subset B^- s_{i_1} s_{i_2} \cdots s_{i_n} B^-$$

by virtue of properties of the Bruhat decomposition.

Proposition 2.16. *[Lus94, Proposition 4.2] (a) $U^+_{\geq 0}$ is a closed subset of U^+*
(b) $U^+_{>0}$ is a dense subset of $U^+_{\geq 0}$

Within the proof of Proposition 4.2 of [Lus94] is also a proof of the following:

Proposition 2.17 (Lusztig)**.** *Given any (not necessarily reduced) word $(i_1, \dots, i_d)$, the map $f_{(i_1,\dots,i_d)}$ is a proper map from $\mathbb{R}^d_{\geq 0}$ to $U^+_{\geq 0}$.*

Using that $\mathbb{R}^d_{>0}$ is locally compact and Hausdorff, one sees that $f_{(i_{j_1},\dots,i_{j_d})}$ is an open map on $\mathbb{R}^d_{>0}$. Combining this with the fact for $(i_1, \dots, i_d)$ reduced that $f_{(i_1,\dots,i_d)}$ is bijective from $\mathbb{R}^d_{>0}$ to its image yields another result of Lusztig:

Theorem 2.18 (Lusztig)**.** *Given any reduced word Q of size d, the map f_Q induces a homeomorphism from $\mathbb{R}^d_{>0}$ to its image.*

Remark 2.19. While the map r from Section 2.17 of [Lus94] that is referenced in the proof of Proposition 2.17 might a priori appear to be defined specifically for w_0, this map r makes equally good sense for all words $(i_1, \dots, i_d)$, including nonreduced words. Likewise, Lusztig's proofs of Proposition 2.17 and Theorem 2.18 both hold in the generality of any word, reduced or otherwise. This added generality will indeed be used in our upcoming results.

Lusztig also generalized beyond type A the following result that he notes was essentially proven by Whitney in [Wh52i] (as observed by Loewner in [Loe55], namely that the result stated below was a consequence of the work of Whitney):

Theorem 2.20. *Given any reduced word Q for the longest element w_0 in the symmetric group, the image of f_Q as a map on $\mathbb{R}^d_{\geq 0}$ is the entire space of real-valued upper triangular matrices with 1's on the diagonal having all minors nonnegative. The analogous statement also holds more generally for the totally nonnegative part of the unipotent radical of a Borel subgroup in a reductive group.*

Lusztig gives a cell decomposition $\{U^+(w)\}_{w\in W}$ of $U^+_{\geq 0}$ based on the map $f_{(i_1,\ldots,i_d)}$ for $(i_1,\ldots,i_d)$ any reduced word for the longest element. His results imply that all choices of reduced word yield the same stratification. This stratification induces a cell stratification of $U^+_{\geq 0}\cap f_{(i_1,\ldots,i_d)}(\Delta^{d-1})$, a space which is compact and reflects all of the topological structure of the original space. (Note that $\Delta^{d-1}\subset\mathbb{R}^d$ consists of the points whose coordinates are nonnegative and sum to 1.) Likewise for each $v<w_0$ there is a subword of $(i_1,\ldots,i_d)$ that is a reduced word for v, allowing us also to deduce from this a cell stratification for $\overline{U_v}=\{U^+(u)\}_{u\leq v}$.

In type A, for example, one way to describe the strata is to note that each matrix $M\in U^+_{\geq 0}$ is assigned to a strata based on which minors in M are strictly positive and which are zero. We will show more generally that for each $p=x_{i_1}(t_1)\cdots x_{i_d}(t_d)$, that the assignment of p to a strata is dictated by taking the subexpression of $x_{i_1}\cdots x_{i_d}$ consisting of exactly those letters x_{i_r} such that $t_r>0$ and then replacing each remaining x_{i_r} by s_{i_r} and calculate the Demazure product to associate to this a Coxeter group element specifying which strata p lies within.

Hersh observed and proved in [Her14] that the Demazure product could be used to make the following statement also for words Q that are not necessarily reduced. The proof of Lemma 2.14 in [Lus94] is also very suggestive of this structure.

Lemma 2.21. *For any word Q of size d (reduced or not), $f_Q(\mathbb{R}^d_{>0})=U^+(\delta(Q))$.*

We will heavily use the following closely related result from [Her14].

Lemma 2.22. *The set $\{x_{i_1}(t_1)x_{i_2}(t_2)\cdots x_{i_d}(t_d)|t_1,\ldots,t_d\geq 0\}$ is equal to the set $\{x_{j_1}(u_1)x_{j_2}(u_2)\cdots x_{j_{d'}}(u_{d'})\}$ if and only if the Demazure product for $(i_1,\ldots,i_d)$ equals the Demazure product for $(j_1,\ldots,j_{d'})$.*

Remark 2.23. Proposition 2.15, part (c), may be used to deduce that the sets $f_Q(\mathbb{R}^d_{>0})$ and $f_{Q'}(\mathbb{R}^{d'}_{>0})$ appearing in Lemma 2.21 are nonintersecting whenever the words Q,Q' have distinct Demazure products.

The above results imply that the image of $f_{(i_1,\ldots,i_d)}$ on the subset of $\mathbb{R}^d_{\geq 0}$ with coordinates summing to a fixed positive constant decomposes into cells that are each the images of one or more cells in the standard cell decomposition of the simplex Δ_{d-1}. In particular, this guarantees that the map $f_{(i_1,\ldots,i_d)}$ of stratified spaces induces a map of face posets. The following combinatorial description of this poset map was introduced implicitly in [Her14] and studied in its own right in [AH11].

Proposition 2.24. *The Demazure product induces a poset map from the face poset of a simplex, namely the Boolean lattice of subwords of a fixed reduced word $(i_1,\ldots,i_d)$ for a Coxeter group element $w\in W$, to the face poset for Y_w, namely the Bruhat order interval $[1,w]$. Each subset $\{j_1,\ldots,j_k\}$ of $\{1,\ldots,d\}$ with $1\leq j_1<\cdots<j_k\leq d$ naturally corresponds to a subword $(i_{j_1},\ldots,i_{j_k})$ of $(i_1,\ldots,i_d)$. Thus, we identify elements of the Boolean lattice with subwords.*

The map f of face posets induced by $f_{(i_1,\ldots,i_d)}$ sends the subword $(i_{j_1},\ldots,i_{j_k})$ to the Bruhat order element $\delta(s_{i_{j_1}},\ldots,s_{i_{j_k}}) \in W$. This is a poset map.

2.4. **Interior dual block complexes.** Each fiber of the map $f_{(i_1,\ldots,i_d)}$ comes with a combinatorial decomposition (see Definition 3.1) induced by the stratification of the simplex (or the nonnegative orthant) by its polyhedral faces. The combinatorics of this decomposition of the fiber precisely matches the block decomposition of the interior dual block complex (see Definition 2.28) of a subword complex, as we will show in Proposition 3.4.

Definition 2.25. For a nonempty face ϕ of a simplicial complex Δ, the *(closed) dual block* of ϕ in Δ is the underlying space of the simplicial complex constructed as follows:

- take the cone from the barycenter β of the face ϕ over the link of ϕ in Δ;
- barycentrically subdivide that cone; and then
- take the star of β (equivalently, delete all vertices in the original link).

Definition 2.26. A *topological n-manifold with boundary* is a Hausdorff space M having a countable basis of open sets, with the property that every point of M has a neighborhood homeomorphic to an open subset of $\mathbb{H}^n$, where $\mathbb{H}^n$ is the half-space of points $(x_1,\ldots,x_n)$ in $\mathbb{R}^n$ with $x_n \geq 0$. The *boundary* of M, denoted ∂M, is the set of points $x \in M$ for which there exists a homeomorphism of some neighborhood of x to an open set in $\mathbb{H}^n$ taking x into $\{(x_1,\ldots,x_n)|x_n = 0\} = \partial\mathbb{H}^n$.

Proposition 2.27. *If Δ is a simplicial PL manifold-with-boundary with a nonempty interior face ϕ (that is, ϕ is not contained in the boundary of Δ), then the dual block of ϕ is homeomorphic to a closed ball.*

Proof. This is a consequence of basic results on PL balls and spheres [BLSWZ99, Theorem 4.7.21]. The link of an interior face ϕ is a PL sphere. Hence the cone over the link from the barycenter β of ϕ is a PL ball, as is the barycentric subdivision, with β as an interior vertex. Thus the star of β in the subdivision is another PL ball. □

Definition 2.28. Fix Δ, a simplicial manifold-with-boundary. The *interior dual block complex* of Δ is

1. the union of the dual blocks of interior faces of Δ, if Δ has nonempty boundary.
2. a ball whose boundary is the dual block complex of Δ, if Δ is a sphere.

(The *dual block complex* of Δ is the union of the dual blocks of its nonempty faces.)

Lemma 2.29. *Fix a simplicial manifold Δ with nonempty boundary. If Δ° is obtained from the barycentric subdivision of Δ by deleting all vertices lying on the boundary and all cells with any of these vertices in their closure, then the underlying spaces of Δ° and the interior dual block complex of Δ coincide.*

Proof. Immediate from Definitions 2.25 and 2.28. □

Proposition 2.30. *The interior dual block complex of any simplicial PL sphere or simplicial PL manifold with nonempty boundary is a regular CW complex.*

Proof. This is immediate from [Mun84, Theorem 64.1], given that the relevant dual blocks are closed cells by Proposition 2.27. ☐

Proposition 2.31. *The interior dual block complex given by any regular CW decomposition of any ball or sphere is contractible.*

Proof. The sphere case is by construction. For balls, use Lemma 2.29 with [Mun84, Lemma 70.1]: removing the full closed subcomplex on a vertex set yields an open subset that retracts onto the full closed subcomplex on the remaining vertices. ☐

For further background and basics on dual blocks and dual block complexes, see [Mun84, §64]. For a brief introduction to *piecewise linear* or *PL topology* that suits our purposes, see [BLSWZ99, Section 4.7(d)]. For a more in-depth introduction to the notion of *manifold-with-boundary*, see [Mun84, §35].

2.5. **A topological interlude.** Now let us review the key topological result that we will use to prove how contractibility of fibers would combine with our other results to yield a new proof of the Fomin-Shapiro Conjecture. That is, we will use this to deduce deduce the homeomorphism type for the image of $f_{(i_1,\ldots,i_d)}$ from contractibility of fibers.

The following beautiful theorem and corollary were stated (in somewhat different language) as Proposition A.1 and Corollary A.2 in [GLMS08].

Theorem 2.32. *A map $f : S^{n-1} \to S^{n-1}$ so that $f^{-1}(y)$ is contractible for all $y \in Y$ can be extended to a map $F : B^n \to B^n$ inducing a homeomorphism $\operatorname{int} B^n \to \operatorname{int} B^n$.*

Corollary 2.33. *Let $\sim$ be an equivalence relation on the closed ball B^n so that*

- *all equivalence classes are contractible,*
- *$S^{n-1}/\sim$ is homeomorphic to S^{n-1},*
- *if $x \sim y$ with $x \in S^{n-1}$, then $y \in S^{n-1}$,*
- *if $x \sim y$ with $x \notin S^{n-1}$, then $y = x$.*

Then B is homeomorphic to $B/\sim$.

Proof of Corollary 2.33. Let $\sim$ be such an equivalence relation on S^{n-1}. Let $f : S^{n-1} \to S^{n-1}$ be the composite of the quotient map $S^{n-1} \to S^{n-1}/\sim$ with a homeomorphism to S^{n-1}. Let $F : B^n \to B^n$ be produced by Theorem 2.32. Let $\sim_F$ be the equivalence relation $x \sim_F y$ if and only if $F(x) = F(y)$. By hypothesis $\sim$ and $\sim_F$ are identical. By the universal property of the quotient topology, there is a continuous bijection $B/\sim_F \to B^n$. Since the domain is compact and the target is Hausdorff, this map is a homeomorphism. ☐

We include a proof of Theorem 2.32 since its elements are not familiar to combinatorialists. The strategy is to argue that the map $S^{n-1} \to S^{n-1}/\sim$ is cell-like and then to apply the cell-like (= CE) approximation theorem as well as the local contractibility of the homeomorphism group of a manifold.

The following definition is taken from the survey of Dydak [Dyd02].

Definition 2.34. A topological space is *cell-like* if any map to a CW complex is null-homotopic. A map $f : X \to Y$ is *cell-like* if f is proper (the inverse image of any compact set is compact) and $f^{-1}(y)$ is cell-like for all $y \in Y$.

The key result in this area is Siebenmann's CE-approximation theorem.

Theorem 2.35. *Let $f : X \to Y$ be a cell-like map between topological manifolds of the same dimension. Then X and Y are homeomorphic.*

Remark 2.36. In fact, if Y is, in addition, a metric space, then for any continuous $\varepsilon : X \to (0, \infty)$, there is a homeomorphism $g : X \to Y$ so that for all $x \in X$, $d(f(x), g(x)) < \varepsilon(x)$.

Remark 2.37. The above theorem was proven by Siebenmann [Si72] in dimensions greater than four, by Armentrout for dimensions less than four, and by Quinn [Qui82] for dimension four.

Proof of Theorem 2.32. We will define a one-parameter family

$$\Phi : [0, 1] \to \mathrm{Map}(S^{n-1}, S^{n-1})$$

of self-maps of S^{n-1} so that $\Phi_0 = f$ and Φ_r is a homeomorphism for $r \in (0, 1]$. Given such a Φ, define $F(rx) = r\Phi_{1-r}(x)$ for $r \in [0, 1]$ and $x \in S^{n-1}$.

The two key ingredients in producing Φ are the CE-Approximation Theorem and local contractibility of the homeomorphism group of a compact manifold, due independently to Černavskiĭ [Če69] and Edwards-Kirby [EK71].

The topology on $\mathrm{Map}(S^{n-1}, S^{n-1})$ and its subspace $\mathrm{Homeo}(S^{n-1})$ is given by the uniform metric $d(g, h) = \sup_{x \in S^{n-1}} \|g(x)-h(x)\|$. Local contractibility of $\mathrm{Homeo}(S^{n-1})$ implies that for every $\epsilon > 0$, there is a $\delta > 0$ so that if $g, h \in B_\delta(\mathrm{Id}) \subset \mathrm{Homeo}(S^{n-1})$, there is a path from g to h whose image lies in $B_\epsilon(\mathrm{Id})$. For $k \in \mathrm{Homeo}(S^{n-1})$, right translation $R_k : \mathrm{Homeo}(S^{n-1}) \to \mathrm{Homeo}(S^{n-1}); R_k(g) = g \circ k$ is an isometry; hence for every $\epsilon > 0$, there is a $\delta > 0$ so that if $g, h \in B_\delta(k)$, there is a path from g to h whose image lies in $B_\epsilon(k)$.

Now for every $i \in \mathbb{Z}_{>0}$, choose $\delta_i > 0$ so that $g, h \in B_{\delta_i}(k)$ implies there is a path from g to h which lies in $B_{1/2^i}(k)$. We also make the choices so that $\delta_i > \delta_{i+1}$ for all i. To define the map $\Phi : [0, 1] \to \mathrm{Homeo}(S^{n-1}); r \mapsto \Phi_r$ we set $\Phi_0 = f$, define $\Phi_{1/2^i}$ using the CE-Approximation Theorem, and then connect the dots using local contractibility. Using the CE-Approximation Theorem, choose homeomorphisms Φ_{2^i} so that $d(f, \Phi_{1/2^i}) < \delta_i/2$. Then by the triangle inequality, $d(\Phi_{2^{i+1}}, \Phi_{2^i}) < \delta_i$. By the choice of δ_i there is a path $\Phi : [1/2^{i+1}, 1/2^i] \to \mathrm{Homeo}(S^{n-1}, S^{n-1})$ from $\Phi_{1/2^{i+1}}$

to $\Phi_{1/2^i}$ which lies in a ball of radius $1/2^i$. Concatenation gives our desired path $\Phi : [0,1] \to \mathrm{Map}(S^{n-1}, S^{n-1})$. $\square$

3. Combinatorics of each fiber

The relevance of interior dual block complexes of subword complexes to Lusztig's parametrizations arises via stratifications, as we discuss now in this section.

Definition 3.1. Fix a word Q and a fiber $F \subseteq \mathbb{R}^d_{\geq 0}$ of Lusztig's parametrization f_Q. For each subword $P \subseteq Q$, let $F_P = F \cap \mathbb{R}^P_{>0}$ be the intersection of F with the strictly positive orthant indexed by P. The *natural stratification* of F has strata F_P for $P \subseteq Q$ and closed strata $\overline{F}_P = F \cap \mathbb{R}^P_{\geq 0}$.

Lemma 3.2. *In the setting of the natural stratification as in Definition 3.1, any nonempty intersection of closed strata is a closed stratum.*

Proof. By definition, $\overline{F}_P \cap \overline{F}_{P'} = \overline{F}_{P \cap P'}$ for any two subwords P and P' of Q. $\square$

Lemma 3.3. *The fiber F of Lusztig's parametrization over any point is a real semialgebraic variety. More precisely, F is obtained by intersecting the nonnegative orthant with the zero set of a family of polynomials with real coefficients.*

Proof. The condition for a point $(t_1, \ldots, t_d)$ with nonnegative coordinates to lie in the fiber over a fixed matrix is polynomial in $t_1, \ldots, t_d$ because the entries of the product matrix $x_{i_1}(t_1) \cdots x_{i_d}(t_d)$ are polynomials in $t_1, \ldots, t_d$. $\square$

Proposition 3.4. *The partially ordered set $\{\overline{F}_P \mid P \subseteq Q\}$ of closed strata of the natural stratification of the fiber F in Definition 3.1 is naturally isomorphic to the face poset of the interior dual block complex of the subword complex $\Delta(Q, w)$, where w indexes the Bruhat cell containing the image of F.*

Proof. Notice that the interior faces of a subword complex $\Delta(Q, w)$ are given exactly by the subwords P of Q whose complementary word $Q \setminus P$ has Demazure product exactly w, as shown in [KM04, Theorem 3.7]. Also recall that this subword complex is a sphere if and only if $\delta(Q) = w$, and recall that the interior dual block complex of a PL-sphere consists of a dual cell to each cell of the original PL-sphere as well as one additional maximal cell having this sphere as its boundary.

With these facts in mind, observe that the interior dual blocks of $\Delta(Q, w)$ are in containment-reversing bijection with the interior faces of $\Delta(Q, w)$, noting that the empty face is an interior face of a subword complex $\Delta(Q, w)$ if and only if $\delta(Q) = w$. The desired stratification of $F = f_Q^{-1}(p)$ for $p \in Y_w^o$ now follows from the description of strata one obtains by combining Proposition 2.7 in [Lus94] with results in [Her14]: that is, the open stratum F_P given by a subword P of Q is nonempty if and only if $\delta(P) = w$. $\square$

Proposition 3.5. *The interior dual block complex $\nabla(Q,w)$ of any subword complex $\Delta(Q,w)$ is a contractible regular CW complex. In particular, the nerve of the cover of $\nabla(Q,w)$ by its closed cells is contractible.*

Proof. The subword complex is a shellable ball or sphere by Theorem 2.12, and therefore it is PL [BLSWZ99, Proposition 4.7.26]. (For a ball, [BLSWZ99, Proposition 4.7.26] a priori only implies directly that it is PL if it has a shelling that can be completed to a shelling of a sphere; but [BLSWZ99, Theorem 4.7.21] implies that every shelling of a ball B can be so extended by adding one closed cell, namely a second copy of B—thought of as a single closed cell—meeting the original copy of B along its boundary.) The result is now a special case of Propositions 2.30 and 2.31. □

In Remark 1.6, we observed that the interior dual block complex of the subword complex $\Delta(Q,w)$ for $Q=(1,3,2,1,3,2)$ and $w=s_1s_3s_2$ is not pure. Specifically, it has a 2-dimensional maximal cell and a one dimensional maximal cell. What leads to the presence of maximal cells of differing dimensions in this case is the existence of an element $u\in W$ that is less than w in Bruhat order but not in weak order. Thus it seems natural to ask:

Question 3.6. Suppose $w\in W$ has exactly the same elements below it in weak order as in Bruhat order. Suppose Q is a word satisfying $\delta(Q)=w$. Does this imply that the interior dual block complex of the subword complex $\Delta(Q,w)$ is pure (namely has all its maximal cells of the same dimension) and is a regular CW closed ball?

4. Cell Decomposition of Each Fiber

In this section, we prove that the fiber of any point has a decomposition into open cells given by intersecting the natural cell decomposition of the simplex with the fiber. First we introduce notions and prove lemmas we will need.

Definition 4.1. The letter i_j is *redundant* in the word $(i_1,\dots,i_d)$ if

$$x_{i_1}\cdots\hat{x}_{i_j}\cdots x_{i_d}=x_{i_1}\cdots x_{i_d}$$

with this equality being as elements in the unsigned 0-Hecke algebra, or equivalently under the Demazure product. On the other hand, the letter i_j is *non-redundant* for $x_{i_1}\cdots\hat{x}_{i_j}\cdots x_{i_d}\neq x_{i_1}\cdots x_{i_d}$.

Example 4.2. The last letter in each of the words $(1,1)$ and $(1,2,1,2)$ is redundant while the last letter of $(1,2,1,2,3)$ is non-redundant.

Remark 4.3. One may show that i_j redundant in $(i_1,\dots,i_d)$ for some $1<j<d$ implies that i_j is redundant in either $(i_1,\dots,i_j)$ or $(i_j,\dots,i_d)$.

The following fundamental fact, which may be deduced from Theorem 2.2, is quite helpful for deducing statements about Demazure products from statements about Coxeter theoretic products. The Coxeter theoretic product has the distinct advantage over the Demazure product (or 0-Hecke algebra product) of a cancellation law due to the presence of inverses of elements. The 0-Hecke algebra does have the following very limited form of cancellation:

Proposition 4.4. *If $x_{i_1}\cdots x_{i_{s-1}}x_{i_s} = x_{j_1}\cdots x_{j_{s-1}}x_{i_s}$ with x_{i_s} non-redundant in both expressions, then $x_{i_1}\cdots x_{i_{s-1}} = x_{j_1}\cdots x_{j_{s-1}}$. Likewise if $x_{i_1}x_{i_2}\cdots x_{i_s} = x_{i_1}x_{j_2}\cdots x_{j_s}$ with x_{i_1} non-redundant in both expressions, then $x_{i_2}\cdots x_{i_s} = x_{j_2}\cdots x_{j_s}$.*

Next we give a relaxation of the notion of deletion pair that will be needed later.

Definition 4.5. In an expression $x_{i_1}\cdots x_{i_d}$ the letters x_{i_r} and x_{i_s} are *deletion partners*, generalizing the notion of deletion pair, if

$$x_{i_r}\cdots x_{i_s} = x_{i_r}\cdots x_{i_{s-1}} = x_{i_{r+1}}\cdots x_{i_s}$$

as elements of the unsigned 0-Hecke algebra and $x_{i_{r+1}}\cdots x_{i_{s-1}}$ is distinct from these.

Equivalently, x_{i_r} and x_{i_s} are *deletion partners* if x_{i_r} and x_{i_s} become a deletion pair after replacing $x_{i_r}\ldots x_{i_{s-1}}$ by a subexpression which (1) has the same Demazure product as $x_{i_r}\cdots x_{i_{s-1}}$, (2) is a reduced expression, and (3) contains x_{i_r}.

For example, the first and last letters in the expression $x_1x_2x_2x_1x_2$ are deletion partners, but they are not a deletion pair.

Lemma 4.6. *The letter i_d (resp. i_1) is non-redundant in the word $(i_1,\ldots,i_d)$ if and only if there is a unique choice for t_d (resp. t_1) within $f^{-1}_{(i_1,\ldots,i_d)}(p)$ for any fixed $p\in Y^o_w$ for $w=\delta(i_1,\ldots,i_d)$.*

Proof. First suppose that i_d is non-redundant, so the cell given by $(i_1,\ldots,i_{d-1})$ is u while the cell for $(i_1,\ldots,i_d)$ is $w=us_{i_d}$ with $\ell(w)>\ell(u)$. Suppose

$$f_{(i_1,\ldots,i_d)}(t_1,\ldots,t_d) = f_{(i_1,\ldots,i_d)}(t'_1,\ldots,t'_d) = p.$$

Without loss of generality, suppose $t_d < t'_d$. Then

$$x_{i_1}(t'_1)\cdots x_{i_{d-1}}(t'_{d-1})x_{i_d}(t'_d-t_d) = px_{i_d}(-t_d) = x_{i_1}(t_1)\cdots x_{i_{d-1}}(t_{d-1}),$$

but then the former is in the cell $Y^o_{us_{i_d}}$ while the latter is in the cell Y^o_u, contradicting the equality of these two points.

On the other hand, if i_d is redundant, then we may apply braid and modified nil-moves to $(i_1,\ldots,i_{d-1})$ to replace it by a reduced word $(j_1,\ldots,j_{d'})$. Now i_d must still be redundant within $(j_1,\ldots,j_{d'},i_d)$, hence must be part of a deletion pair within this word. This implies that we can now apply braid moves to move a letter forming a deletion pair with i_d to its immediate left and then shift value to t_d from this position, implying non-uniqueness of t_d in this case.

The proof is analogous but mirrored for the statement involving i_1 in place of i_d. ☐

Corollary 4.7. *Suppose i_d (resp. i_1) is redundant in $(i_1,\ldots,i_d)$. Consider a fixed $p\in Y^o_w$ for $w=\delta(i_1,\ldots,i_d)$. Then the open cells of the simplex in which t_d (resp.*

t_1) achieves its maximal possible value within $f^{-1}_{(i_1,\dots,i_d)}(p)$ are exactly those given by subexpressions $x_{i_{j_1}} \cdots x_{i_{j_r}}$ of $x_{i_1} \cdots x_{i_d}$ such that (1) $j_r = d$ (resp. $j_1 = 1$), (2) $\delta(i_{j_1}, \dots, i_{j_r}) = w$, and (3) $x_{i_{j_r}}$ (resp. $x_{i_{j_1}}$) is non-redundant in $x_{i_{j_1}} \cdots x_{i_{j_r}}$.

Lemma 4.8. *If i_d (resp. i_1) is non-redundant in $(i_1, \dots, i_d)$, then $f^{-1}_{(i_1,\dots,i_d)}(p) \cong f^{-1}_{(i_1,\dots,i_{d-1})}(p')$ (resp. $f^{-1}_{(i_1,\dots,i_d)}(p) \cong f^{-1}_{(i_2,\dots,i_d)}(p')$) via the projection map $(t_1, \dots, t_d) \mapsto (t_1, \dots, t_{d-1})$ (resp. $(t_1, \dots, t_d) \mapsto (t_2, \dots, t_d)$) for $p \in Y^o_w$ for some $p' \in Y^o_u$ with $w = us_{i_d}$ (resp. $w = s_{i_1}u$) and $l(u) < l(w)$.*

Proof. The solutions to the equation

$$x_{i_1}(t_1) \cdots x_{i_{d-1}}(t_{d-1}) x_{i_d}(t_d) = p$$

all have $t_d = k_d$ for a unique value k_d by Lemma 4.6. Thus, these solutions correspond via a homeomorphism obtained as the restriction of the map $(t_1, \dots, t_d) \to (t_1, \dots, t_{d-1})$ to the set of solutions to the equation

$$x_{i_1}(t_1) \cdots x_{i_{d-1}}(t_{d-1}) = p x_{i_d}(-k_d).$$

This gives the result for i_d.

The corresponding result for i_1 is proven completely analogously. □

Lemma 4.9. *Suppose $(t_1, \dots, t_d) \in f^{-1}_{(i_1,\dots,i_d)}(p)$ with $p \in Y^o_w$ such that*

$$x_{i_{l-1}}(-t_{l-1}) \cdots x_{i_2}(-t_2) x_{i_1}(-t_1) p \in Y^o_w,$$

and suppose that x_{i_l} is redundant in $x_{i_l} \cdots x_{i_d}$. Then for any choice of t'_l satisfying $0 \le t'_l \le t_l$, there exists $t'_{l+1}, \dots, t'_d$ such that $(t_1, \dots, t_{l-1}, t'_l, t'_{l+1}, \dots, t'_d) \in f^{-1}_{(i_1,\dots,i_d)}(p)$.

Proof. Let $A = x_{i_1}(t_1) \cdots x_{i_{l-1}}(t_{l-1})$ and let $B = x_{i_{l+1}}(t_{l+1}) \cdots x_{i_d}(t_d)$. Then

$$p = A x_{i_l}(t_l) B = A x_{i_l}(t'_l) x_{i_l}(t_l - t'_l) B$$

for any $0 \le t'_l \le t_l$. We may apply braid moves and modified nil-moves to $x_{i_{l+1}} \cdots x_{i_d}$ to produce a reduced expression $x_{j_{l+1}} \cdots x_{j_{d'}}$. The redundancy of x_{i_l} within $x_{i_l} \cdots x_{i_d}$ implies that x_{i_l} forms a deletion pair with some letter x_{j_s} within $x_{i_l} x_{j_{l+1}} \cdots x_{j_{d'}}$. But then we may apply braid moves to $x_{j_{l+1}} \cdots x_{j_s}$ yielding $x_{j'_{l+1}} \cdots x_{j'_s}$ with $j'_{l+1} = i_l$. This implies the existence of parameters $u_{l+1}, \dots, u_s$ with

$$x_{j'_{l+1}}(u_{l+1}) \cdots x_{j'_s}(u_s) = x_{j_{l+1}}(t'_{l+1}) \cdots x_{j_s}(t'_s).$$

Thus, we have

$$p = A x_{i_l}(t'_l) x_{j'_{l+1}}(t_l - t'_l + u_{l+1}) x_{j'_{l+2}}(u_{l+2}) \cdots x_{j'_s}(u_s) x_{j_{s+1}}(t'_{s+1}) \cdots x_{j_{d'}}(t'_{d'}).$$

In particular, this exhibits the existence of a choice of values for the parameters when t_l is replaced by any t'_l satisfying $0 \le t'_l \le t_l$, as the braid and modified-nil moves may be reversed with appropriate changes of coordinates in the parameters for each move. □

Definition 4.10. Let cf (short for *"change-fiber"*) be a map taking as its input a choice of point $p \in Y_w^o$ together with values $k_1, \ldots, k_r$ for an initial segment of parameters $t_1, \ldots, t_r$, with cf outputting a point $q \in Y_u^o$ for some $u \leq w$ as follows. Let $cf(p; k_1, \ldots, k_r) := x_{i_r}(-k_r) \cdots x_{i_1}(-k_1)p$.

In Theorem 4.12, we will generalize Lusztig's result that the map $f_{(i_1,\ldots,i_d)}$ given by a reduced word $(i_1, \ldots, i_d)$ is an embedding from $\mathbb{R}^d_{>0}$. First we give a helpful lemma.

Lemma 4.11. *Consider $(i_1, \ldots, i_d)$ with $\delta(i_1, \ldots, i_d) = w$. Choose $S \subseteq \{1, \ldots, d\}$ whose complement $S^C = \{j_1, \ldots, j_{d-s}\}$ indexes the rightmost subword $(i_{j_1}, \ldots, i_{j_{d-s}})$ of $(i_1, \ldots, i_d)$ that is a reduced word for w. Then $S = \{j'_1, \ldots, j'_s\}$ has the following characterization: a letter $i_{j'_r}$ is redundant in $(i_{j'_r}, i_{j'_r+1}, \ldots, i_d)$ if and only if $j'_r \in S$.*

Proof. The proof is straightforward from the definitions. □

Theorem 4.12. *Let $(i_1, \ldots, i_d)$ be a word with $w = \delta(i_1, \ldots, i_d)$. Let $S = \{j_1, \ldots, j_s\}$ be the subset of $\{1, \ldots, d\}$ whose complement $S^C = \{j'_1, \ldots, j'_{d-s}\}$ indexes the rightmost subword $(i_{j'_1}, \ldots, i_{j'_{d-s}})$ of $(i_1, \ldots, i_d)$ that is a reduced word for w.*

Consider the restriction of $f_{(i_1,\ldots,i_d)}$ to the domain D defined as follows:

$$D = \{(t_1, \ldots, t_d) \in \mathbb{R}^d_{\geq 0} | t_{j'_l} > 0 \text{ for } l = 1, \ldots, d-s; \sum_{i=1}^{d} t_i = K; t_{j_r} = k_{j_r} \text{ for } 1 \leq r \leq s\},$$

for any fixed choice of constants $k_{j_1}, \ldots, k_{j_s} \geq 0$ satisfying $\sum_{i=1}^{s} k_i < K$. Then $f_{(i_1,\ldots,i_d)}|_D$ is to a homeomorphism $h : D \to im(h) \subseteq Y_w^o$. Moreover, $im(h) \cong Y_w^o$.

Proof. Consider $p \in im(h)$. Let us now prove $p = h(x)$ for unique x. By definition of the domain D, this is equivalent to proving uniqueness of $x|_{S^C} = (t'_1, \ldots, t'_{d-s}) \in \mathbb{R}^{d-s}_{>0}$. To this end, we proceed from left to right through all of the parameters $t_1, \ldots, t_d$ in x, whether or not they are in S^C, showing that each such parameter t_i is uniquely determined by the point $p_0 := p$ and the values of the parameters $t_1, \ldots, t_{i-1}$ to its left. At each of the steps in which we encounter some t_{j_r} for $j_r \in S$, t_{j_r} has already been set to some constant k_{j_r}, leaving no choice for t_{j_r}. We then adjust the set-up so as to make the next parameter t_{j_r+1} leftmost as follows: we replace the point p_{j_r-1} whose fiber we have been considering at the current stage by $p_{j_r} = x_{j_r}(-k_{j_r})p_{j_r-1}$, and we replace $f_{(i_{j_r},\ldots,i_d)}$ by $f_{(i_{j_r+1},\ldots,i_d)}$. That is, we let $p_{j_r} = cf(p; k_1, \ldots, k_{j_r})$ and turn next to considering the fiber $f^{-1}_{(i_{j_r+1},\ldots,i_d)}(p_{j_r})$ with leftmost parameter t_{j_r+1}. Lemma 4.8 (applied repeatedly) assures that if t_{j_r+1} takes a unique value in $f^{-1}_{(i_{j_r+1},\ldots,i_d)}(p_{j_r}$ then it will take this same unique value in $f^{-1}_{(i_1,\ldots,i_d)}(p)$.

On the other hand, when we encounter a parameter t_l for $l \in S^C$, we apply Lemma 4.6 to deduce uniqueness of the value for t_l within in our suitably modified fiber $f^{-1}_{(i_l,\ldots,i_d)}(p_{l-1})$ for $p_{l-1} = cf(p; k_1, \ldots, k_{l-1})$, using that it is the leftmost parameter there; more specifically, we use the alternate characterization for S^C given in Lemma 4.15, which shows that x_{i_l} is non-redundant in $x_{i_l} \cdots x_{i_d}$ for $l \in S^C$. Again, we use Lemma 4.8 (again applied repeatedly) to see that this unique value k_l for t_l for points in $f^{-1}_{(i_l,\dots,i_d)}(p_{l-1})$ is also the unique value taken by t_l for points in $f^{-1}_{(i_1,\dots,i_d)}(p)$. After determining k_l, we again change the set-up so as to make t_{l+1} the new leftmost parameter. That is, we multiply p_{l-1} on the left by $x_{i_l}(-k_l)$ to obtain p_l, and we replace $f_{(i_l,\dots,i_d)}$ by $f_{(i_{l+1},\dots,i_d)}$. In this manner, we proceed through all parameters from left to right, showing that each parameter in turn is uniquely determined by p together with the values for the parameters to its left. Thus, we deduce injectivity of h, hence that h is a bijection from D to $im(h)$.

We now invoke Lusztig's result that $f_{(i_1,\dots,i_d)}$ is continuous and proper on $\mathbb{R}^d_{\geq 0}$ to deduce from this that h is continuous and proper on D by virtue of being a restriction of $f_{(i_1,\dots,i_d)}$. See the proof of Proposition 4.2 in [Lus94] for these assertions for $f_{(i_1,\dots,i_d)}$ on the domain $\mathbb{R}^d_{\geq 0}$ as well as proofs of these assertions. Properness of h on a locally compact, Hausdorff space, namely on D, ensures that h is a closed map on D. This combines with bijectivity of h to yield that h is an open map, completing the proof that h is a homeomorphism from D to $im(h)$.

To show $im(h) \cong Y^o_w$, we will use the fact that the proof above applies equally well for any choice of nonnegative real values for $k_1,\dots,k_s$ satisfying $\sum_{i=1}^s k_i < K$, including the case with $k_1 = k_2 = \cdots = k_s = 0$. Setting $k_1 = k_2 = \cdots = k_s = 0$ yields exactly Y^o_w as $im(h)$. Changing the nonnegative real choices of values for $k_1,\dots,k_s$ (subject to our $\sum_{i=1}^s k_i < K$ requirement) does not change the homeomorphism type of the domain of h. Thus, we get $im(h) \cong Y^o_w$ by composing three homeomorphisms, namely one homeomorphism of domains and two homeomorphisms h given by two different choices of domains (namely one map h based on the given set of values $k_1,\dots,k_s$ for the parameters $t_{j_1},\dots,t_{j_s}$ and the other map h given by setting all these parameters $t_{j_1},\dots,t_{j_s}$ to 0). □

Now we define a map *rtn* to be used later. While the non-maximality requirement for each parameter in turn in Definition 4.13 may seem cumbersome, it is exactly what will be needed; in particular, this will ensure combinatorial structure is independent of our choices of parameter values for parameters indexed by S.

Definition 4.13. Given a word $(i_1,\dots,i_d)$ with $\delta(i_1,\dots,i_d) = w$, consider the set $S = \{j_1,\dots j_{d'}\} \subseteq \{1,2,\dots,d\}$ such that $S^C := \{1,2,\dots,d\} \setminus S$ indexes a subword of $(i_1,\dots,i_d)$ that is the rightmost subword that is a reduced word for w.

Define a map *rtn* (short for *"redundant-to-nonredundant"*) that takes as its input a point $p \in Y^o_w$ together with a choice of values $t_{j_i} = k_{j_i}$ determined from left to right for $j_1,\dots,j_{d'} \in S$, subject to the requirement for each t_{j_i} that k_{j_i} is not the maximal possible value within the part of $f^{-1}_{(i_1,\dots,i_d)}(p)$ satisfying the given choice of values $k_{j_1},\dots,k_{j_{i-1}}$ already made for the parameters to its left also indexed by S.

Given this input $(p, k_{j_1},\dots,k_{j_{d'}})$, the map *rtn* outputs the vector of unique values that the parameters at positions not in S are forced to take (when given the choices of $t_{j_r} = k_r$ for $j_r \in S$) to obtain a point indeed in the fiber given by p. This map rtn has the explicit formula described next. Let π denote the projection map sending $(t_1, \dots, t_d)$ to the vector comprised of just those coordinates not indexed by S. Let h be the map given by the choices $t_i = k_i$ above, as defined and proven to be bijective (and hence invertible) in the statement and proof of Theorem 4.12. Then we define rtn by the formula $rtn(p, k_{j_1}, \dots, k_{j_{d'}}) = \pi(h^{-1}(p, k_{j_1}, \dots, k_{j_{d'}}))$ for each $p \in im(h)$ for h as given by the constants $k_{j_1}, \dots, k_{j_{d'}}$.

Corollary 4.14. *The map rtn from Definition 4.13 is a well-defined homeomorphism.*

Proof. This follows from these same properties for h^{-1} and the projection map π. These properties for h^{-1} are confirmed within the proof of Theorem 4.12. These properties for π follow from our choice of domain D in Theorem 4.12. $\square$

Now we give a series of results that will combine to yield as a corollary a cell decomposition for $f^{-1}_{(i_1,\dots,i_d)}(p)$.

Lemma 4.15. *Let $S \subseteq \{1, \dots, d\}$ be a set of size $d' \leq d$ whose complement S^C indexes the rightmost subword of $(i_1, \dots, i_d)$ that is a reduced word for $w = \delta(i_1, \dots, i_d)$. Then this set $S = \{j_1, \dots, j_{d'}\}$ for $j_1 < j_2 < \cdots < j_{d'}$ may equivalently be described as follows: j_1 is the smallest index with the property that t_{j_1} takes more than one value within $f^{-1}_{(i_1,\dots,i_d)}(p)$. Once $j_1, j_2, \dots, j_{s-1}$ have inductively been determined for a given $s - 1 \geq 1$, then j_s is defined to be the smallest index with $j_s > j_{s-1}$ such that there exists $(k_1, \dots, k_{j_s-1}) \in \mathbb{R}^{j_s-1}_{>0}$ such that t_{j_s} takes more than one value within $f^{-1}_{(i_1,\dots,i_d)}(p) \cap \{(t_1, \dots, t_d) \in \mathbb{R}^d_{\geq 0} | t_1 = k_1; t_2 = k_2; \dots ; t_{j_s-1} = k_{j_s-1}\}$.*

Proof. Lemmas 4.6 and 4.9 combine to show in this case that t_{j_1} will take a range $[0, t^{\max}_{j_1}]$ of values for some $t^{\max}_{j_1} > 0$; that is, we proceed from left to right through the parameters, using Lemma 4.6 to reduce to the case where $j_1 = 1$ and then apply Lemma 4.9. The same argument likewise applies for each $j_l \in S$, as we proceed from left to right through the parameters, using the choice of values for the parameters to the left of a given parameter t_{j_r} for which the associated $t^{\max}_{j_r}$ is being determined. $\square$

Lemma 4.16. *Consider any fiber $f^{-1}_{(i_1,\dots,i_d)}(p)$ for $p \in Y^o_w$, together with any choice of strata F given by restricting the fiber to the open cell of the simplex indexed by a subword Q of $(i_1, \dots, i_d)$ with $\delta(Q) = w$. Let $|Q|$ denote the number of letters in Q. Let v be the vertex contained in $\overline{F}$ whose support is the subword of Q that is the rightmost reduced word for w. Then there is a well-defined map*

$$f_F : [0, 1)^{|Q|-l(w)} \to \bigcup_{v \subseteq \overline{\sigma} \subseteq \overline{F}} \sigma.$$

Proof. Let $d' = |Q| - l(w)$ where $\delta(Q) = w$. Let us now define the map $f_F : [0, 1)^{d'} \to f^{-1}_{(i_1,\dots i_d)}(p)$. First we choose a subset S of size d' of the set of indices for the parameters of points in $\overline{F} \subseteq f^{-1}_{(i_1,\dots,i_d)}(p)$; choose S such that S^C is exactly the support of v. We have $S = \{j_1, \dots, j_{d'}\}$ with $1 \le j_1 < \cdots < j_{d'} \le d$ indexing positions from left to right in $(i_1, \dots, i_d)$. We will use the fact that these indices $j_1, \dots, j_{d'}$ may equivalently be defined as in Lemma 4.15 to justify the applicability of Lemma 4.9 shortly.

Now we define $f_F : [0,1)^{d'} \to f^{-1}_{(i_1,\dots,i_d)}(p)$ as follows. Let $f_F(u_1, \dots, u_{d'}) = (t_1, \dots, t_d)$ for $t_1, \dots, t_d$ determined from left to right for each $j_r \in S$ in turn, using u_r together with the values $k_1, \dots, k_{j_r-1}$ for $t_1, \dots, t_{j_r-1}$ to determine t_{j_r} as follows. Let $t^{\max}_{j_r}$ be the largest value t_{j_r} takes within the set:

$$\overline{F}_{k_1,\dots,k_{r-1}} = f^{-1}_{(i_1,\dots,i_d)}(p) \cap \{(t_1, \dots, t_d) \in \mathbb{R}^d_{\ge 0} | t_i = k_i \text{ for } i < j_r; t_i = 0 \text{ for } i \notin supp(Q)\}.$$

Lemma 4.9 allows us to set $t_{j_r} = u_r \cdot t^{\max}_{j_r}$ for any $u_r \in [0,1)$ and be sure this is still consistent with obtaining a point in a strata $\sigma \subseteq \overline{F}$ having $v \in \overline{\sigma}$. Whenever we encounter a parameter indexed by S^C as we proceed from left to right, we use Theorem 4.12 to guarantee the value for this parameter is uniquely determined, given that all parameters to its left have already been determined. □

Lemma 4.17. *The map f_F from Theorem 4.16 is invertible, hence is a bijection from $[0,1)^{d'}$ to $\bigcup_{v \subseteq \overline{\sigma} \subseteq \overline{F}} \sigma$.*

Proof. It will suffice to show how to invert f_F. Define $t^{\max}_{j_r}$ as in the proof of Theorem 4.16. Observe that we may calculate $f_F^{-1}(t_1, \dots, t_d)$ by setting $u_r = \frac{t_{j_r}}{t^{\max}_{j_r}}$ for each r in turn, proceeding from right to left, provided that $t^{\max}_{j_r}$ is nonzero for $r = 1, \dots, d'$. Corollary 4.7 guarantees that $t^{\max}_{j_r} = 0$ implies $u_i = 1$ for some $i \ne r$ (using the equivalency of $u_i = 1$ to $t_{j_i} = t^{\max}_{j_i}$). Our choice of domain $[0,1)^{d'}$ for f_F precludes having $u_i = 1$ for any i. Thus, f_F is invertible, hence is a bijection as desired. □

Lemma 4.18. *Let $t^{\max}_{j_r}$ be the maximal value t_{j_r} takes within $f^{-1}_{(i_1,\dots,i_d)}(p)$ subject to the constraints $t_{j_i} = k_{j_i}$ for $i = 1, \dots, r-1$ for any choice of constants $k_{j_1}, \dots, k_{j_{r-1}} \ge 0$ with $k_{j_i} < t^{\max}_{j_i}$ for $i = 1, \dots, r-1$. Then $t^{\max}_{j_r}$ is a continuous function of $k_{j_1}, \dots, k_{j_{r-1}}$.*

Proof. We proceed from left to right. Upon reaching t_{j_r}, we may assume that all of the parameters to the left of t_{j_r}, both those indexed by S and those not indexed by S, have been determined already. By induction we may assume that each k_{j_i} for $j_i < j_r$ is a continuous function of $k_{j_1}, \dots, k_{j_{i-1}}$.

The quantity $t^{\max}_{j_r}$ in the statement of this result is precisely the maximal value that is taken by the leftmost parameter among all points within the fiber $f^{-1}_{(i_{j_r},\dots,i_d)}(p'_r)$ for $p'_r := x_{i_{j_r-1}}(-k_{j_r-1}) \cdots x_{i_2}(-k_2) x_{i_1}(-k_1) p$. But then setting t_{j_r} equal to $t^{\max}_{j_r}$ would force t_s to equal 0 for each t_s that is given by a deletion partner i_s to the right of i_s, namely for each deletion partner i_s for i_{j_r} satisfying $s > j_r$. This would effectively replace the word $(i_{j_r}, \dots, i_d)$ by the subword Q' obtained by deleting all these deletion partners for i_{j_r} to the right of i_{j_r}. Within Q' the letter i_{j_r} is nonredundant, implying $t^{\max}_{j_r}$ takes the unique value for t_{j_r} in $f^{-1}_{Q'}(p'_r)$. The unique value that the parameter

t_{j_r} must take for all points in $f_{Q'}^{-1}(p'_r)$ will be the same if we replace Q' by a subword Q'' having $\delta(Q'') = \delta(Q')$. In particular, we may choose such a Q'' that is a reduced word. Thus, it suffices to prove that the unique value for the leftmost parameter for points in $f_{Q''}^{-1}(p'_r)$ is a continuous function of the parameters $t_1, \ldots, t_{j_r-1}$ to deduce that $t_{j_r}^{\max}$ is also thereby a continuous function of $t_1, \ldots, t_{j_r-1}$.

To this end, notice that $\delta(Q')$ is independent of the choices we made for $t_1, \ldots, t_{r-1}$, provided that a non-maximal value was chosen for each parameter. This allows us to choose the same subword Q'' for each $(u_1, \ldots, u_{r-1}) \in [0,1)^{r-1}$. Thus, we deduce the desired continuity of $t_{j_r}^{\max}$ from the pair of facts (1) that $f_{Q''}$ is a homeomorphism from $\mathbb{R}_{>0}^{|Q''|}$ to its image and (2) that the point $p'_r = x_{i_{j_r-1}}(-k_{j_r-1}) \cdots x_{i_1}(-k_1)p$ whose fiber we are taking is by definition a continuous function of $k_1, \ldots, k_{j_r-1}$. $\square$

Theorem 4.19. *For any fiber $f_{(i_1,\ldots,i_d)}^{-1}(p)$ with $p \in Y_w^o$ and for any choice of cell F given by a subword Q of $(i_1, \ldots, i_d)$ with $\delta(Q) = w$ and wordlength $|Q|$, let v be the unique vertex in $\overline{F}$ indexing the rightmost subword of Q that is a reduced word for w. Then the map f_F from $[0,1)^{|Q|-l(w)}$ to $f_{(i_1,\ldots,i_d)}^{-1}(p)$ defined in Lemma 4.16 is a homeomorphism from $[0,1)^{|Q|-l(w)}$ to the union of those open cells $\sigma \subseteq f_{(i_1,\ldots,i_d)}^{-1}(p)$ having $v \subseteq \overline{\sigma} \subseteq \overline{F}$.*

Proof. Our construction of f_F in Lemma 4.16 and its inverse map in the proof of Lemma 4.17 together demonstrate that f_F is a bijection which has as its image the desired union of cells. What remains is to check continuity of f_F and f_F^{-1}. This will follow from the following three facts whose justification is discussed next:

1. Each parameter whose index is not in S is uniquely determined by our choices of values for the parameters indexed by S (by virtue of being determined only by parameters to its left due to our choice for S^C as being rightmost possible).
2. These parameters indexed by S^C are continuous functions of the parameters indexed by S (by virtue of being continuous functions of the point whose fiber is being taken together with all parameters to the left of the parameter being determined).
3. The quantities $t_{j_r}^{\max}$ for $1 \leq r \leq d'$, each calculated in turn proceeding from left to right through positions indexed by S, are continuous functions of the values chosen for those parameters to their left that are indexed by S.

The first two claims are justified in Theorem 4.12. The third claim is verified in Lemma 4.18. $\square$

Now we deduce the desired cell stratification for each fiber:

Theorem 4.20. *Given any point $p \in Y_w^o$, the intersection of the fiber $f_{(i_1,\ldots,i_d)}^{-1}(p)$ with the open face of the simplex corresponding to a subword Q of $(i_1, \ldots, i_d)$ is either empty or a single point or an open cell of dimension $|Q| - \ell(w)$. The nonempty cases occur precisely when $\delta(Q) = w$. The case of a single point occurs exactly when Q*

is a reduced word for w. The standard cell stratification of a simplex induces a cell stratification of $f^{-1}_{(i_1,\dots,i_d)}(p)$.

Proof. This follows from Theorem 4.19 by noting that $f_F|_{(0,1)^{\dim F}}$ has image exactly the open cell F. The requirement needed to get not just a cell decomposition but also a cell stratification is automatic from out set-up. $\square$

5. Fiber conjecture implies Fomin–Shapiro Conjecture

We now conclude the paper by showing how a proof of Conjecture 1.4 would yield as a corollary a new, short proof of the Fomin-Shapiro Conjecture. In fact, what we show is how a consequence of Conjecture 1.4 would suffice, namely we show how to derive a new proof of the Fomin-Shapiro Conjecture from contractibility of fibers, by way of assorted results we have just proven.

Remark 5.1. It should be noted that the original proof of the Fomin-Shapiro Conjecture in [Her14] seemingly does prove contractibility of fibers along the way within the proofs of other results (though this is never explicitly stated in that paper and not formally verified that indeed this is a corollary of the proofs there). Nonetheless, we thought this implication and the potential it provides for a new, independent proof of the Fomin-Shapiro Conjecture based on a more conceptual understanding of the fibers (particularly if Conjecture 1.4 were proven) could be enlightening.

Theorem 5.2. *Let Y_w for $w \in W$ be the closure of any cell Y_w^o in the Bruhat decomposition of the link of the identity in the totally nonnegative, real part of the unipotent radical of a Borel subgroup in a semisimple, simply connected algebraic group over $\mathbb{C}$ defined and split over $\mathbb{R}$, and let $(i_1,\dots,i_d)$ be any reduced word for any element of W. Then contractibility of $f^{-1}_{(i_1,\dots,i_d)}(p)$ for each $p \in Y_w^o$ for each $w \in W$ satisfying $w \le_{Bruhat} \delta(i_1,\dots,i_d)$ implies that each cell closure Y_w for $w \in W$ with $w \le \delta(i_1,\dots,i_d)$ is a regular CW complex homeomorphic to a closed ball.*

Proof. We interpret Y_w as the image of a closed simplex under the map $f_{(i_1,\dots,i_d)}$ given by a reduced word $(i_1,\dots,i_d)$. We will prove that under our contractibility hypothesis that all of the conditions needed to apply Corollary 2.33 are met.

First recall from [FS00] that the image of $f_{(i_1,\dots,i_d)}$ endowed with our given stratification (namely with cells being the images of the cells of the simplex) has closure poset the Bruhat order. This is known to be a CW poset (by virtue of the shelling of Björner and Wachs from [BW82] or of Dyer from [Dy93] together the fact that it is thin, which is clear by virtue of its definition). We assume by induction the desired result for all reduced words strictly shorter than length d. This inductive hypothesis ensures that each closed cell in the boundary of the image is a ball, and that its stratification resulting from a reduced subword of $(i_1,\dots,i_d)$ of strictly shorter length is a regular CW decomposition.

In particular, this implies that the boundary of $im(f_{(i_1,\ldots,i_d)})$ is a regular CW complex. This in turn implies that the boundary of the image is homeomorphic to the order complex of its closure poset (after removal of the element $\hat{0}$ representing the empty face). But this is a sphere whose dimension equals the dimension of the complex, by virtue of each open interval of Bruhat order having order complex homeomorphic to such a sphere. Thus, the restriction of $f_{(i_1,\ldots,i_d)}$ to the boundary of the simplex has image a sphere of appropriate dimension.

By a result of Lusztig which is recalled in Theorem 2.18, the restriction of $f_{(i_1,\ldots,i_d)}$ to the interior of the simplex is a homeomorphism. We have assumed that the preimage of each point in the boundary of the image is contractible. By Lemma 2.21, the image of the interior of the simplex is nonintersecting with the image of the boundary of the simplex. Finally, note that the preimage of $f_{(i_1,\ldots,i_d)}$ is a ball by virtue of being a closed simplex. Combining with our contractibility hypothesis for fibers, we have all of the hypotheses needed to apply Corollary 2.33. Thus we may conclude that the image of $f_{(i_1,\ldots,i_d)}$ is a closed ball. Since this argument works for any reduced word $(i_1,\ldots,i_d)$ for any $w \in W$, this completes the proof that contractibility of fibers would yield a new proof of the Fomin-Shapiro Conjecture. $\square$

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