A LEXICOGRAPHIC SHELLABILITY CHARACTERIZATION OF GEOMETRIC LATTICES

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Abstract. Geometric lattices are characterized as those finite, atomic lattices such that every atom ordering induces a lexicographic shelling given by an edge labeling known as a minimal labeling. Equivalently, geometric lattices are shown to be exactly those finite lattices such that every ordering on the join-irreducibles induces a lexicographic shelling. This new characterization fits into a similar paradigm as McNamara's characterization of supersolvable lattices as those lattices admitting a different type of lexicographic shelling, namely one in which each maximal chain is labeled with a permutation of $\{1, \ldots, n\}$.

1. Introduction

In [7], McNamara proved that supersolvable lattices can be characterized as lattices admitting a certain type of EL-labeling known as an S_n-EL-labeling. Each maximal chain is labeled by the set of labels $\{1, \ldots, n\}$ with each label used exactly once in each maximal chain. Previously, Stanley had proven that all supersolvable lattices admit such EL-labelings in [10]. Thus, McNamara's result gave a new characterization of supersolvable lattices: that a finite lattice is supersolvable if and only if it has an S_n-EL-labeling.

This paper gives a result of a similar spirit for geometric lattices – a new characterization of geometric lattices as the lattices admitting a family of lexicographic shellings induced by the various possible orderings on the join-irreducibles. Geometric lattices are well-known to have the property that every atom ordering induces an EL-labeling by labeling each cover relation $u \lessdot v$ with the smallest atom that is below v but not u. Our main result is that this is a characterization of geometric lattices, i.e. that all finite atomic lattices in which every atom

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ordering induces an EL-labeling are geometric lattices. We also prove a reformulation in which the atomicity requirement is removed (following a suggestion of Peter McNamara): that if every ordering of the join-irreducibles in a finite lattice induces an EL-labeling are geometric lattices.

There is an extensive literature on the notion of lexicographic shellability. Most of the emphasis is on proving that important classes of partially ordered sets admit lexicographic shellings. For example, upper semimodular lattices (Garsia [6]), geometric lattices (Stanley [11], Björner [3]) and semilattices (Wachs-Walker [13]), supersolvable lattices (Stanley [10]), subgroup lattices of solvable groups (Shareshian [9], Woodroofe [14]), and Bruhat order (Björner-Wachs [5]) are all known to be lexicographically shellable. Our aim is to take things in the opposite direction, namely to use the types of lexicographic shellings (induced by EL-labelings) that geometric lattices are known to have as a way of characterizing geometric lattices.

One of the primary combinatorial motivations for the notion of lexicographic shellability is as a tool to compute Möbius functions. In particular, geometric lattices arise as the intersection lattices of real, central hyperplane arrangements, and there is special interest in knowing their Möbius functions resulting from this interpretation in terms of hyperplane arrangements, for the following reason. Zaslavsky expressed the number of regions in the complement of a real hyperplane arrangement in terms of Möbius functions of geometric lattices and semi-lattices in [15].

McNamara's characterization of supersolvable lattices has given a useful new way of proving new classes of lattices to be supersolvable. See e.g. [1] for one such result. Our results will imply there is a similar potential for geometric lattices.

2. Background and terminology

Let P be a finite poset. Let $E(P)$ denote the set of edges of the Hasse diagram of P. We write $x \lessdot y$ to indicate that y covers x in P, namely $x \le z \le y$ implies $z = x$ or $z = y$. If $\lambda : E(P) \to \mathbb{N}$ is an edge labeling of the Hasse diagram of P and $x \lessdot y$, then we write $\lambda(x, y)$ to indicate the label given to the edge from x to y. Recall that λ is an *EL-labeling* for P if for every interval $[x, y]$ of P, there is a unique *rising chain* $C := x = x_1 \lessdot x_2 \lessdot \cdots \lessdot x_j = y$ where $\lambda(x, x_2) \le \lambda(x_2, x_3) \le \cdots \le \lambda(x_{j-1}, y)$, and the label sequence of C is lexicographically smaller than the label sequence of every other saturated chain in the interval $[x, y]$ (cf [3]). It is well-known that an

EL-labeling gives a shelling order for the facets (maximal faces) of the order complex $\Delta(P)$ of P.

We now review the notion of geometric lattice as well as the types of EL-labelings which they are already known to possess. An *atom* in a poset P with unique minimal element $\hat{0}$ is any $a \in P$ such that a covers $\hat{0}$. A *lattice* is a poset such that any pair of elements x, y has a unique least upper bound $x \vee y$ and a unique greatest lower bound $x \wedge y$. A lattice is *atomic* if every element is a join of atoms. A lattice is *semimodular* if it has a rank function ρ this satisfies

$$(i)\ \rho(x \wedge y) + \rho(x \vee y) \leq \rho(x) + \rho(y).$$

A finite lattice is *geometric* if it is atomic and semimodular. An element x in a lattice is a *join-irreducible* if $x = y \vee z$ implies $y = x$ or $z = x$. See [12] for further background on posets.

Our interest is in using the existence of a certain family of edge labelings for a poset P to show that P fits into an important class of posets, namely the geometric lattices. Therefore, let us now introduce these types of labelings we will use.

Let L be a finite lattice with n join-irreducibles. Let $J(L)$ denote the set of join-irreducibles of L. For $x \in L$, let

$$J(x) = \{w \leq x | w \in J(L)\}.$$

Further, given a bijection $\gamma : J(L) \to [n]$, let $\gamma(J(x))$ denote the set $\{\gamma(w) | w \in J(x)\}$. The map γ induces a *minimal labeling* $\lambda_\gamma : E(L) \to [n]$ by the rule $\lambda_\gamma(x, y) = \min\{\gamma(J(y)) \setminus \gamma(J(x))\}$.

Let $\mathcal{A}(L)$ denote the atoms of L, and for $x \in L$ let $A(x) = \{a \leq x | a \in \mathcal{A}(L)\}$. Note that in an atomic lattice L we have $J(L) = \mathcal{A}(L)$. So for an atomic lattice, our definition coincides with Björner's original definition of a minimal labeling.

Theorem 1 (Björner [3])**.** *The minimal labeling resulting from any ordering of the atoms in a geometric lattice is an EL-labeling.*

The following proposition, which appears as Corollary 1, p. 81, in [2], gives a convenient property often referred to as the 'diamond property'. It implies not only the existence of a rank function, but also the inequality (i) above.

Proposition 2 (Birkhoff)**.** *Let L be a finite lattice. The following two conditions are equivalent:*

- *L is graded, and the rank function ρ of L satisfies the semimodularity condition (i) above.*
- *If x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y.*

3. Lexicographic shellability characterizations of geometric lattices

The section is devoted to developing two new characterization of geometric lattices, one based on atom orderings and the other based on orderings on join-irreducibles. In both cases, we prove that finite lattices in which every ordering of the atoms (resp. join-irreducibles) induces a so-called minimal labeling which is an EL-labeling are geometric lattices. To this end, we first develop some helpful properties of minimal labelings.

Lemma 3. *Let L be a finite atomic lattice and let λ_γ be a minimal labeling of $E(L)$. Then for each chain $C = x_1 \lessdot \cdots \lessdot x_k$, $\lambda_\gamma(x_i, x_{i+1}) \neq \lambda_\gamma(x_j, x_{j+1})$ whenever $i \neq j$. In other words, the labels on any particular saturated chain are distinct.*

Proof. This is immediate from the fact that $A(x_{j+1}) \setminus A(x_j)$ is by definition disjoint from $A(x_{i+1}) \setminus A(x_i)$ for $i \neq j$. □

Lemma 4. *Let L be a finite atomic lattice. If $A(u) \subseteq A(v)$, then $u \leq v$.*

Proof. This follows since L is atomic and every element is written as a join of atoms. □

Remark 1. In fact, we will make use of the following statement that is equivalent to Lemma 4: if v is not less than or equal to u then there exists $a_v \in A(v)$ such that $a_v \notin A(u)$.

Lemma 5. *Let L be a finite atomic lattice. Suppose that $x, y \in L$ both cover $x \wedge y$, but that $x \vee y$ does not cover x. Given any atom a_y such that $y = (x \wedge y) \vee a_y$ then $a_y \notin A(z)$ for all z such that $x \lessdot z$.*

Proof. Assume $a \vee (x \wedge y) = y$ and $a \leq z$. Since $x \wedge y \leq z$, we have $y = a \vee (x \wedge y) \leq z$, implying $x \vee y \leq z$, a contradiction. □

Now to our two characterizations of geometric lattices.

Theorem 6. *Let L be a finite, atomic lattice such that every atom ordering induces a minimal labeling that is an EL-labeling. Then L is geometric.*

Proof. Since a geometric lattice is a finite, semi-modular, atomic lattice, it will suffice to prove that L is graded with a rank function ρ satisfying $\rho(x \wedge y) + \rho(x \vee y) \leq \rho(x) + \rho(y)$, i.e. with rank function satisfying (i) above. Then we will take any pair of elements x, y both covering $x \wedge y$ but not both covered by $x \vee y$ and construct an atom ordering in terms of x and y whose minimal labeling will not be an EL-labeling.

Assume by way of contradiction that for all γ, λ_γ is an EL-labeling but the "diamond property" does not hold. In other words, suppose that there exist $x, y \in L$ such that x and y both cover $x \wedge y$ but $x \vee y$ does not cover at least one of x and y. Note this immediately implies that L has at least three atoms, because the structure of a two-atom atomic lattice must be a diamond. Assume without loss of generality that $x \vee y$ does not cover x.

Assume by way of contradiction that $x \wedge y \lessdot x$ and $x \wedge y \lessdot y$, but $x \vee y$ does not cover x. By Lemma 4, we can choose some atom $a_x \in A(x) \setminus A(x \wedge y)$ such that $a_x \notin A(y)$. Since L is an atomic lattice, there must exist $a_y \in A(y)$ such that $(x \wedge y) \vee a_y = y$. By Lemma 5, $a_y \notin A(z)$ for any z such that $x \lessdot z$. Clearly $a_y \neq a_x$, or else we would have $a_y \in A(x) \subseteq A(z)$ for any z covering x, a contradiction. Let $\gamma : \mathcal{A}(L) \to [n]$ be any atom ordering such that $\gamma(a_x) = 1$ and $\gamma(a_y) = 2$.

Since $a_x \in A(x) \setminus A(x \wedge y)$ and $\gamma(a_x) = 1$, we know that $\gamma(a_x) = \min\{\gamma(a) | a \in A(x) \setminus A(x \wedge y)\}$ and therefore $\lambda_\gamma(x \wedge y, x) = 1$. Then the lexicographically smallest chain in the interval $[x \wedge y, x \vee y]$ is of the form

$$C := \quad x \wedge y = x_0 \lessdot x = x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k = x \vee y.$$

By Lemma 5, $a_y \notin A(x_2)$, because x_2 covers x. Therefore, $\lambda_\gamma(x_1, x_2) \neq 2$. By Lemma 3 there is no repetition in the label sequence, implying $\lambda_\gamma(x_1, x_2) > 2$. For some $2 < j \leq k$, we must have $a_y \in A(x_j) \setminus A(x_{j-1})$. Now $1 \notin \{\gamma(a) | a \in A(x_j) \setminus A(x_{j-1})\}$ for any $j > 2$, so $\gamma(a_y) = 2 = \min\{\gamma(a) | a \in A(x_j) \setminus A(x_{j-1})\}$.

But $\min\{\gamma(a) | a \in A(x_2) \setminus A(x_1)\} \geq 3$, so $\lambda_\gamma(x_1, x_2) > \lambda_\gamma(x_{j-1}, x_j)$. This contradicts the fact that the lexicographically smallest chain in the interval $[x \wedge y, x \vee y]$ must be increasing for λ_γ to be an EL-labeling. Thus, whenever x and y cover $x \wedge y$, then $x \vee y$ covers both x and y. By Proposition 2, this means that L is a geometric lattice. □

We now conclude this section with another characterization of geometric lattices which avoids assuming the lattices are atomic. The essence will be a reduction to the theorem we have just proven.

Theorem 7. *Let L be a finite lattice with n join-irreducibles. If for every ordering of the join-irreducibles, i.e. every bijective map $\gamma : J(L) \to \{1, \ldots, n\}$, the labeling λ_γ is an EL-labeling then L is a geometric lattice.*

Proof. Assume by way of contradiction that L and λ satisfy all of the hypotheses of Theorem 6, but that there exists $v \in J(L)$ that is not an atom. Without loss of generality we can take v such that if $a \leq v$ and $a \in J(L)$ then a is an atom. It is well known (see [12] p. 286) that in a finite lattice, the join-irreducibles are precisely the elements that cover exactly one other element. So there exists exactly one u such that $u \lessdot v$. Furthermore, the set $J(u)$ is entirely composed of atoms and $|J(u)| = k$ for some $1 \leq k \leq n-1$. Let γ be such that $\gamma(v) = 1$ and $\gamma(J(u)) = \{2, 3, \ldots, k+1\}$. (Note $2 = k+1$ holds when u is an atom). Then the lexicographically smallest label sequence for all chains in the interval $[\hat{0}, v]$ has a descent because the label $\lambda_\gamma(u, v) = 1$ and $1 < \lambda_\gamma(x, y)$ for all $x \lessdot y$ in the interval $[\hat{0}, v]$. This contradicts the fact that every bijection $\gamma : J(L) \to [n]$ induces an EL-labeling, so no such v can exist. Thus $J(L) \subset \mathcal{A}(L)$. Since for any finite poset $\mathcal{A}(L) \subset J(L)$, L is atomic. □

4. Lexicographic shellability characterization of semimodular lattices

Let L be a finite lattice with n join-irreducibles. Let $J(L)$ denote the set of join-irreducibles of L. Let $E(L)$ denote the edges of the Hasse diagram of L. For $x \in L$, let

$$J(x) = \{w \leq x | w \in J(L)\}.$$

Further, given a map $\gamma : J(L) \to [n]$, let $\gamma(J(x))$ denote the set $\{\gamma(w) | w \in J(x)\}$. Then the map γ induces a *minimal labeling* $\lambda_\gamma : E(L) \to [n]$ by the rule $\lambda_\gamma(x, y) = \min\{\gamma(J(y)) \setminus \gamma(J(x))\}$.

Remark 2. Let L be a finite lattice and let $u, v \in L$. Then if $u \leq v$, $J(u) \subseteq J(v)$.

Lemma 8. *Let L be a finite lattice, let $x, y \in L$ be such that x and y both cover $x \wedge y$, but $x \vee y$ does not cover x. If j is a join-irreducible satisfying $y = (x \wedge y) \vee j$, then $j \notin J(z)$ for any z covering x.*

Proof. Assume by way of contradiction that $j \vee (x \wedge y) = y$ and $j \leq z$ where $x \lessdot z$ and x and y satisfy the condition x and y both cover $x \wedge y$, but $x \vee y$ does not cover x. Since $x \lessdot z$ we know that $x \wedge y \leq z$. Then $x \wedge y \leq z$ and $j \leq z$ imply that $y = (x \wedge y) \vee j \leq z$ by the definition of a least upper bound. Now we have $x \leq z$ and $y \leq z$. But then $x \vee y \leq z$, contradicting the fact that z covers x. So, $j \notin J(z)$ for any $x \lessdot z$. □

Lemma 9. *Let L be a finite lattice satisfying $|J(L)| = n$ and let $x \in L$. If $|J(x)| = k$ and $\hat{\gamma} : J([\hat{0}, x]) \to [k]$ is a linear extension of the subposet of join-irreducibles of the interval $[\hat{0}, x]$, then there exists a* *linear extension $\gamma : J(L) \to [n]$ of the subposet $J(L)$ that restricts to the map $\hat{\gamma}$.*

Proof. By definition a linear extension $\hat{\gamma} : [\hat{0}, x] \to [k]$ is a map satisfying the condition that if $u \leq v$ in $[\hat{0}, x]$ then $\hat{\gamma}(u) < \hat{\gamma}(v)$ in the set $[k]$. Let $\overline{\gamma} : J(L \setminus [\hat{0}, x]) \to [n] \setminus [k]$ be a linear extension map of the induced subposet of join-irreducibles in the complement of the interval $[\hat{0}, x]$.

If $u \in [\hat{0}, x]$ satisfies $u \leq w$ for $w \in L \setminus [\hat{0}, x]$, then no possible choice of $\overline{\gamma}(w) \in [n] \setminus [k]$ can violate the condition $\gamma(u) < \gamma(w)$. Note there does not exist $z \in L \setminus [\hat{0}, x]$ satisfying $z \leq u$ for any $u \in [\hat{0}, x]$ by the definition of an interval. So define the map $\gamma : J(L) \to [n]$ by $\gamma(u) = \overline{\gamma}(u)$ for $u \in J(L \setminus [\hat{0}, x])$ and $\gamma(u) = \hat{\gamma}(u)$ for $u \in J([\hat{0}, x])$. Then γ is the required linear extension. □

Theorem 10. *Let L be a finite lattice with set of join-irreducibles $J(L)$ satisfying $|JI(L)| = n$. If for every linear extension $\gamma : J(L) \to [n]$ of the subposet $J(L)$, the minimal labeling λ_γ is an EL-labeling, then L is (upper) semimodular.*

Proof. Recall that for a finite lattice L is (upper) semimodular if and only if whenever s and t cover $s \wedge t$, then $s \vee t$ covers both s and t. Assume by way of contradiction that x and y cover $x \wedge y$ but $x \vee y$ does not cover x. Now $|J(x \wedge y)| = k \geq 0$. Note that if $k = 0$ then x and y are atoms. By Remark 2 there exists a join-irreducible $j_x \in J(x) \setminus J(x \wedge y)$ and a join-irreducible $j_y \in J(y) \setminus J(x \wedge y)$.

Let γ be a linear extension of $J(L)$ satisfying the following: let γ restrict to a linear extension of the interval $[\hat{0}, x \wedge y]$ on the k join-irreducibles in $[\hat{0}, x \wedge y]$, e.g. there is a linear extension $\hat{\gamma} : [\hat{0}, x \wedge y] \to [k]$ that extends to γ (this is possible by Lemma 9) and set $\gamma(j_x) = k + 1$ and $\gamma(j_y) = k + 2$.

Then, the lexicographically smallest chain in the interval $[x \wedge y, x \vee y]$ must be of the form

$$C = x \wedge y \lessdot x \lessdot x_2 \lessdot \cdots x_{m-1} \lessdot x_m = x \vee y$$

for $m > 2$, since by hypothesis $x \vee y$ does not cover x. Now we claim that $\lambda_\gamma(x, x_2) = r \geq k + 3$: first, $\lambda_\gamma(x, x_2) = \min \gamma(J(x_2 \setminus J(x))$, and the labels $1, 2, ..., k + 1$ are taken by elements of $J(x)$ which cannot be in the set $J(x_2) \setminus J(x)$. Furthermore, by Lemma 8, $j_y \notin J(x_2)$, so the label $k + 2$ cannot be in the set $\gamma(J(x_2) \setminus J(x))$. So $r \geq k + 3$. But since $J(y) \subset J(x \vee y)$, $j_y \in J(x_\ell) \setminus J(x_{\ell-1})$ for some $2 < \ell \leq m$. So $\min(\gamma(J(x_\ell) \setminus J(x_{\ell-1})) \leq k + 2$, and we have $\lambda_\gamma(x_{\ell-1}, x_\ell) \leq k + 2 < r = \lambda_\gamma(x, x_2)$ and the chain C must have a descent. This contradicts the fact the the lex smallest chain in an EL-labeling is increasing. So, no such pair x and y can exists. Therefore L is a semimodular lattice. □

5. Concluding remarks

Before turning to concluding remarks, we review a little bit more background that we will need regarding matroids. If $M = M(S)$ is a matroid of rank r on a finite set S, the *independence complex* $IN(M)$ is the $(r-1)$-dimensional simplicial complex formed by the family of all independent sets in M. On the other hand, a geometric lattice is the lattice of *flats*, or closed sets, of a matroid. Using our terminology, the matroid structure of a geometric lattice L is the matroid with ground set $\mathcal{A}(L)$ and the closure operator $cl(W)$ on a subset $W \subseteq \mathcal{A}(L)$ is $cl(W) = \{a \in A(x) | x \text{ is the join of the atoms in } W\}$.

Remark 3. In [4], there is a result of a similar flavor (Theorem 7.3.4) concerning matroid complexes: a simplicial complex $\triangle$ is the independence complex of a matroid if and only if $\triangle$ is pure and every ordering of the vertices induces a shelling of $\triangle$. Geometric lattices are also connected to matroids in that a geometric lattice is exactly the lattice of flats of a matroid. Though it seems interesting to note the analogy between the necessary hypotheses for Theorem 7.3.4 of [4] and those of our characterization of geometric lattices, our result is a fundamentally different result.

Remark 4. Our initial motivation was the question of whether posets admitting certain types of edge labelings were in fact more general than geometric lattices, so as to see whether it made sense to generalize results of [8] from geometric lattices to more general posets with the types of edge-labelings one has for geometric lattices. Our main result says that these two classes of lattices are in fact exactly the same, i.e., the latter is not actually any larger.

Remark 5. Axel Hultman has informed us that our result may also be modified to the following statement: Let L be a finite lattice with set of join-irreducibles P and $k = |P|$. Then the labeling λ_γ induced by each choice of order-preserving bijection $\gamma : P \to [k]$ is an R-labelling if and only if L is semimodular.

Remark 6. Notice finally that we can restate our first characterization of geometric lattices as a corollary of our characterization of semimodular lattices. First of all, if L is atomic, the set of all atom orderings of $A(L)$ is the same as the set of all linear extensions of $J(L)$ because

$A(L)$ is an antichain. And we simply showed before that requiring all atom orderings to induce EL labelings precludes the existence of join-irreducibles that are not atoms.

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References

[1] Riccardo Biagioli and Frederic Chapoton, Supersolvable LL-lattices of binary trees, Discrete Math, **296** (2005), no. 1, 1–13.

[2] G. Birkhoff, Lattice Theory. Third edition. American Mathematical Society Colloquium Publications, Vol. XXV, *American Mathematical Society, Providence, R.I.* 1967, vi + 418 pp.

[3] Anders Björner. Shellable and Cohen-Macaulay partially ordered sets. *Transactions of the American Mathematical Society* **260** (1980), 159-183.

[4] Anders Björner. Homology and Shellability of Matroids and Geometric Lattices. *Matroid Applications*. Ed. Ned White. Cambridge University Press, 1992.

[5] Anders Björner, Michelle Wachs. Bruhat order of Coxeter groups and Shellability. *Advances in Mathematics* **43** (1982), 87–100.

[6] Adriano Garsia, Combinatorial methods in the theory of Cohen-Macaulay rings, *Adv. in Math.* **38** (1980), no. 3, 229–266.

[7] Peter McNamara. EL-labelings, supersolvability, and 0-Hecke algebra actions on posets. *Journal of Combinatorial Theory Series A* **101** (2003), 69–89.

[8] Kathryn Nyman and Ed Swartz. Inequalities for the h-Vectors and Flag h-Vectors of Geometric Lattices. *Discrete and Computational Geometry* **32** (2004), 533–548.

[9] John Shareshian, On the shellability of the order complex of the subgroup lattice of a finite group, *Trans. Amer. Math. Soc.* **353** (2001), no. 7, 2689–2703.

[10] Richard Stanley. Supersolvable lattices. *Algebra Universalis* **2** (1972), 197–217.

[11] Richard Stanley, Finite lattices and Jordan-Hölder sets, *Algebra Universalis* **4** (1974), 361–371.

[12] Richard Stanley. *Enumerative Combinatorics Volume I.* Cambridge University Press, 1997.

[13] M. Wachs and J. Walker, *Order* **2** (1986), 367–385.

[14] Russ Woodroofe, An EL-labeling of the subgroup lattice, *Proc. Amer. Math. Soc.* **136** (2008), no. 11, 3795–3801.

[15] Thomas Zaslavsky. "Facing up to Hyperplane Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes." Memoirs of the American Mathematical Society **154** (1975), 102 pp.

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