GRÖBNER BASIS DEGREE BOUNDS ON $\mathrm{Tor}_\bullet^{k[\Lambda]}(k,k)_\bullet$ AND DISCRETE MORSE THEORY FOR POSETS

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Abstract. The purpose of this paper is twofold.

▷ We give combinatorial bounds on the ranks of $\mathrm{Tor}_\bullet^R(k,k)_\bullet$ in the case where $R = k[\Lambda]$ is an affine semi-group ring, and in the process provide combinatorial proofs for bounds by Eisenbud, Reeves and Totaro on which Tor groups vanish. In addition, we show that if the bounds hold for a field k then they hold for $\mathbb{K}[\Lambda]$ and any field $\mathbb{K}$. Moreover, we provide a combinatorial construction for a free resolution of $\mathbb{K}$ over $\mathbb{K}[\Lambda]$ which achieves these bounds.

▷ We extend the lexicographic discrete Morse function construction of Babson and Hersh for the determination of the homotopy type and homology of order complexes of posets to a larger class of facet orderings that includes orders induced by monomial term orders.

Since it is known that the order complexes of finite intervals in the poset of monomials in $k[\Lambda]$ ordered by divisibility in $k[\Lambda]$ govern the Tor-groups, the newly developed tools are applicable and serve as the main ingredients for the proof of the bounds and the construction of the resolution.

1. Introduction.

Let Λ be a submonoid of $\mathbb{N}^e$ which is finitely generated by $\alpha_1, \ldots, \alpha_n$, and denote by $k[\Lambda]$ the affine semi-group ring of Λ generated over the field k by monomials $\underline{\mathbf{x}}^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_e^{\alpha_{ie}}$, $1 \le i \le n$. Thus, $k[\Lambda] \cong k[z_1, \ldots, z_n]/I_\Lambda$ is the coordinate ring of an affine, not necessarily normal, toric variety. The isomorphism results from sending z_i to $\underline{\mathbf{x}}^{\alpha_i}$, $1 \le i \le n$, and letting the toric ideal I_Λ record the syzygies among the generators. The monoid Λ is endowed with a partial order given by $\mu \le \lambda$ if and only if $\lambda - \mu \in \Lambda$. Denote by $\Delta(\mu, \lambda)$ the simplicial complex of linearly ordered subsets $\mu < \mu_0 < \cdots < \mu_i < \lambda$ of the interval $[\mu, \lambda] := \{\gamma \in \Lambda \mid \mu \le \gamma \le \lambda\}$.

Based on work by Laudal and Sletjøe [LS] and Peeva, Reiner, Sturmfels [PRS], several recent papers (see e.g. [HRW], [BW]) have used this

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partial order on Λ and the simplicial homology of order complexes $\Delta(\mu, \lambda)$ for $\mu \leq \lambda$ in Λ as a tool for understanding minimal free resolutions of the field k as a $k[\Lambda]$-module. In general, the minimal free resolution of the field k over a k-algebra R is still a mysterious object (see [Av]) and even results known to hold by algebraic arguments pose hard and interesting combinatorial questions when $R = k[\Lambda]$ (see [BjWe]). Notably the Koszul property has attracted a lot of interest (for background motivating the study of the Koszul property see [Fr]). This property is equivalent by work of Peeva, Reiner, Sturmfels [PRS] to the property of all intervals in the poset Λ being Cohen-Macaulay over k. In general, it is known that a standard graded k-algebra is Koszul whenever its defining ideal has a quadratic Gröbner basis. In Peeva, Reiner, Sturmfels [PRS] and subsequent work [HRW] a combinatorial understanding of this implication is developed. In [BW] it is shown that if each interval in Λ is shellable then it is actually possible to construct a minimal free resolution for k as a $k[\Lambda]$-module.

In Section 5 we give an alternative combinatorial approach based on a discrete Morse function that also explains all these phenomena related to the Koszulness of $k[\Lambda]$ without requiring a shelling. The main idea behind our combinatorial approach is quite natural, and is explained in Remark 2.8 and the discussion that follows, after suitable notation is introduced. Section 6 uses the discrete Morse function of Section 5 to provide a minimal free resolution for k as a $k[\Lambda]$-module when I_Λ has a quadratic Gröbner basis, whether or not each interval in Λ is shellable.

In Section 7 we give the proof of our main result, a discrete Morse function on the order complex of Λ which provides combinatorial upper bounds on all the multigraded Tor groups for $k[\Lambda]$, most notably yielding the following:

Theorem 1.1. *Let $\Lambda \subseteq \mathbb{N}^e$ be an affine semi-group generated by n elements of $\mathbb{N}^e$. Assume there is a field k such that for $k[\Lambda] \cong k[z_1, \ldots, z_n]/I_\Lambda$ the ideal I_Λ has a Gröbner basis of degree d, then*

(i) $\tilde{H}_i(\Delta(\hat{0}, \lambda); \mathbb{K}) = 0$ *for $i < -1 + \frac{\deg(\lambda)-1}{d-1}$ and any field $\mathbb{K}$.*
(ii) $\mathrm{Tor}_i^{k[\Lambda]}(\mathbb{K}, \mathbb{K})_\lambda = 0$ *for $i < 1 + \frac{\deg(\lambda)-1}{d-1}$ and any field $\mathbb{K}$.*

Moreover, the vanishing of Tor*-groups is achieved by a free cellular resolution resulting from a discrete Morse function on Λ.*

Here we denote by $\deg(\lambda)$ the length, i.e. cardinality minus one, of a saturated chain in the poset interval $[\hat{0}, \lambda]$. Note that if all generators of Λ lie on an affine hyperplane then this grading actually makes Λ a graded poset and $k[\Lambda]$ a standard graded k-algebra. In general, $\deg(\lambda)$ is the degree of the image of $\underline{\mathbf{x}}^\lambda$ in the associated graded ring of R.

By results of [LS], Theorem 1.1 (i) will immediately imply Theorem 1.1 (ii). The vanishing of Tor-groups in the case $\mathbb{K} = k$ also follows for general standard graded k-algebras, by a flat degeneration argument, from a result of Eisenbud, Reeves and Totaro (cf. [ERT]) about monomial ideals. Our arguments are completely combinatorial.

The main tool for the proof of Theorem 1.1 is discrete Morse theory, which was developed in the mid 90's by Forman [Fo]. Discrete Morse theory is a tool for determining the homology and homotopy type of a simplicial complex, or more general a regular CW-complex. In [BH] the authors develop tools that facilitate the use of discrete Morse theory in the case when the simplicial complex is the order complex of a poset whose edges are labeled. The edges $\mu < \lambda$ of the Hasse diagram of Λ are naturally labeled by $\lambda - \mu$ which by the definition of the order relation is one of the generators $\lambda_1, \ldots, \lambda_n$ of Λ. Thus, to any saturated chain $\mu_0 < \cdots < \mu_i$ in λ there is associated the monomial $z_{j_1} \cdots z_{j_n}$ in $k[z_1, \ldots, z_n]$, where $\mu_i - \mu_{i-1} = \lambda_{j_i}$. In particular, any term order on $k[z_1, \ldots, z_n]$ induces a partial order on the finite saturated chains in Λ. In the case of Gröbner bases with properties analogous to requirements made in [PRS] (called 'supporting a poset"), we can apply the results from [BH] on lexicographic discrete Morse function which then easily give a degree d analogue of a lexicographic shelling, and imply the desired connectivity bound. In this Morse function, collections of at most d labels, given by descents and Gröbner basis leading terms, will play the role traditionally filled by the descents in a lexicographic shelling. In order to able to handle arbitrary Gröbner bases, whether or not they support a poset, we extend (Sections 3 and 4) the applicability of the methods from [BH] by using critical cell cancellation via gradient path reversal. Most notably, we introduce the notion of a content-lex facet ordering, which has also recently proven useful in work of [HHS]. The discrete Morse function on $\Delta(\Lambda) := \bigcup_{\lambda \in \Lambda} \Delta(\hat{0}, \lambda)$ also yields a free resolution of $\mathbb{K}$ over $\mathbb{K}[\Lambda]$ whose multigraded Betti-numbers can be read off from the number of critical cells of given dimension and given multidegree, i.e. from the Morse numbers, and from the gradient paths governing incidence among critical cells. Sections 6 and 8 will describe the critical cells in the Morse function as follows:

Theorem 1.2. *The critical cells of the discrete Morse function on $\Delta(\Lambda)$ are in bijection with the words of a language accepted by a finite state automaton, i.e. the words of a regular language. Thus, the* *generating function for Morse numbers is a rational function whose coefficients give upper bounds on all of the Betti numbers.*

In the quadratic Gröbner basis case, this generating function is exactly the Poincaré-Betti series, yielding yet another proof of its rationality in this case. There are known rational function bounds on the Poincaré-Betti series (see Proposition 3.3.2 in [Av]). But in contrast to these bounds our rational function comes close to the actual Betti numbers, in the sense that it exhibits the vanishing of Betti numbers as in Theorem 1.1.

We conclude the paper in Section 9 with remarks and open questions.

2. Background

2.1. **Posets and Order Complexes.** Let P be a poset with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$. We denote by $\Delta(P)$ the simplicial complex whose i-simplices are the chains $\hat{0} < p_0 < \cdots < p_i < \hat{1}$ in P. The maximal chains – with respect to inclusion – in P are sometimes called *saturated chains*. Notice that the saturated chains in P give rise to the facets (maximal faces) in $\Delta(P)$, and sometimes we will speak of saturated chains of P and facets of $\Delta(P)$ interchangeably. For $x \leq y$ in P, let $[x, y]$ be the closed interval $\{z \mid x \leq z \leq y\}$ and $(x, y) := [x, y] - \{x, y\}$ the open interval. We write $\Delta(x, y)$ for $\Delta([x, y])$.

For an arbitrary simplicial complex we denote by $H_i(\Delta; R)$ and $\widetilde{H}_i(\Delta; R)$ the non-reduced and reduced simplicial homology of Δ with coefficients in the ring R. The i-th Betti number b_i of Δ (with respect to $\mathbf{Z}$) is the rank of the free part of the i-th non-reduced homology group of Δ with coefficients in $\mathbf{Z}$. In order to calculate the non-reduced homology $H_i(\Delta; R)$ or the reduced homology $\widetilde{H}_i(\Delta; R)$ for the relevant simplicial complexes Δ we will use two basic facts from algebraic topology. First, if Δ is homotopy equivalent to a topological space X then the simplicial homology of Δ and the cellular/singular homology of X coincide. In our situation X will always be a CW-complex. The second important fact is, that if a CW-complex X has m_i cells of dimension i then the i-th Betti number b_i satisfies $b_i \leq m_i$. A poset P if called *homotopically Cohen-Macaulay* if the order complexes $\Delta(x, y)$ of all intervals $[x, y]$ in P are homotopy equivalent to a wedge of spheres of dimension $\dim(\Delta(x, y))$. In particular this holds if for each $\Delta(x, y)$ either $\dim(\Delta(x, y)) \leq 0$ or $\dim(\Delta(x, y)) \geq 1$ and $\Delta(x, y)$ is homotopy equivalent to a CW-complex with no cell in dimension $0 < i < \dim(\Delta(x, y))$ and a single cell in dimension 0.

Our main tool for the construction of a CW-complex X homotopy equivalent to a given simplicial complex Δ is discrete Morse theory.

2.2. **Discrete Morse Theory: General Theory.** This section reviews discrete Morse theory results we will need from [Fo], [Ch], [Jo], [BH], [He2] and [BW], along with some other requisite background. Forman [Fo] defines a function f which assigns real values to the cells in a regular CW-complex X to be a *discrete Morse function* if for each cell $\sigma \in X^{(*)}$ the sets

$$\left\{\tau \subseteq \overline{\sigma} \;\middle|\; \tau \in X^{(*)}, \dim(\tau) = \dim(\sigma) - 1, f(\tau) \geq f(\sigma)\right\}$$

and

$$\left\{\overline{\tau} \supseteq \sigma \;\middle|\; \tau \in X^{(*)}, \dim(\tau) = \dim(\sigma) + 1, f(\tau) \leq f(\sigma)\right\}$$

each have cardinality at most one. Here $X^{(*)}$ denotes the collection of open cells in X and $\overline{\sigma}$ denotes the closure of σ in X for $\sigma \in X^{(*)}$. The condition implies that for each σ, at most one of two sets is non-empty. When both are empty, then σ is called a *critical cell*. The main result on discrete Morse functions is the following:

Theorem 2.1 ([Fo])**.** *If f is a discrete Morse function on the regular CW-complex X then X is homotopy equivalent to a (not necessarily regular) CW-complex X^M, such that for any given i the number of cells of dimension i in X^M equals the number of critical cells of dimensions i of the Morse function of f. Moreover, incidences among cells in X^M are governed by a collapsing procedure that leads from X to X^M while preserving homotopy type at each step.*

In [Ch], Chari reformulated discrete Morse functions for regular CW-complexes in terms of certain types of face poset matchings. Recall that the *face poset* of a CW-complex X is the partial order $F(X)$ on the cells in $X^{(*)}$ defined by $\tau \leq \sigma$ whenever τ is contained in the closure $\overline{\sigma}$ of σ. If X is the geometric realization of an abstract simplicial complex Δ then this order is just the inclusion relation between the simplices of Δ. The *Hasse diagram* of a poset is the graph whose vertices are the poset elements and whose edges are the covering relations $x \prec y$, i.e. pairs $x < y$ such that $x \leq z \leq y$ implies $z = x$ or $z = y$.

Definition 2.2 ([Ch])**.** A matching on the Hasse diagram of the face poset $F(X)$ of a regular CW-complex X is called *acyclic* if the directed graph obtained by directing matching edges upward and all other poset edges downward has no directed cycles.

Notice that the non-critical cells of a discrete Morse function f come in pairs that prevent each other from being critical. Hence, this pairing gives a matching on the face poset of the CW-complex. Furthermore, this matching is acyclic, because Chari's edge orientation will orient all edges in the direction in which f weakly decreases. Conversely, many different (but in some sense equivalent) discrete Morse functions may be constructed from any face poset acyclic matching. For instance, one may obtain f by choosing a monotone function on any total order extension of the partial order given by the acyclic directed graph. The face poset elements that are left unmatched by an acyclic matching are exactly the critical cells in any corresponding discrete Morse function. We will work exclusively in terms of acyclic matchings rather than discrete Morse functions, but at times it is helpful to have both points of view in mind.

Denote by m_i the number of critical cells of dimension i in a discrete Morse function on a regular CW-complex X. As usual b_i is the i-th Betti number of X. By virtue of Theorem 2.1 X is homotopy equivalent to a complex X^M constructed from m_i cells of dimension i. The first of the following two results is an immediate corollary from Theorem 2.1, the second was first proved in [Fo]. Both results exhibit a strong analogy with traditional Morse theory:

$$m_j \geq b_j \text{ for } 0 \leq j \leq \dim(X) \tag{1}$$

$$\sum_{i=0}^{\dim(X)} (-1)^i m_{\dim(X)-i} = \sum_{i=0}^{\dim(X)} (-1)^i b_{\dim(X)-i} = \chi(X) \tag{2}$$

The inequality (1) will be used in later sections in oder to obtain bounds on Betti numbers. For our applications also the following special situation which we already mentioned in Section 2.1 will be of importance.

Corollary 2.3. *Let X be a regular CW-complex and f a discrete Morse functions on X with m_i critical cells of dimension i. If $m_i = 0$ for $0 < i < j$ and $m_0 = 1$ then X is $(j-1)$-connected. In particular, if for all order-complexes $\Delta(x,y)$ of intervals $[x,y]$ in a poset P such that $\dim(\Delta(x,y)) > 0$ there is a Morse function f_{xy} with no critical cell in dimension $0 < i < \dim(\Delta(x,y))$ and a single critical cell in dimension 0, then each interval is homotopy equivalent to a wedge of spheres of dimension $\dim((\Delta(x,y))$ and P is homotopically Cohen-Macaulay.*

Definition 2.4. Let X be a regular CW-complex and f a discrete Morse function on X. A *gradient path* from a critical cell τ to another critical cell σ of dimension $\dim(\tau) - 1$ is a directed path upon which the Morse function weakly decreases.

It is a simple consequence of the definition that a gradient path between τ and σ will alternate between cells of dimension $\dim(\tau)$ and $\dim(\tau) - 1$.

It will turn out that for our purposes we need to find a discrete Morse function on $\Delta(x, y)$ for intervals $x \leq y$ in an affine semigroup Λ such that inequality (1) becomes an equality. We will not always be able to achieve this goal and indeed it is open whether this is even possible. In our approach we construct a discrete Morse function on $\Delta(x, y)$ and then try to optimize the function in order to make (1) tight. For the latter we will employ the following observation.

Observation 2.5 ([Fo])**.** *If there is a unique gradient path from τ to σ, then reversing the orientation of each edge in this path yields a new acyclic matching for which τ and σ are no longer critical.*

We refer to the procedure described in Observation 2.5 as "cancelling critical cells."

The following lemma from [Jo] will be useful for combining several acyclic matchings to a single acyclic matching.

Lemma 2.6 (Cluster Lemma)**.** *Let X be a regular CW-complex which decomposes into collections X_p of cells indexed by the elements p in a partial order P with unique minimal element $\hat{0}$ as follows:*

(1) *X decomposes into the disjoint union $\cup_{p \in P} X_p$, that is, each cell belongs to exactly one X_p*
(2) *For each $p \in P$, $\cup_{q \leq p} X_p$ is a subcomplex of X*

For each $p \in P$, let M_p be an acyclic matching on the subposet $F(X) \cap X_p$ of $F(X)$ consisting of the cells in X_p. Then $\cup_{p \in P} M_p$ is an acyclic matching on $F(X)$.

2.3. **Discrete Morse Theory: The Case of Poset Order Complexes.** Let P be a poset with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$. Assume that the edges of the Hasse diagram of P are labeled by a labelling $\mathfrak{L}$ which takes values in a linearly ordered set. Via this labelling we can assign to each saturated chain $p_0 < p_1 < \cdots < p_{i-1}$ in P of cardinality i an $(i-1)$-tuple $(\mathfrak{L}(p_0, p_1), \ldots, \mathfrak{L}(p_{i-1}, p_i))$. Thus any linear extension of the lexicographic ordering on the tuples induces a linear order on the saturated chains in P.

In our situation where $P = [x, y]$ for $x \leq y$ in an affine semigroup Λ generated by $\alpha_1, \ldots, \alpha_n$ the labelling $\mathfrak{L}$ is given by sending a cover relation $\lambda \prec \mu$ to $\alpha_i = \mu - \lambda$ or if we consider $k[\Lambda]$ as a quotient of $k[z_1, \ldots, z_n]$ we can equivalently label $\lambda \prec \mu$ with z_i. Note, that in the latter case the product over the labels of a saturated chain is a monomial in $[z_1, \ldots, z_n]$. In either case we can choose an arbitrary linear order on the sets $\{\alpha_1, \ldots, \alpha_n\}$ or $\{z_1, \ldots, z_n\}$. Since the two labellings are equivalent we will not distinguish between z_1 and λ_i in the rest of the paper.

It is shown in [BH] how to construct a discrete Morse function on $\Delta(P)$, respectively an acyclic matching on the face poset $F(\Delta(P))$ of $\Delta(P)$ for a labeled poset P, from the lexicographic order on the saturated chains of P. If this lexicographic ordering happens to be a shelling order (see [BjW]) then it is possible to infer directly from the constructed Morse function that the poset is homotopically Cohen-Macaulay – just as it can be deduced from the lexicographic shelling itself. We will refer to a Morse function resulting from a lexicographic ordering on the saturated chains as a *lexicographic discrete Morse function*. We will show in Section 3 that the construction of [BH] applies to a larger class of orders on saturated chains. This will allow us to construct a discrete Morse function on $\Delta(\Lambda)$ for an affine semigroup Λ generated by $\alpha_1, \ldots, \alpha_n$ from a facet order based on an arbitrary monomial term order on $k[z_1, \ldots, z_n]$.

Let $\mathfrak{L}$ be a labelling of the poset covering relations such that $\mathfrak{L}(u, v) \neq \mathfrak{L}(u, w)$ for $v \neq w$. Then the ordering of the label sequences on the saturated chains by the lexicographic order already gives a total order on saturated chains, i.e. an ordering $F_1, \ldots, F_r$ on facets in $\Delta(P)$. To describe the corresponding discrete Morse function from [BH], we will need to speak of the ranks of elements of a saturated chain, whether or not a poset is graded. We will do this by assigning to an element of a saturated chain the rank of the element within the chain and speak of the rank with respect to the chain. Indeed, we do not require consistency of the notion of rank between different saturated chains. Also we will identify a set of chains in P, resp. faces of $\Delta(P)$, with the simplicial complex generated by the chains.

Each maximal face in $F_j \cap (\cup_{i<j} F_i)$ has rank set (with respect to F_j) of the form $1, \ldots, i, j, \ldots, n$, i.e. it consists of all ranks in F_j except for a single interval $i+1, \ldots, j-1$ of consecutive ranks that are omitted; this follows from the use of a lexicographic order on facets. Call each such list $i+1, \ldots, j-1$ of ranks a *minimal skipped interval* of F_j, and say the interval has *height* $j-i-1$. Following [BiH] and [BH], call the collection of minimal skipped intervals for F_j the *interval system* or *set of I-intervals* of F_j.

For each facet F_j, [BH] constructs an acyclic matching on the set of faces in $F_j \setminus \cup_{i<j} F_i$ in terms of the interval system for F_j. This is done in such a way that the union (over all F_j) of these matchings is acyclic, and each $F_j \setminus (\cup_{i<j} F_i)$ includes at most one critical cell. We say F_j *contributes a critical cell* if $F_j \setminus (\cup_{i<j} F_i)$ contains a critical cell. F_j will contribute a critical cell if and only if the homotopy type changes with the attachment of F_j.

Description of critical cells:

(Case 1) If the I-intervals of F_j do not collectively have support covering all the ranks in F_j, then F_j does not contribute a critical cell.

(Case 2) If the I-intervals of F_j cover all ranks and have disjoint support, then the critical cell consists of the lowest rank from each of the I-intervals.

(Case 3) If there is some overlap in the minimal skipped intervals of F_j, then iterate the following procedure to obtain a potential critical cell, ordering I-intervals so that their minimal ranks are increasing:

(1) Include the lowest rank from I_1 in the potential critical cell.
(2) Truncate all the remaining minimal. skipped intervals by chopping off any ranks that they share with I_1.
(3) Discard I_1 and any skipped intervals that are no longer minimal.
(4) Re-index the remaining truncated minimal skipped intervals to begin with a new I_1.
(5) Repeat until there are no more minimal skipped intervals.

The non-overlapping intervals obtained by the above truncation procedure are called the *J-intervals* of F_j.

Remark 2.7. *F_j contributes a critical cell if and only if its J-intervals cover all ranks of F_j. In this case, the dimension of the critical cell is one less than the number of J-intervals.*

In order to cancel critical cells by reversing gradient paths, we will also need some information about the matching itself.

Description of the acyclic matching on $F_j \setminus \cup_{i<j} F_i$:

- If the I-intervals do not cover all ranks, then there is at least one cone point in $F_j \cap (\cup_{i<j} F_i)$. In this situation we match by including/excluding cone point of lowest rank.
- Otherwise, match any non-critical cell based on the lowest I-interval of F_j where the cell differs from the potential critical cell. Specifically, match by including/excluding the lowest element of the I-interval, since the cell must include at least one element of the I-interval other than this lowest possible element, in order to cover the I-interval but differ from the critical cell.
- If the I-intervals cover all ranks, but the J-intervals do not, then match all remaining cells (i.e. the cells which agree with the potential critical cell on all J-intervals) by including/excluding the lowest rank not in any J-interval

Remark 2.8. *If there is some d such that every I-interval in a lexicographic discrete Morse function has height at most $d-1$, then the above construction immediately implies that each poset interval (x, y) is at least $(-1+\frac{\mathrm{rk(y)-rk(x)-1}}{d-1})$-connected. In fact, it suffices for the average interval height to be at most $d-1$.*

In our setting, where $P = \Delta(x, y)$ for $x \leq y$ in an affine semigroup Λ Remark 2.8 will turn out to be applicable for d the degree of a Gröbner basis of the ideal I_Λ (i.e. the maximal total degree of a polynomial in the Gröbner basis). We will use the fact that every I-interval results from a label sequence descent or from a syzygy leading term, by virtue of our use of a monomial term order to order the saturated chains. Furthermore, all leading terms will be divisible by Gröbner basis leading terms of degree at most d, which will allow us to show that the average interval height is at least $d-1$, yielding the desired lower bound on connectivity. In a sense, this gives a new combinatorial explanation for the connection between Gröbner basis degree and complexity, in which a shelling is the special case with $d = 2$. However, just as in the shelling in [PRS], only certain types of monomial term orders and Gröbner bases will immediately yield the degree d analogue of a shelling.

Example 2.9. *Consider $k[z_1, z_2, z_3, z_3]/(z_1z_4 - z_2^2)$ with a term order $\preceq$ such that $\mathrm{in}_\preceq(z_1z_4 - z_2^2) = z_1z_4$. If z_1z_4 is a Gröbner basis leading term, but z_1z_3 and z_3z_4 are not Gröbner basis leading terms, Then $z_3(z_1z_4 - z_2^2) = 0$ precludes the shelling from [PRS], since z_1z_3 and z_3z_4 are not leading terms,*

In order to be able to allow completely general monomial term orders, i.e. to deal with situations such as in Example 2.9, we need to extend the tools from [BH] described in this section by performing critical cell cancellation. This extension will be done in Section 3. The the method of critical cell cancellation is explained in the next section.

2.4. **Discrete Morse Theory: Optimizing Discrete Morse Functions.** This section reviews tools from [He2] for eliminating critical cells by cancelling pairs by a gradient path reversal. Later we will construct a lexicographic discrete Morse function for monoid posets, and then use these Morse function optimization tools to eliminate all of the low-dimensional critical cells.

Let Δ be a simplicial complex and f a discrete Morse function on Δ. Define the *multi-graph face poset*, denoted $F(\Delta)^M$, for the complex Δ^M of critical cells as follows:

(1) The vertices in $F(\Delta)^M$ are the cells in Δ^M, or equivalently the critical cells in the discrete Morse function on Δ.
(2) There is one edge between a pair of cells σ, τ of consecutive dimension $\dim(\tau) = \dim(\sigma) + 1$ for each gradient path from τ to σ.

Theorem 2.10. *Any acyclic matching on $F(\Delta)^M$ specifies a collection of gradient paths in $F(\Delta)$ that may simultaneously be reversed to obtain a discrete Morse function M' whose critical cells are the unmatched cells in the matching on $F(\Delta)^M$.*

To cancel cells, we will need to know that a gradient path from a critical cell τ to a critical cell σ is the only gradient path from τ to σ.

Definition 2.11. Let $u \prec v \prec w$ be covering relations in a poset P labeled by $\mathfrak{L}$. Assume we are given a linear order on the saturated chains in $[u, w]$. The labels $\mathfrak{L}(u, v)$ and $\mathfrak{L}(v, w)$ on covering relations $u \prec v \prec w$ are said to *commute* if the least saturated chain in $[u, w]$ is labeled by $\mathfrak{L}(u, v), \mathfrak{L}(v, w)$ arranged in ascending order.

Let $\mathfrak{L}(u, v)$ denote the sequence of edge labels on the least saturated chain from u to v.

Definition 2.12. Let P be a poset labelled by $\mathfrak{L}$ and assume that within each interval the saturated chains are linearly ordered by an order depending only on the label sequence.

(i) The weakly increasing rearrangement of the label sequence of a saturated chain is called the *content* of the chain.
(ii) The labelling $\mathfrak{L}$ is called *least-increasing* if every interval has a (weakly) increasing chain as its least saturated chain.
(iii) The labelling $\mathfrak{L}$ is is called *least-content-increasing* if it is east-increasing and in addition the label sequence of the least chain equals or precedes the content of the label sequence of every other saturated chain in the interval.

Note, that since the definition assumes the linear order on the saturated chains only depends on the label-sequences, Definition 2.12(iii) we can consider this order as an order on label sequences. If the linear order on the saturated chains is given by the lexicographic order then the condition least-increasing is weaker than being an EL-labelling (see [BjWa]), in that intervals may have several increasing chains.

Remark 2.13. *If two critical cells τ, σ in a least-content-increasing labelling are contributed by saturated chains of equal content, then every downward step in any gradient path from τ to σ must preserve content,* *and in fact must sort labels on the interval where the chain element was deleted.*

Combining results (see Theorem 6.6) in [He2] yields the following.

Theorem 2.14. *Let P be a poset labelled by a least-content-increasing labelling $\mathfrak{L}$. Let M be the lexicographic discrete Morse induced by $\mathfrak{L}$ and let τ, σ, $\dim(\tau) = \dim(\sigma) + 1$, be critical cells resulting from saturated chains whose label sequences $\mathfrak{L}(\tau), \mathfrak{L}(\sigma)$ have equal content. Suppose further that the permutation transforming $\mathfrak{L}(\tau)$ to $\mathfrak{L}(\sigma)$ is 321-avoiding. If there is a gradient path γ from τ to σ such that each downward step swaps a pair of consecutive labels by deleting an element v from a chain which also includes elements covering and covered by v, then γ is the unique gradient path from τ to σ.*

Remark 2.15. *Our upcoming discrete Morse function will use a content-lex facet order, as introduced in Section 3. Theorem 2.14 also applies in that setting without requiring any modification.*

Theorem 4.2 will generalize the above to deal with non-saturated chain segments, as needed for cancelling critical cells in our upcoming Morse function.

2.5. **Discrete Morse Theory: Application to Cellular Resolutions.** Let M be a module over a commutative ring R. A *free resolution* of M over R is a complex of free R-modules F_i and R-module homomorphisms ∂_i

$$\mathcal{F} : \cdots \stackrel{\partial_i+1}{\to} F_i \stackrel{\partial_i}{\to} F_{i-1} \stackrel{\partial_{i-1}}{\to} \cdots \stackrel{\partial_1}{\to} F_0.$$

which is exact in all degrees $\neq 0$ (i.e., $\mathrm{Im}(\partial_i) = \mathrm{Ker}(\partial_{i-1})$ for $i \geq 2$) and $\mathrm{Coker}(\partial_1) \cong M$. In our case $R = k[\Lambda]$ is a k-algebra and carries an additional multigraded structure, Recall that a k-algebra $R = k[z_1, \ldots, z_n]/I$ is called $\mathbb{N}^d$*-multigraded* if $R = \bigoplus_{\alpha \in \mathbb{N}^d} R_\alpha$ as k-vector spaces and $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$. If $d = 1$ and R is generated in degree 1 over $k = R_0$ then R is called *standard graded*. Analogously defined are $\mathbb{N}^d$-graded R-modules. In this situation we consider multigraded free resolutions $\mathcal{F}$. In addition to being a resolution one demands that the F_i are free multigraded R-modules and that the ∂_i are $\mathbb{N}^d$-homogeneous. A free multigraded module F is a direct sum $\oplus_{\alpha \in \mathbb{N}^d} R(-\alpha)^{\beta_\alpha}$ of free R-modules $R(-\alpha)$ of rank one whose grading is defined by assigning α as the degree of the unit element 1.

A $\mathbb{N}^d$-graded free resolution

$$\mathcal{F} : \cdots \stackrel{\partial_i+1}{\to} \bigoplus_{\alpha \in \mathbb{N}} R(-\alpha)^{\beta_{i,\alpha}} \stackrel{\partial_i}{\to} \bigoplus_{\alpha \in \mathbb{N}} R(-\alpha)^{\beta_{i-1,\alpha}} \stackrel{\partial_{i-1}}{\to} \cdots \stackrel{\partial_1}{\to} \bigoplus_{\alpha \in \mathbb{N}} R(-\alpha)^{\beta_{0,\alpha}}.$$

is called cellular, if there is a CW-complex X and a map $\mathrm{gr} : X^{(*)} \to \mathbb{N}^d$ from the set $X^{(*)}$ of its cells to $\mathbb{N}^d$ such that:

- There is a basis e_c of $F_i = \bigoplus_{\alpha \in \mathbb{N}} R(-\alpha)^{\beta_{i,\alpha}}$ indexed by the i-cells of X in such a way that if e_c belongs to $R(-\alpha)$ then $\mathrm{gr}(c) = \alpha$.
- For the i-cell c of X and its cellular differential $\delta_i(c) = \sum_{c'} [c : c']c'$, we have $\partial_i(e_c) = \sum_{c'} [c : c'] \underline{\mathbf{x}}^{\mathrm{gr}(c') - \mathrm{gr}(c)} e_{c'}$.

In this situation we say that X *supports* the resolution $\mathcal{F}$.

Consider the face poset $P = F(X)$ of a CW-complex X supporting a resolution. [BW, Proposition 2.2] shows that an acyclic matching A on P leads to a chain homotopy between the original resolution and a smaller cellular resolution, given by the smaller CW complex X^M of critical cells in a discrete Morse function given by A, if A matches only cells that have the same value under gr.

It is well known that for any $\mathbb{N}^d$-graded module M there exists a *minimal* multigraded free resolution (i.e., a resolution that uses the least number of free modules in each degree). Now the results from [BW], as described above, allow one to construct smaller resolutions from a given resolution. It is also clear (see [BW]) that this process will not always allow one to produce the minimal free resolution.

Let us consider a 'big' cellular resolution in the situation treated in this paper (i.e, $R = k[\Lambda]$ and $M = k$). It is well known that the simplicial complex $\Delta(\Lambda)$ of all finite chains $\lambda_0 < \cdots < \lambda_r$ in Λ together with the grading $\mathrm{gr}(\lambda_0 < \cdots < \lambda_r) = \lambda_r$ gives a multigraded free cellular resolution of the maximal ideal $\mathbf{m} = \mathrm{Ker}(k[\Lambda] \to k)$ – the normalized Bar resolution. Since a free minimal resolution of k over $k[\Lambda]$ starts with $k[\Lambda]$, minimizing the normalized Bar resolution is equivalent to minimizing a resolution of k. A well known criterion for a resolution to be minimal is that no unit elements of R occur in the matrices representing the differentials. If in addition R is standard graded and all matrix entries are either 0 or elements of degree 1, then the resolution is called *linear*. If k has a linear resolution then R is called *Koszul*.

In our case, where $R = k[\Lambda]$, we know that R is standard graded if and only if the generators of Λ lie on an affine hyperplane. In this situation R carries two gradings, the standard grading and a multigrading given by Λ. The Tor-groups $\mathrm{Tor}_i^R(k,k) \cong k^{\beta_i}$, where $\beta_i = \sum_\alpha \beta_{i,\alpha}$, also carry a multigraded structure $\mathrm{Tor}_i^R(k,k)_\alpha = k^{\beta_{i,\alpha}}$. Thus Koszulness can be read off from the Tor-groups. Namely, R is Koszul if and only if $\mathrm{Tor}_i^R(k,k)_\alpha = 0$ for $\underline{\mathbf{x}}^\alpha$ with standard grading not equal to i.

A well known sufficient condition for a k-algebra $R = k[z_1, \ldots, z_n]/I$ which is standard graded to be Koszul is that I has a quadratic Gröbner basis. Recall that for a monomial order $\preceq$ on $k[z_1, \ldots, z_n]$ a *Gröbner basis* of I is a generating set $\mathcal{G}$ of polynomials in I such that the *initial ideal* $\mathrm{in}_{\preceq}(I) := \langle \mathrm{in}_{\preceq}(f) | f \in I \rangle$ is equal to the ideal generated by $\{\mathrm{in}_{\preceq}(f) | f \in \mathcal{G}\}$. Recall that the *leading monomial*, denoted $\mathrm{in}_{\preceq}(f)$, for a polynomial f is the largest monomial with respect to $\preceq$ occurring in f; we write $\mathrm{in}(I)$ and $\mathrm{in}(f)$ if the monomial order is clear from the context. Finally, a *monomial term order* on $k[z_1, \ldots, z_n]$ is a linear order $\preceq$ on the monomials in the ring such that (1) $1 \preceq m$ for all monomials m, and (2) $m \preceq m'$ implies $mn \preceq m'n$ for all monomials n, m, m'.

2.6. **Basic Facts on Finite State Automata, Regular Languages and Rational Generating Functions.** A central question in the theory of infinite resolutions is 'Which conditions on a module M imply that Poincare'-Betti series of its minimal free resolution is rational ?' (see [Av]). When $R = k[\Lambda]$ is a standard graded and multigraded k-algebra and $M = k$, the (graded and multigraded) *Poincare'-Betti series* is given by $\sum_{i,\alpha} \beta_{i,\alpha} t^i \underline{\mathbf{x}}^\alpha z^{\deg(\alpha)}$, where $\deg(\alpha)$ is the degree of $\underline{\mathbf{x}}^\alpha$ in the standard grading on R.

We will not be able to give a new criterion for the rationality of the Poincare'-Betti series. But we will be able to give in Section 8 a rational series which bounds the Poincare'-Betti series from above, i.e. all coefficients are greater or equal to the ones in the Poincare'-Betti series. In order to prove rationality of our series we will resort to the theory of regular languages. It is well known (see [BR]) that the generating series of a regular language is rational. A language L over a finite *alphabet* Σ is called *regular* if there is a finite state automaton which accepts exactly the words in L. The generating series of L is given by $\sum_{w \in L} t^{|w|}$, where $|w|$ is the number of letters in w. See for instance [BR] for additional information.

3. Monomial term orders and discrete Morse functions resulting from (not-necessarily-lexicographic) facet orders

This section will show how the lexicographic discrete Morse function construction of [BH] generalizes easily to a larger class of facet orders for poset order complexes; this will include facet orders for monoid posets based on arbitrary monomial term orders. First observe that the [BH] construction applies without modification to any facet order $F_1, \ldots, F_r$

which yields an interval system structure on each $F_j \setminus (\cup_{i<j} F_i)$. Equivalently, the construction will work for facet orderings satisfying the crossing condition, as introduced in [He1] and defined below.

Crossing condition. *Let $\leq$ be a linear order on the saturated chains in a partially ordered set P of rank n and rank function* rk. *Let F be a saturated chain, $G \leq F$ and $\sigma = F \cap G$. Suppose that $[n] - \{\mathrm{rk}(p) \mid p \in \sigma\}$ is not an interval of natural numbers. Then there is some facet $G' \leq F$ such that $F \cap G \subsetneq F \cap G'$.*

The crossing condition implies that for a saturated chain F, maximal faces in $F \cap (\cup_{G<F} G)$ are supported on a set of ranks whose complement is a single interval of consecutive ranks.

Theorem 3.1. *If a facet ordering on an order complex satisfies the crossing condition, then the acyclic matching construction of [BH] applies to this facet ordering.*

PROOF. The effect of the crossing condition for a particular facet order $F_1, \dots, F_k$ is to ensure that each maximal face of $F_j \cap (\cup_{i<j} F_i)$ for $1 < j \leq k$ skips a single interval of consecutive ranks. Thus, the faces in $F_j \setminus (\cup_{i<j} F_i)$ are the ones that "hit" each of these intervals, implying that the matching construction from [BH] still applies. □

Definition 3.2. A facet order on a poset order complex which satisfies the crossing condition is called *lex-like facet order*. The discrete Morse function obtained by applying the construction from [BH] to such a facet order is called a *lex-like discrete Morse function*.

Let $[x, v]$ be an interval in a labelled poset P. For a saturated chain in $[x, y]$, the set of all saturated chains in $[x, y]$ having the same content is called the fibre of the content. Now again turn to intervals in affine semigroups Λ. Let $\mathfrak{L}$ be the usual labelling of a covering relation $\lambda \prec \mu$ by the generator $\mu - \lambda$ of the semigroup. We assume that we are given a monomial term order on $k[z_1, \dots, z_n]$ and again as usual identify the generators of Λ with the variables z_i. This identification allows us to order the saturated chains by the given monomial term order. Recall, that we identify a label sequence with the monomial which is the product over the labels, which in turn can be seen as the content of the saturated chain with the given label sequence. Lemma 3.3 considers linear orderings of saturated chains in intervals of Λ obtained by combining the (commutative) monomial term ordering with a lexicographic order on each fibre; that is, we extend the order given by the monomial term order on the content by the lexicographic ordering on chains that have the same content. The lexicographic order uses the monomial term order on degree 1 monomials to order the labels.

Lemma 3.3. *Let $P = [x, y]$ be an interval in an affine semigroup Λ and assume that the saturated chains in P are ordered by a monomial term order refined by the lexicographic order. Then this ordering satisfies the crossing condition. In particular, we can construct a discrete Morse function just as in the case of a lexicographic order.*

PROOF. Let $\equiv$ be the equivalence relation on the set of saturated chains such that $m \equiv n$ if and only if m and n are two saturated chains in the same closed interval in P. We abuse notation and say $m = n$ if the two chains are labeled by the same commutative monomial, i.e. the labels on one saturated chain are a permutation of the labels on the other. Suppose that for two saturated chains m_1m_2 and n_1n_2 we have $n_1 \neq m_1$, $n_2 \neq m_2$ but $n_1 \equiv m_1$ and $n_2 \equiv m_2$. Assume further that $n_1m_1 <_{monom} n_2m_2$ in the given monomial order $<_{monom}$. This is one situation where a saturated chain has an overlap face with earlier saturated chains such that the complement of the ranks in the overlap face is disconnected.

We check that either $m_1n_2 <_{monom} n_1n_2$ or $n_1m_2 <_{monom} n_1n_2$, as follows. Suppose $m_1n_2 >_{monom} n_1n_2$, which implies $m_1 >_{monom} n_1$. Suppose $n_1m_2 >_{monom} n_1n_2$ also holds, implying $m_2 >_{monom} n_2$. Combining these inequalities yields

$$m_1m_2 >_{monom} n_1m_2 >_{monom} n_1n_2,$$

a contradiction. Hence, at least one of the monomials m_1n_2 or n_1m_2 precedes n_1n_2 in our term order. This ensures that the maximal face shared by m_1m_2 and n_1n_2 is not a maximal face in the simplicial complex of faces shared by n_1n_2 and facets that precede it in lexicographic order, just as needed.

Now suppose there is a saturated chain F_j not dealt with above that has an overlap face with an earlier saturated chain F_i such that the complement of the ranks in the overlap face is disconnected. Then F_i, F_j are labeled $m_1m_2m_3, \pi(m_1)n_2\sigma(m_3)$, respectively, where π, σ are permutations on the labels in m_1, m_3 and $m_2 \equiv n_2$ but $m_2 \neq n_2$. Consider F_i' which is labeled $\pi(m_1)m_2\sigma(m_3)$. Since $F_i \cap F_j \subsetneq F_i' \cap F_j$, we are done. □

Remark 3.4. Examples suggest that the following procedure may be convenient for posets with no particularly well-behaved global labelling.

1. Label edges in a poset Hasse diagram in a natural way (or more generally, give a chain-labelling on saturated chains).
2. Partition the set of saturated chains into groups called *content classes* according to the content of their label sequences.

(3) Put an ordering on these content classes.
(4) Within each content class, order saturated chains lexicographically.
(5) Prove that the resulting facet order satisfies the crossing condition.
(6) Cancel pairs of critical cells which have the same content, i.e. pairs in the same content class.

This approach can make critical cell cancellation manageable for posets with no particularly nice global labelling, because gradient paths that begin and end in the same content class must never leave that content class. This will be essential to our analysis of monoid posets and to arguments in [HHS].

Definition 3.5. A *content-lex facet order* is a lex-like facet order such that:

- The ordering is constructed from a labelling by refining a linear order on fibres by a lexicographic order.
- The least saturated chain in each interval has weakly increasing labels.

One way content-lex facet orders arise is when each content class individually has an EL-labelling. This is the situation for our upcoming facet order on monoid posets as well as a $GL_n(q)$-analogue of the partition lattice examined in [HHS].

Remark 3.6. *Content-lex facet orders behave as least-content-increasing labellings for purpose of applying Theorem 2.14 to verify gradient path uniqueness.*

4. Uniqueness of 321-avoiding gradient paths in discrete Morse functions from content-lex facet orders

Now we generalize Theorem 2.14 to allow non-saturated chain segments in gradient paths between critical cells in the same content class in a content-lex facet order, under certain additional assumptions. Remark 9.2 suggests that the 321-avoiding assumption is probably necessary for any general result about gradient path uniqueness for lex-like discrete Morse functions.

Upcoming sections will construct a non-optimal discrete Morse function for monoid posets, then use the following theorem to improve it.

Definition 4.1. A *delinquent chain* in a least-increasing labelling (or in a content-lex facet order) is an increasing chain that is not lexicographically smallest on an interval. A content-lex facet order is *consistently* *delinquent* if the existence of a delinquent chain labeled $a_1 \cdots a_k$ implies that any chain segment labeled $b_1 \cdots b_{k'}$ for $b_1 = a_1, b_{k'} = a_k$ and $\{a_1, \dots, a_k\} \subseteq \{b_1, \dots, b_{k'}\}$ is also delinquent.

Denote by $e\alpha$ the earliest facet containing a cell α, and let $\mathfrak{L}(\alpha)$ be the label sequence on $e\alpha$. We say that a chain element *covers* a delinquent chain if it is in the interior of the delinquent chain, preventing the poset chain from belonging to an earlier content class.

Theorem 4.2. *Let τ, σ, $\dim(\tau) = \dim(\sigma) + 1$, be critical cells in the same content class in a content-lex facet order which is consistently delinquent. Suppose $\mathfrak{L}(\sigma)$ differs from $\mathfrak{L}(\tau)$ by a 321-avoiding permutation π which either shifts a single group of one or more consecutive ascending labels upward, or shifts a single label downward. Then there is at most one gradient path from τ to σ.*

PROOF. Since τ is critical, every pair of consecutive labels in $\mathfrak{L}(\tau)$ is either a descent or part of a minimal delinquent chain. Gradient paths from τ to σ can never uncover any delinquent chains, since that would cause the gradient path to pass to an earlier content class, from which it could never reach σ. Since the facet order is also least-increasing, each downward step must either preserve the label sequence or eliminate a descent by deleting an element v_r from a chain τ_i of the form $v_1 < \cdots < v_{r-1} < v_r < v_{r+1} < \cdots < v_s$, causing the labels on the segments of $e\tau_i$ from v_{r-1} to v_r and from v_r to v_{r+1} to be sorted into a single ascending list. The least-increasing property Thus, any inversions present in $\mathfrak{L}(\sigma)$ must be preserved throughout the gradient path, since they can never be re-introduced. With these observations in hand, we will describe the only possible gradient path from τ to σ, showing at each stage there is only one choice for how to proceed.

Suppose $\mathfrak{L}(\sigma)$ is obtained from $\mathfrak{L}(\tau)$ by shifting a collection of consecutive, ascending labels upward. Let μ be the label to be shifted upward to the highest destination in $\mathfrak{L}(\sigma)$, and let b be the label immediately above μ in $\mathfrak{L}(\tau)$. By virtue of the Morse function construction of [BH], τ includes exactly the ranks where $e\alpha$ has descents as well as exactly one rank that covers each of the minimal delinquent chains, namely the lowest ranks in the J-intervals. The first gradient path downward step must eliminate a descent since it is not allowed to uncover a delinquent chain. The only choice that will not eliminate an inversion that is present in $\mathfrak{L}(\sigma)$ is to eliminate a descent between μ and the label above it, i.e. deleting an element v_r from a chain $v_1 < \cdots < v_{r-1} < v_r < v_{r+1} < \cdots < v_s$. This yields a chain σ_1 with $e\sigma_1 = F_j$ having a cone point in $F_j \cap (\cup_{i<j} F_i)$ between v_{r-1} and v_{r+1}

in F_j, but no lower cone points. Thus, the subsequent upward step must insert some $v_{r'}$ above v_{r-1} and below the label μ, with the labels between v_{r-2} and $v_{r'}$ now comprising either a descent or minimal delinquent chain. In the latter case, the delinquent chain just below $v_{r'}$ must include the label b as its highest label. Labels now below b can no longer shift upward, since any such labels to be shifted upward in $\mathfrak{L}(\sigma)$ must be smaller than b. If there were instead a descent at v_{r-1}, only one label is allowed between v_{r-1} and $v_{r'}$, so either it is the label b defined above, which does not shift upward, or it is a label smaller than b, which is now prevented from moving upward by virtue of being smaller than b.

Continuing in this fashion, there is only one viable downward step at any given stage until μ reaches its destination, namely the chain deletion which shifts μ upward, since no J-interval is ever covered by more than one chain element while μ is shifting upward. By the argument above, labels can only shift upward while they are shifting as part of a block of consecutive labels which includes μ, so all label shifting is complete once μ has reached its destination, and there is a unique way for this to happen. All that remains is to consider additional gradient path steps which preserve label sequence.

Suppose the label above μ in $\mathfrak{L}(\sigma)$ is larger than μ. Then μ must be within a delinquent chain, since σ is critical. After inserting a cone point below μ, the gradient path must take a downward step deleting a chain element above μ, since the J-interval for the delinquent chain which includes μ will now be covered by at least two chain elements. At this point, no upward step is possible, so the gradient path must have reached σ. Finally, we show that μ cannot form a descent with the label above it in $\mathfrak{L}(e\sigma)$. Otherwise, μ would be the highest label in a minimal delinquent chain labeled $\nu_1, \dots, \nu_k, \nu, \mu$, but μ could not be shifted upward to this position from below without passing to an earlier content class.

The case where a single label μ shifts downward to obtain $\mathfrak{L}(\sigma)$ from $\mathfrak{L}(\tau)$ is quite similar, so is essentially left to the reader. The first gradient path step must again eliminate a descent so as to preserve content class, and the only allowable choice is a descent between μ and the label below it. Similar reasoning to above will show that the only possible gradient path will progressively shift μ downward to its destination, concluding once μ reaches the interior of a delinquent chain. $\square$

5. The Cohen-Macaulay property for monoid posets with quadratic Gröbner bases

Throughout this section, we assume the toric ideal $I_\Lambda = \mathrm{Ker}(\phi)$ has a quadratic Gröbner basis B. However, Section 5.1 will apply to higher degree Gröbner bases with essentially no modification needed, and large parts of Sections 5.2– 5.4 will also generalize easily to higher degree Gröbner bases.

Definition 5.1. *Denote by* $\mathrm{in}(I_\Lambda)$ *the initial ideal of* I_Λ *with respect to the term order giving rise to* B*, i.e. the ideal generated by leading terms of elements of* B*.*

This section will show that each interval $(\hat{0}, m)$ in the resulting monomial divisibility poset has the homotopy type of a wedge of spheres of top dimension. Our approach will be to construct a lex-like discrete Morse function based on a content-lex facet order in Theorem 5.2, and then to cancel all but some of the top-dimensional critical cells in Theorem 5.21.

We will employ the simple fact that any leading term of a polynomial in a toric ideal is divisible by a Gröbner basis leading term. In particular, leading terms of degree greater than the degree of the Gröbner basis will include variables that in some sense are non-essential. We will use these non-essential variables to cancel critical cells. In the case of quadratic Gröbner bases, this approach will allow us to cancel all critical cells that are not saturated chains. Later, we will use a similar (but somewhat more intricate) analysis for Gröbner bases of degree d.

5.1. A non-optimal Morse function.

The first step will be to give a content-lex facet ordering.

Theorem 5.2. *The monoid poset interval* $(\hat{0}, m)$ *has a lex-like discrete Morse function resulting from a content-lex facet order. Its minimal skipped intervals are the saturated chain segments with label sequences of the following two types:*

(1) *descents*
(2) *sequences of weakly increasing labels* $\lambda_i, \ldots, \lambda_k$ *such that* $\lambda_i\lambda_k \in \mathrm{in}(I_\Lambda)$*, but* $\lambda_{i'}\lambda_j \notin \mathrm{in}(I_\Lambda)$ *for every other pair* $i \le i' < j \le k$*.*

PROOF. The finite saturated chains on intervals in a monoid poset correspond naturally to pairs (m_i, π) where m_i is a monomial in the ring $k[z_1, \ldots, z_n]$, and π is an ordering on the content of m_i. This is equivalent to labelling saturated chains by non-commutative monomials, in $k\langle y_1, \ldots, y_n\rangle$, the viewpoint taken in [PRS]. Following [PRS], we order saturated chains in an interval $(\hat{0}, m)$ by using the monomial term order which led to the Gröbner basis B to order the factorizations $m_i \in \phi^{-1}(m)$, and then lexicographically ordering label sequences of any fixed content, with our label order given by the monomial term order applied to monomials of degree one. Lemma 3.3 confirms the crossing condition for this facet order, implying it gives rise to a lex-like discrete Morse function from a content-lex facet order. Next, we characterize its minimal skipped intervals.

Notice that a descent on the saturated chain segment $u \prec v \prec w$ implies a lexicographically smaller ascend $u \prec v' \prec w$, obtained by reversing the order in which semi-group generators are multiplied, so descents always give minimal skipped intervals. On the other hand, any label sequence $a_1, \nu_1, \nu_2, \ldots, \nu_r, a_2$ as in (2) will give rise to a minimal skipped interval because the Gröbner basis element $a_1a_2 - b_1b_2$ with leading term a_1a_2 implies the existence of an earlier saturated chain on the interval labeled by the increasing rearrangement of the monomial $b_1n_1 \cdots n_jb_2$; its minimality follows from the lack of descents and of Gröbner basis leading terms not requiring both a_1 and a_2.

Definition 5.3. The second type of minimal skipped interval in the statement of the theorem is called a *syzygy interval.*

To see there are no other minimal skipped intervals, note that a label sequence $a_1, \ldots, a_r$ on any other minimal skipped interval must be ascending to avoid descents which would preclude its minimality; to have an earlier saturated chain on the interval $a_1, \ldots, a_r$ must be a leading term, hence divisible by a Gröbner basis leading term m. But minimality ensures a_1, a_r divide m, and the fact that the Gröbner basis is quadratic implies $m = a_1a_r$. □

Corollary 5.4. *Theorem 4.2 may be applied to the above Morse function to cancel pairs of critical cells that belong to the same fibre.*

Definition 5.5. A minimal skipped interval is *non-trivial* if it has height greater than one. Notice that only syzygy intervals may be non-trivial.

Example 5.6. Consider the interval $(1, x_1^2x_2^2x_3x_4)$, or equivalently, $(1, z_1z_2z_3z_4)$, in the ring $k[x_1x_2, x_1^2, x_3, x_4, x_2^2] \cong k[z_0, z_1, z_2, z_3, z_4]/(z_1z_4 - z_0^2)$, with $\mathrm{in}(I_\Lambda) = (z_1z_4)$. Saturated chains are labeled by indices of the generators $z_0, \ldots, z_4$. Figure 1 shows four saturated chains on this interval, the leftmost and rightmost of which will contribute critical cells. Notice that $1, 4$ labels a syzygy interval, while there are descents at ranks 1 and 2 in the leftmost saturated chain, so it has a critical

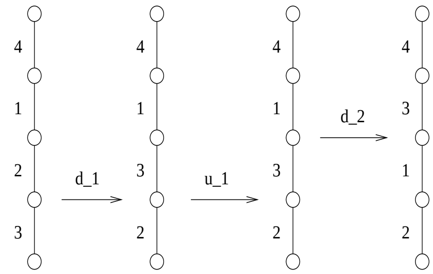

FIGURE 1. A gradient path shifting the label 3 into a syzygy interval

cell τ comprised of ranks 1,2,3. On the other hand, the label sequence $1, 3, 4$ in the rightmost saturated chain also labels a syzygy interval, and this chain has a descent at rank 1, so it has a critical cell σ comprised of ranks 1,2.

The remainder of Section 5 is devoted to cancelling pairs of critical cells by gradient path reversal, so as to eliminate all critical cells not given by saturated chains. This will require an acyclic matching on $F(\Delta)^M$ consisting of pairs of critical cells to be cancelled.

Definition 5.7. We call the critical cells that remain after all this cancellation the *surviving critical cells.*

5.2. **Syzygy intervals and their non-essential sets.** For now we assume all monomials on our monoid poset interval are square-free. The general case is dealt with in Theorem 5.21.

Remark 5.8. *For convenience, we will refer interchangeably to a critical cell and the saturated chain which contributes it.*

Figure 1 gives an example of a gradient path from a critical 2-cell to a critical 1-cell in the order complex resulting from the semigroup ring $k[z_0, z_1, z_2, z_3, z_4]/(z_1z_4 - z_0^2)$. This gradient path shifts the label z_3 to the interior of a syzygy interval, using the fact that z_3 is not essential to $z_3(z_1z_4 - z_0^2) = 0$. This is the only gradient path between these two critical cells. Our goal will be to systematically cancel many such pairs of critical cells simultaneously.

Definition 5.9. Denote by $I(a_1, a_2)$ the syzygy interval with ascending labels $a_1, \lambda_1, \ldots, \lambda_k, a_2$ in a saturated chain, and refer to the Gröbner basis leading term a_1a_2 with $a_1 \leq a_2$ as an *increasing leading term*, or *ILT* for short.

We will soon use ILTs to collect critical cells into Boolean algebras within $F(\Delta)^M$.

Example 5.10. In the affine semi-group ring $k[z_1,\ldots,z_6]/(z_2z_6-z_1^2)$, consider the saturated chain F_j that is labeled $z_4z_3z_2z_5z_6$. F_j contributes the critical cell $\sigma = z_4 < z_3z_4 < z_2z_3z_4$. By Theorem 2.14, there is a unique gradient path from the critical cell $\tau = z_5 < z_4z_5 < z_3z_4z_5 < z_2z_3z_4z_5$ to σ, given by the reduced expression $s_1 \circ s_2 \circ s_3$. More generally, each $T \subseteq S = \{z_3, z_4, z_5\}$ gives rise to a critical cell $Crit(T)$ contributed by a facet F_T, as follows. F_T has label sequence $z_{(S\setminus T)^{rev}}z_2z_Tz_6$, where z_T is the list of members of T in increasing order, and $z_{(S\setminus T)^{rev}}$ is the list of members of $S \setminus T$ listed in decreasing order. $S = \{z_3, z_4, z_5\}$ is the non-essential set of the interval. Theorem 2.14 will show that the set of critical cells $\{Crit(T)|T \subseteq S\}$ sits inside the multi-graph face poset $F(\Delta)^M$ as a Boolean algebra, depicted in Figure 2.

This Boolean algebra has covering relations $Crit(T \cup \{z_i\}) \prec Crit(T)$ for each $T \subseteq S$ and each $z_i \in S \setminus T$.

Remark 5.11. *A gradient path cannot swap non-commuting labels (in the sense of Definition 2.11) without passing to an earlier fibre, so cells to be cancelled will agree up to allowable label commutation.*

Definition 5.12. A label λ in a saturated chain M is *upward-shiftable* into a syzygy interval $I(a_1, a_2)$ if it satisfies all the following conditions:

(1) $a_1 <_{monom} \lambda <_{monom} a_2$
(2) λ appears below $I(a_1, a_2)$
(3) all labels between λ and $I(a_1, a_2)$ are smaller than λ and commute with λ
(4) all labels within $I(a_1, a_2)$ commute with λ
(5) λ is not the top of some $I(\mu, \lambda)$ with either non-empty interior or such that the label ν immediately above λ would neither form a descent with μ nor be part of an ILT together with μ

Likewise, λ is *downward-shiftable* from $I(a_1, a_2)$ to just above λ' if λ commutes with all labels separating it from λ' and is larger than all such labels.

Remark 5.13. *Each label has at most one syzygy interval into which it is upward-shiftable, because the lowest such interval will separate it from all higher ones.*

Definition 5.14. If λ appears within $I(a_1, a_2)$, then the *topologically decreasing position* for λ below $I(a_1, a_2)$ is the highest position below $I(a_1, a_2)$ to which λ is downward-shiftable so as to obtain the label sequence for a critical cell with λ not in the interior of any ILT, if such a position exists.

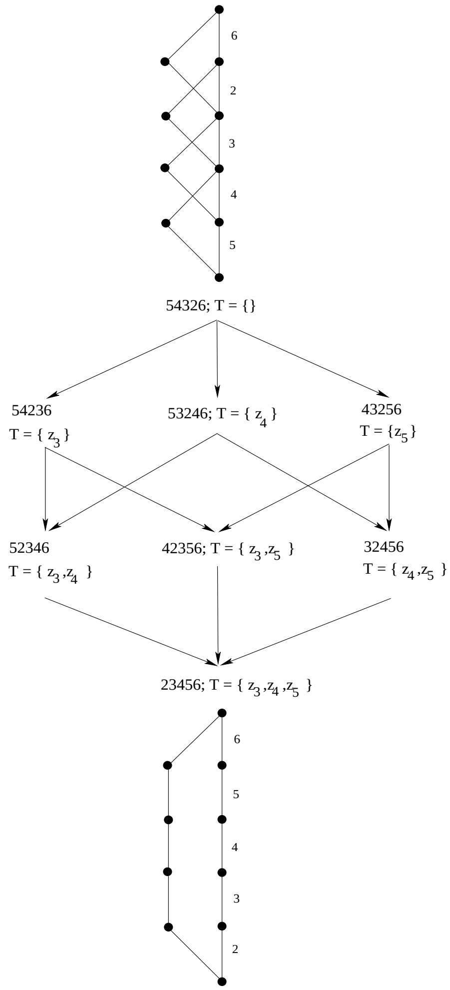

FIGURE 2. A Boolean algebra in $F(\Delta)^M$ indexed by subsets of $\{z_3, z_4, z_5\}$

In its topologically decreasing position below an ILT, λ must form descents or ILTs with the labels above and below it. Lemma 5.23 will construct gradient paths that shift labels from their topologically decreasing positions below ILTs to the interior of ILTs. In some circumstances, we will also speak of the topologically decreasing position of λ above $I(a_1, a_2)$, by which we mean the lowest position above $I(a_1, a_2)$ to which λ is upward-shiftable to yield the label sequence for a critical cell.

Definition 5.15. The *non-essential set* of a syzygy interval $I(a_1, a_2)$ that appears in the label sequence $\mathfrak{L}(\sigma)$ for a critical cell σ will be a collection of labels that appear in $\mathfrak{L}(\sigma)$ either within $I(a_1, a_2)$ or in topologically decreasing positions below $I(a_1, a_2)$. This set of labels (to be defined precisely in the remainder of Section 5.2 and Section 5.3) is denoted $S(a_1, a_2)$.

To try to convey the intuition for $S(a_1, a_2)$, we now give an oversimplified definition. Section 5.3 will modify this into a much more technical definition that accomplishes exactly what is needed. Initially, let us include in $S(a_1, a_2)$ those labels that appear within $I(a_1, a_2)$ that are downward-shiftable to topologically decreasing positions below $I(a_1, a_2)$. Denote these labels by $n_1, \ldots, n_j$. Also include in $S(a_1, a_2)$ those labels $m_1, \ldots, m_r$ that are upward-shiftable into $I(a_1, a_2)$, chosen in order from highest to lowest topologically decreasing position below $I(a_1, a_2)$.

We call critical cells which are not maximal faces in the order complex $\Delta(\hat{0}, m)$ *unsaturated*. In the case of a quadratic Gröbner basis, we will match and cancel all unsaturated critical cells, using the fact that each must have one or more syzygy intervals with non-empty interior.

5.3. **The matching on critical cells.** In this section, we precisely define non-essential sets and show that the resulting matching on critical cells is well-defined. We also show that every critical cell which has at least one syzygy interval with non-empty interior is indeed matched and cancelled, by showing it has at least one syzygy interval with non-empty non-essential set. The fact that pairs of critical cells to be matched do indeed comprise covering relations in $F(\Delta)^M$ will be verified in Section 5.4.

Definition 5.16. A label λ is *preferable* to a label μ within a label sequence if either λ is in the non-essential set of a higher syzygy interval than μ is in, or $\lambda, \mu \in S(a_1, a_2)$ with the topologically decreasing position for λ higher than for μ.

Using our oversimplified definition of non-essential set from the previous section, let $I(a_1, a_2)$ be the highest syzygy interval in a saturated chain C such that $S(a_1, a_2) \neq \emptyset$, and let λ be the label in $S(a_1, a_2)$ with highest topologically decreasing position below $I(a_1, a_2)$. The theorem below will sometimes include in a non-essential set $S(a_1, a_2)$ a single label that shifts downward into $I(a_1, a_2)$ from above. When this happens, denote this label m_0, and eliminate from $S(a_1, a_2)$ any labels below $I(a_1, a_2)$ that do not commute with m_0.

Theorem 5.17. *Every critical cell with a syzygy interval with an internal label μ is matched. Specifically, if μ is excluded from the non-essential set of the interval, then there must be another label that allows the cell to be matched and cancelled. Moreover, the matching choices are made consistently.*

PROOF. We will typically match by shifting λ from inside $I(a_1, a_2)$ to its topologically decreasing position below $I(a_1, a_2)$, or vice versa. However, special care is needed in four circumstances described below. When λ is excluded from a non-essential set, then the matching instead shifts the label with highest preferability among those belonging to some non-essential set.

In each of these circumstances, we will show that either λ may be included in $S(a_1, a_2)$ or that there is an alternative label to λ allowing the cell to be matched. Moreover, when λ is excluded from a non-essential set, it will be excluded for all critical cells in the Boolean algebra within which our critical cell is matched.

(1) $\lambda \in I(a_1, a_2)$ shifts downward to a topologically decreasing position which is higher than some label μ with which λ does not commute; however, μ may shift upward to the interior of a syzygy interval $I(b_1, b_2)$ to obtain another critical cell when λ appears in $I(a_1, a_2)$ but not when λ has shifted downward to its topologically decreasing position.
(2) $\lambda \in I(a_1, a_2)$ shifts downward to just above a label μ with which λ does not commute to form an ILT $I(\mu, \lambda)$ which then has some label $\nu \in S(\mu, \lambda)$.
(3) $\lambda \in I(a_1, a_2)$ cannot shift downward to a topologically decreasing position without first encountering a label μ with which it does not commute.
(4) $\lambda \in I(a_1, a_2)$, but shifting λ downward causes a_2 to be in the non-essential set of a higher ILT, or more generally the shifting of all labels within $I(a_1, a_2)$ downward to topologically decreasing positions, cumulatively causes a_2 to belong to a higher non-essential set.

In the first case above, exclude λ from $S(a_1, a_2)$. When λ appears within $I(a_1, a_2)$, then $\mu \in S(b_1, b_2)$, ensuring the cell may still be matched. Matching by shifting μ will clearly give a cell which would also exclude λ from $S(a_1, a_2)$. Shifting a label μ' that is preferable to μ also gives a cell that excludes λ, either by virtue of μ, or if μ', λ do not commute, then by virtue of μ'.

In the second case, note that $\lambda \in I(a_1, a_2)$ implies either (a) ν is in the non-essential set of some $I(a_1', a_2')$ above μ, (b) ν is separated from the lowest such $I(a_1', a_2')$ by a label with which it does not commute, or (c) ν does not commute with some label in the interior of the lowest such $I(a_1', a_2')$; this follows from $\mu < \nu < \lambda < a_2$ together with the fact that μ and the label immediately above it must form either a descent or an ILT. In case 2(a), ν provides an alternative label for matching, and λ may be excluded from $S(a_1, a_2)$. Notice that the partner cell in which ν or a preferable label ν' has been shifted may also exclude λ by the following reasoning. Since λ commutes with ν, the critical cell with $\nu \in I(a_1', a_2')$ and λ immediately above μ may be matched by shifting ν downward to just above λ, instead of by shifting λ.

For 2(b), include λ in $S(a_1, a_2)$, since the cell with $\nu \in I(\mu, \lambda)$ may be matched by shifting ν upward to its topologically decreasing position above $I(\mu, \lambda)$, or by shifting a label that is preferable to ν. Notice that we this also deals with case 4, by considering it from a different viewpoint.

Definition 5.18. When such a label ν is matched by such upward-shifting, we say that ν *blocks* a_2 from belonging to a higher non-essential set.

Furthermore, observe that shifting ν upward, or shifting a preferable label, still gives a critical cell which excludes λ from $S(a_1, a_2)$; this is because either $\nu \in I(\mu, \lambda)$, or ν appears above λ and would form an ascend with μ if λ were shifted upward. For 2(c), if $I(a_1', a_2')$ has non-empty interior, then this gives an alternative to λ, allowing λ to be excluded from $S(a_1, a_2)$. When this alternative label is shifted to outside $I(a_1', a_2')$, it is still preferable to λ, by the conventions from case 2(a). If there are no interior labels in $I(a_1', a_2')$, then ν does not commute with a_2'; λ is included in $S(a_1, a_2)$, noting that the cell with ν shifted into $I(\mu, \lambda)$ will be matched by shifting ν upward to just below a_1', similarly to case 2(b).

Now we turn to the third case. First notice that μ must appear in a lower ILT, either as its lowest label, or in its proper interior. If μ appears in the interior of some $I(a_1', a_2')$, then exclude λ from $S(a_1, a_2)$ and apply our argument to μ or a preferable label, proceeding downward until we find a way of matching. By virtue of case (1), the matching partner which has shifted μ or a preferable label will also have excluded λ from $S(a_1, a_2)$. If μ is the lowest label of some $I(\mu, \mu')$ with non-trivial interior, then again use a label from the interior or a preferable label for matching. On the other hand, if $I(\mu, \mu')$ has no interior, then consider the descending labels $\nu_j, \nu_{j-1}, \ldots, \nu_1$ immediately above μ', up through the lowest label ν_1 in the next lowest ILT above $I(\mu, \mu')$. Let T be the subset of $\{\lambda, \nu_2, \ldots, \nu_j, \mu'\}$ which consists of those μ' along with those labels which commute with μ', but not with μ. If all labels in T belong to the non-essential sets of higher ILTs, then there is a Boolean algebra of critical cells in which each $T' \subset T$ specifies which labels to leave in decreasing order immediately above μ, rather than shifted upward into the interior of various ILTs; however, the empty set is missing from this Boolean algebra unless μ forms a descent with the label just above it when all labels in T are shifts upward into interiors of ILTs.

Thus, allowing $\lambda \in S(a_1, a_2)$ and matching by shifting λ to just above μ gives nearly a complete matching on this Boolean algebra of critical cells, but there is no matching partner for the cell indexed by $T' = \{\lambda\}$. However, the cell indexed by $\{\lambda\}$ may instead be matched based on any $\nu_i \notin T$, since such a ν_i either belongs to the non-essential set of some higher ILT, or may be shifted downward into $I(\mu, \lambda)$; in the latter case, we are in a situation where ν_i blocks λ from belonging to $S(a_1, a_2)$, in the sense described above. In any event, all cells in question are matched, and it is clear that the matching partners are also matched in the same fashion.

The fourth case was already handled within the argument for the second case. $\square$

5.4. **The Cohen-Macaulay Property.** In this section we verify that the matching of the previous section consists of covering relations in $F(\Delta)^M$, and that these comprise an acyclic matching on $F(\Delta)^M$ with only top-dimensional surviving critical cells. We begin with an important special case which captures most of the idea.

Definition 5.19. The *expanding interval* of a saturated chain C, denoted $I(a_1, a_2)$, is the highest syzygy interval with non-empty non-essential set in C..

Theorem 5.20. *If I_Λ has a quadratic Gröbner basis and each $\phi^{-1}(m)$ is square-free, then the poset interval $(0, m)$ has a discrete Morse function whose critical cells are all saturated, implying the interval is homotopy equivalent to a wedge of spheres.*

PROOF. In a syzygy interval labeled $a_1 n_1 \ldots n_j a_2$, recall that $a_1 a_2$ is a Gröbner basis leading term. Any unsaturated critical cell has at least one syzygy interval with $j > 0$. We described in the previous section how to match all such cells so that partner cells differ by exactly one in their number of minimal skipped intervals. Lemma 5.22 verifies that they in fact differ in dimension by exactly one. See Figure 1 for an example of a gradient path from one such critical cell to its matching partner.

Lemma 5.23 shows for the expanding interval $I(a_1, a_2)$ that $S(a_1, a_2)$ gives rise to a Boolean algebra of critical cells within $F(\Delta)^M$, indexed by the subsets of $S(a_1, a_2)$. $S(a_1, a_2)$ is chosen so that each $T \subseteq S(a_1, a_2)$ gives rise to a unique such cell, denoted $Crit(T)$. $Crit(T)$ is contributed by a saturated chain $M(T)$, which has exactly the labels in T inside $I(a_1, a_2)$, and each of the labels in $S(a_1, a_2)\setminus T$ shifted to its topologically decreasing position outside $I(a_1, a_2)$. All labels in $M(T)$ other than a_1, a_2 and the members of $S(a_1, a_2)$ will appear in the same relative order for all choices of $T \subseteq S(a_1, a_2)$.

Any Boolean algebra has a complete acyclic matching simply by matching by including/excluding any fixed set element. We assign each critical cell to the Boolean algebra given by the non-essential set of its expanding interval, then take a union of complete acyclic matchings on these Boolean algebras. Lemma 5.26 checks that when one critical cell is assigned to a particular Boolean algebra, then all critical cells in that Boolean algebra are assigned to it, ensuring the matching is well-defined. Section 5.3 already showed that we match all unsaturated critical cells.

The final step is to show that this union of complete matchings on Boolean algebras is an acyclic matching on $F(\Delta)^M$. By Theorem 2.10, this would imply that we may simultaneously reverse all these gradient paths to cancel all but some top-dimensional critical cells. To get acyclicity, we show two things: (1) Lemma 2.6 ensures that cycles cannot involve Boolean algebras from distinct fibres $\phi^{-1}(m)$, due to the filtration $m_1 \subseteq m_1 \cup m_2 \subseteq \dots$ based on the monomial term order $m_1, m_2, \dots$, and (2) Lemma 5.27 verifies that cycles cannot involve multiple Boolean algebras in the same fibre. Thus, we will produce a discrete Morse function whose critical cells are all top-dimensional, implying the order complex has the homotopy type of a wedge of spheres of top dimension. $\square$

Next we deal with the possibility that not all monomials are square-free. The lemmas that follow do not use the square-free assumption, so they apply to the general case.

Theorem 5.21. *If I_Λ has a quadratic Gröbner basis, then the monoid poset Λ has a discrete Morse function whose critical cells are all top-dimensional, implying Λ is homotopically Cohen-Macaulay.*

PROOF. The only issue left to address is repetition of labels. To this end, we adjust the definition of non-essential set and make sure surviving critical cells still do not have any syzygy intervals with non-empty interior. When multiple copies of a letter appear inside $I(a_1, a_2)$ or are upward-shiftable into it, only include one copy in $S(a_1, a_2)$ that shifts downward to below $I(a_1, a_2)$; we cannot shift more than one copy outside the interval and still get a critical cell, since consecutive identical labels not within a syzygy interval give a saturated chain rank not covered by any minimal skipped interval. Including one copy of the repeated letter in the non-essential set is enough to ensure the Boolean algebra B_n has $n \geq 1$, hence has a complete matching. A letter cannot initiate or conclude a syzygy interval and also appear in its interior, since then the syzygy interval would not be a minimal skipped interval. We may have a syzygy interval which begins and ends with the same label a_1, but then there cannot be any interior labels at all. $\square$

Lemma 5.22. *Critical cells that are matched differ in dimension by exactly one.*

PROOF. If none of the I-intervals are discarded in their conversion to J-intervals, then there is no issue (see Section 2.3 for definitions). When there is discardment, this means there are three or more overlapping I-intervals such that a middle one is unnecessary for covering all ranks by I-intervals, so Gröbner basis leading term elements for these intermediate I-intervals will each belong to the non-essential set of a higher I-interval, ensuring matching by shifting such an individual label to outside the collection of overlapping ILTs. This matching operation preserves the number of J-intervals from ILTs and alters by exactly one the number of J-intervals coming from descents. Thus, dimension changes by exactly one. $\square$

Lemma 5.23. *The critical cells indexed by subsets of $S(a_1, a_2)$, for $I(a_1, a_2)$ the expanding interval of a saturated chain $M(T)$, have the same incidences in $F(\Delta)^M$ as a Boolean algebra of subsets of $S(a_1, a_2)$. That is, there is a unique gradient path from $Crit(T)$ to $Crit(T \cup \{i\})$ for each $T \subseteq S(a_1, a_2)$ and each $i \in S(a_1, a_2) \setminus T$, and these are the only gradient paths among critical cells in $B_{S(a_1,a_2)}$.*

PROOF. Since all saturated chains in a fibre have equal content, and our facet order is content-lex, downward steps in a gradient path must sort labels. The critical cell $Crit(T)$ is obtained by arranging labels in T in increasing order within $I(a_1, a_2)$, and labels in $S \setminus T$ in unique topologically decreasing positions below $I(a_1, a_2)$ (or above $I(a_1, a_2)$, in the special circumstance that a_2 must be "blocked" from upward-shifting into another non-essential set), from which they may shift into $I(a_1, a_2)$. First we exhibit for each pair $T' = T \cup \{i\}$ with $i \in S \setminus T$, that there is a gradient path from $Crit(T)$ to $Crit(T')$. Choose r and t so that the t-th element in the chain $Crit(T)$ is just above the lowest label of $I(a_1, a_2)$, and the r-th element of $Crit(T)$ is just above the label i in $Crit(T)$, for i satisfying $T' = T \cup \{i\}$. In Example 5.24, let $i = 3, t = 3$ and $r = 1$.

Example 5.24. *Figure 1 depicts a gradient path from a critical 2-cell of rank set* $\{1,2,3\}$ *to a critical 1-cell with rank set* $\{1,2\}$ *, based on a ring with* $z_1z_4 \in \mathrm{in}(I_\Lambda)$*. We have* $S = \{2,3\}$*,* $T = \emptyset$ *and* $T' = \{3\}$*.*

There is a gradient path from $Crit(T)$ to $Crit(T')$ of the form

$$d_r \circ u_r \circ d_{r+1} \circ u_{r+1} \circ \cdots \circ d_{t-1} \circ u_{t-1} \circ d_t,$$

because i commutes with all labels separating it from $I(a_1,a_2)$, i is larger than all these separating labels, and our discrete Morse function comes from a least-increasing facet order. Since the resulting permutation on labels is 321-avoiding, Theorem 4.2 ensures that this gradient path is unique, whether or not non-saturated chain segments are encountered in it.

To show that there are no other covering relations in $F(\Delta)^M$, i.e. none between other pairs of critical cells corresponding to subsets of $S(a_1,a_2)$, we use the fact that gradient paths can never introduce inversions. Thus, a gradient path from $Crit(T)$ to $Crit(T')$ would imply $T \subseteq T'$, since any $j \in T \backslash T'$, would imply an inversion (j, a_1) in $Crit(T)$ that is not present in $Crit(T')$. ☐

Next we verify that critical cells are indeed partitioned into Boolean algebras.

Remark 5.25. *Critical cells with label sequences of distinct content or with non-commuting labels in opposite order are assigned to distinct Boolean algebras.*

Lemma 5.26. *Whenever one critical cell is assigned to a Boolean algebra, then all critical cells in that Boolean algebra are assigned to it.*

PROOF. We must show that if $I(a_1,a_2)$ is the expanding interval for a saturated chain $M(T)$ for some $T \subseteq S(a_1,a_2)$, then the saturated chain $M(T')$ for each $Crit(T')$ with $T' \subseteq S(a_1,a_2)$ also has $I(a_1,a_2)$ as its expanding interval. Each label in a saturated chain belongs to the non-essential set of at most one syzygy interval, since it cannot pass through the lowest such syzygy interval above it to reach higher ones via a gradient path; a label is only assigned to the non-essential set of a syzygy interval below it when it cannot shift into one above it. Shifting labels belonging to $S(a_1,a_2)$ from within $I(a_1,a_2)$ to their topologically decreasing positions or vice versa cannot cause a higher non-essential set to become non-empty, by virtue of the choices made in Theorem 5.17. ☐

Finally, let us confirm that these complete matchings on Boolean algebras collectively give an acyclic matching on $F(\Delta)^M$.

Lemma 5.27. *The matching on critical cells in $F(\Delta)^M$ is acyclic.*

PROOF. Lemma 2.6 ensures there are no directed cycles involving multiple fibres. Suppose there were a directed cycle C in a single fibre. Any such C must alternate upward (matching) steps with downward steps. Our matching consists of a union of complete matchings on Boolean algebras. Since the upward steps in a fixed Boolean algebra all insert the same fixed element i, each downward step must take us to a different Boolean algebra, to avoid yielding the top of an upward-oriented edge in the same Boolean algebra, from which the cycle could not have continued.

Suppose a matching step in C shifts a label μ upward from within an ILT $I(a_1, a_2)$ to above it. Then by virtue of our matching, a_2 must belong to the non-essential set of a higher ILT in the cell which has μ and all other labels within $I(a_1, a_2)$ shifted to below $I(a_1, a_2)$. Furthermore, μ must not be upward-shiftable into a higher ILT. To pass to a distinct Boolean algebra, the downward step immediately after this upward-shifting of μ must either (1) shift a_2 upward into a higher ILT, (2) shift a label λ downward from a topologically decreasing position into the interior of an ILT, or (3) shift a label μ' upward into the interior of an ILT it then blocks (cf. Definition 5.18). (1) is impossible because a_1, μ form a non-inversion after a_2 is shifted upward, implying an ascend between consecutive commuting labels somewhere between a_1 and μ, causing the cell not to be critical. In case (3), we can never un-do this shifting of μ', since a matching step will not shift it downward from $I(a_1, a_2)$, since μ blocks a_2 from shifting upward, but μ is too large to shift below a_2 without a_1 also present. Case (2) is allowed, but eventually we still would need to shift μ downward into $I(a_1, a_2)$, at which point we would have a downward step keeping us in the same Boolean algebra, making it impossible for the cycle to continue. Thus, we can rule out upward-shifting gradient path steps in a cycle.

Next suppose there is a step that shift labels downward either creating or eliminating an ILT. Consider the lowest ILT $I(a_1, a_2)$ ever created/destroyed. It must be destroyed by a downward step shifting a_1 into the interior of a lower ILT $I(b_1, b_2)$. Eventually we have an upward (matching) step, shifting a_1 back upward from within $I(b_1, b_2)$ to below a_2. But we have already eliminated the possibility of such upward-shifting matching steps within a cycle.

Finally, if all upward (matching) steps shift labels downward from within ILTs to between them, and all downward (non-matching) steps shift labels downward from between ILTs into lower ILTs, preserving the set of ILTs at each step, then labels not initiating or concluding

ILTs move progressively downward and may never return upward, making completing a cycle impossible. To be precise we create inversions between labels initiating/concluding ILTs and other labels, but we may never eliminate these inversions. □

6. Applications: minimal free cellular resolution and a finite state automaton which computes Poincare'-Betti series

This section describes the surviving critical cells in the quadratic Gröbner basis case in two ways:

(1) as the words generated by a finite state automaton, implying the generating function for Morse numbers is rational
(2) as representatives of the J'-non-stuttering, J'-commuting equivalence classes of words, as developed in [HRW] to count Betti numbers.

Theorem 6.1. *The discrete Morse function of the previous section gives a minimal free cellular resolution of k as a $k[\Lambda]$-module.*

PROOF. Results of [BW] imply that the complex of critical cells from our Morse function supports a free cellular resolution of k as a $k[\Lambda]$-module, because our acyclic matching preserves multi-grading. Furthermore, there are no incidences among critical cells of equal multidegree in this Morse function, because all critical cells of multidegree λ come from saturated chains with highest element λ, making gradient paths from one critical cell to another of the same multi-degree impossible, despite the fact that critical cells need not be concentrated in a single dimension. If we consider the complex obtained by tensoring the complex of critical cells with k, this implies that all its boundary maps are 0 maps. This implies that the resolution supported by the complex of critical cells is a minimal free resolution.

Alternatively, one may see that the resolution is minimal by checking that Morse numbers equal Betti numbers. Theorem 6.4 does this by constructing a bijection between the critical cells in our Morse function and the J'-non-stuttering, J'-commuting equivalence classes of words of [HRW]; these equivalence classes of words were shown in [HRW] to index a basis for $\mathrm{Tor}(\mathrm{k}, \mathrm{k})$, because those of fixed multidegree λ index a homology basis for $(0, \lambda)$.

□

Remark 6.2. *The fact that the discrete Morse function gives a minimal free cellular resolution implies Morse numbers equal Betti numbers.*

Thus, the upcoming generating function for Morse numbers also computes the Poincare'-Betti series.

The next theorem will construct a finite state automaton that generates exactly the label sequences for the surviving critical cells. The list of states in this finite state automaton is far from minimal in general among all finite state automata generating this language. Specifically, we keep track of more data in each state than is strictly necessary, in order to greatly simplify the description of our automaton.

Theorem 6.3. *The label sequences for saturated chains which contribute surviving critical cells are exactly the words of a regular language. Thus, the generating function for Morse numbers, which in this case equals the Poincare'-Betti series, is rational.*

PROOF. The alphabet for the language is the set of labels on covering relations, i.e. of generators for the monoid. For convenience, we view label sequences on saturated chains as words by reading them from top to bottom. Since all surviving critical cells are saturated chains, the dimension of each such critical cell is two less than the length of the word labelling it. Thus, the Morse number m_i counts words of length $i+2$ in the language of label sequences. We will describe a set of states and of legal transitions between states that comprise a finite state automaton that generates exactly the language of label sequences for surviving critical cells. The existence of such an automaton will imply that the language is regular, and hence the generating function for Morse numbers is rational (cf. Section 2.6). In fact, the rational generating function may be determined from the finite state automaton (see [BR]). The remainder of the proof describes how to construct such an automaton.

The automaton has a unique initial state, and each time a label is read, a transition is made from one state to another state if the label sequence read so far could be the initial segment for a label sequence of a surviving critical cell. To decide which labels give valid transitions, each state must keep track of enough data about previously read labels to decide whether concatenating a newly read label will

- give a label sequence for a surviving critical cell, in which case a transition is made to a final state, or
- give a label sequence for a critical cell which is cancelled, but one where reading additional labels could again yield a surviving critical cell, in which case a transition is made to a non-final state, or

- give a label sequence not meeting either of the above forms, in which case there is no valid transition, so the word is not generated by the automaton.

Specifically, for there to be a transition labeled λ out of a state S, λ must form either a descent or an ILT with the most recently read label, and there are further constraints related to non-essential sets. The requirement about descents and ILTs is necessary because every pair of consecutive labels for a surviving critical cell must take this form.

Each state will contain the following data: the list of previously encountered ILTs and individual labels, together with the order of the most recent occurrences of these ILTs and labels. Thus, each state has associated to it a subset of the finite set of monoid generators and leading terms in our Gröbner basis, together with a permutation on the elements of this subset. Earlier occurrences of the same ILTs or individual labels are unnecessary for deciding whether all non-essential sets are empty, or else would have already caused the word to be unproducable by the automaton at an earlier stage. Thus, we have a finite list of states.

If a partial label sequence w concludes with a label μ and leads to a final state S, then the next label λ to be read gives a legal transition from S to another final state if and only the following conditions are all met:

(1) λ, μ comprise a descent or ILT
(2) λ is not in the non-essential set of any earlier ILT. That is, every previously encountered ILT $I(a_1, a_2)$ with $a_1 < \lambda < a_2$ either has (a) $\lambda\nu \in \text{in}(I_\Lambda)$ for some label ν read more recently than $I(a_1, a_2)$, (b) $\lambda a_1 \in \text{in}(I_\Lambda)$, (c) $\lambda a_2 \in \text{in}(I_\Lambda)$, or (d) λ smaller than some label ν read after $I(a_1, a_2)$
(3) If λ, μ comprise an ILT, then there is no previously encountered label μ' in its non-essential set. That is, there is no previously encountered μ' with all the following properties: (a) μ' satisfies $\lambda < \mu' < \mu$, (b) μ' is smaller than all labels read after it and before λ, (c) μ' commutes with all labels read after it, and (d) deleting μ' would cause μ to be in the non-essential set of a previously encountered ILT.

When the first and third conditions hold but the second one fails, there is still a transition to a non-final state U. However, the only legal transitions from such a non-final state U are given by labels λ' such that λ', λ form an ILT which causes λ no longer to belong to a non-essential set, i.e. when we are in one of the following circumstances:

- $\lambda' < \mu$ and $\lambda'\mu \notin \mathrm{in}(I_\Lambda)$, because then shifting λ upward would yield a non-critical cell
- λ' would also belong to the non-essential set of some ILT once λ is shifted upward into an ILT (i.e. Theorem 5.17, case 1), implying the critical cell with ILT λ', λ is not matched by shifting λ upward.

The necessity of these constraints on allowable words is immediate from the description of critical cells and the matching to cancel them in earlier sections. These constraints on legal transitions are also sufficient to produce a surviving critical cell because any such label sequence will label a critical cell whose ILTs all have empty non-essential set, i.e. a critical cell that is not cancelled. □

Following [HRW], let $J = \mathrm{in}(I_\Lambda)$, and let J' be the complement of J. Labels a, b commute if and only if $ab \notin J$, in which case we say they are *J'-commuting*. Define a *J'-commuting equivalence class* of label sequences to be a set of label sequences which agree up to J'-commutation. A label sequence is *J'-stuttering* if it has consecutive labels a_1, a_1 where $a_1^2 \notin J$. A J'-commuting equivalence class C is *J'-non-stuttering* if none of the label sequences in C are J'-stuttering.

Theorem 6.4. *There is a bijection between the J'-non-stuttering, J'-commuting equivalence classes of a given content and the label sequences of the same content for critical cells that survive cancellation. Moreover, exactly one member of each J'-non-stuttering, J'-commuting equivalence class is a label sequence for a critical cell surviving cancellation.*

PROOF. We will show that each J'-non-stuttering J'-commuting equivalence class contains exactly one label sequence for a critical cell surviving cancellation. First we show the existence of such a label sequence within each such J'-non-stuttering J'-commuting equivalence class by providing an algorithm which applies a series of J'-commutation relations to transform any member of such a class into the label sequence for a surviving critical cell. Then we show that each such class has at most one label sequence from a critical cell surviving cancellation. Finally, we show that J'-stuttering anywhere within a J'-commuting equivalence class of a label sequence implies that the label sequence either does not come from a critical cell or is cancelled.

The algorithm sequentially processes the labels, proceeding from smallest to largest label value, and in the case of repetition, proceeds from highest to lowest initial location for each value. The algorithm terminates because it processes a finite number of labels and will use a finite number of steps to process each label. If the label μ immediately below a label ν to be processed is smaller than ν, then the algorithm would have processed μ before ν, and we will soon see that the pair must form an ILT. In this case, we say that μ is *attached* to ν at the time ν is processed.

A label ν is processed as follows. If ν is not attached to a label immediately below it, then ν is shifted upward until ν either encounters a label with which it does not commute, reaches the top of the label sequence, or encounters a label smaller than it such that all current ILTs $I(a_1, a_2)$ above ν with $a_1 < \nu < a_2$ and $a_1\nu, \nu a_2 \notin \mathrm{in}(I_\Lambda)$ are currently separated from ν by labels with which ν does not commute. Notice that ν will not encounter another copy of ν before reaching such a position, because the J'-commuting equivalence class is J'-non-stuttering. If ν is shifted to just below a label λ_2 with which ν does not commute, such that $\nu < \lambda_2$, then ν is now attached to λ_2. It in addition the label λ_1 previously below λ_2 had formed an ILT with λ_2, then $\nu < \lambda_1 < \lambda_2$, and we may detach λ_1 from λ_2 at the same time that we attach ν to λ_2 to form a new ILT $I(\nu, \lambda_2)$. If a label ν to be processed is attached to a label μ just below it, then the pair is shifted upward as a unit past labels larger than ν that commute with both μ and ν, with the following special rules:

- if μ, ν encounter a label $\lambda > \nu$ which commutes with ν but not with μ, then detach ν from μ, shift ν past λ, and attach λ to μ, and continue processing ν as an unattached label
- if the label immediately above ν forms either a descent or ILT with μ and ν can be shifted upward into the interior of and ILT $I(a_1, a_2)$ with $a_1 < \nu < a_2$, $a_1\nu, \nu a_2 \notin \mathrm{in}(I_\Lambda)$, and ν commuting with all labels between ν and $I(a_1, a_2)$, then ν is detached from μ and shifted upward to above $I(a_1, a_2)$ and continues its processing.

The fact that the only ascends in the output are between non-commuting pairs ensures it labels a critical cell. To see it also is one that survives cancellation, one may check that all non-essential sets are empty. At the time μ was processed, μ could not be further shifted upward into an ILT for which it would belong to the non-essential set, or else μ would have been shifted farther upward in its processing. The fact that all smaller values had already been processed by the time μ was processed ensures that this property is preserved throughout the algorithm. We may also eliminate the possibility of non-essential set members that shift downward into ILTs, because these only arise when the top of some ILT is capable of shifting upward without its partner, but our algorithm would have actually performed this shifting, and again the fact that we process smaller labels before larger ones means this property is preserved throughout the rest of the algorithm. Thus, all non-essential sets are empty, so the algorithm indeed outputs the label sequence of a surviving critical cell.

To show that there is at most one label sequence surviving critical cell cancellation in each J'-commuting equivalence class, first note that pairs of consecutive labels that commute must appear in descending order to avoid either having an ascend not appearing within an ILT (implying the saturated chain does not contribute a critical cell) or having an ILT with non-empty interior (implying the critical cell is cancelled). Now suppose there are two label sequences in the same J'-commuting equivalence class, both from critical cells surviving cancellation. Then there must be some pair of J'-commuting labels μ_1, μ_2 with $\mu_1 < \mu_2$, such that the pair are inverted in one label sequence and not the other. One may use the intermediate value theorem to show that the non-inverted pair μ_1, μ_2 must be separated by at least one ILT $I(a_1, a_2)$ with $a_1 < \mu_1 < a_2$ and by at least one ILT $I(b_1, b_2)$ with $b_1 < \mu_2 < b_2$, such that $I(a_1, a_2)$ either equals $I(b_1, b_2)$ or occurs before $I(b_1, b_2)$. This implies either that $\mu_1 \in S(a_1, a_2)$, ensuring cancellation, or that some label not commuting with μ_1 separates μ_1 from $I(a_1, a_2)$, implying μ_2 must commute with all labels between it and a_1, in order for it to be possible to swap μ_1, μ_2. But then the critical cell with μ_2, μ_1 forming an inversion must have $\mu_2 \in S(b_1, b_2)$, and so cannot also survive cancellation.

Now we turn to the issue of non-stuttering. Any two consecutive identical labels must appear within a syzygy interval to avoid comprising an ascend which would make the cell non-critical. But we showed that any critical cell with a syzygy interval with non-empty interior is cancelled. If a label sequence ω is in the J'-commuting equivalence class of a label sequence which has consecutive identical labels, then some label λ appears more than once in ω, separated by one or more syzygy intervals. But then one of these syzygy intervals will have the lower copy of λ in its non-essential set, implying the critical cell is cancelled, unless λ does not commute with some separating label. But this label also does not commute with the other copy of λ, making it impossible for the two labels to be shifted to consecutive positions, as needed for stuttering, a contradiction. Thus, if a label sequence is J'-commuting equivalent to one with stuttering, then the label sequence does not survive cancellation. $\square$

7. Gröbner bases of higher degree

In this section, we extend results of Section 5 from the quadratic Gröbner basis case to the degree d to prove:

Theorem 7.1. *If I_Λ has a Gröbner basis of degree d, then $\tilde{H}_i(\Delta(\hat{0},\lambda)) = 0$ for $i < -1 + \frac{\deg(\lambda)-1}{d-1}$, with* $\deg(\lambda)$ *defined as below. Hence, we have $Tor_i^{k[\Lambda]}(k,k)_\lambda = 0$ for $i < 1 + \frac{\deg(\lambda)-1}{d-1}$. Moreover, this vanishing is achieved by a free cellular resolution resulting from a discrete Morse function on Λ.*

In the standard-graded case, $\deg(\lambda)$ is given by the grading. In general, let $\deg(\lambda)$ be one more than the length of the shortest saturated chain on the poset interval $(\hat{0},\lambda)$, i.e. the degree of the image of $\underline{x}^\lambda$ in the associated graded ring.

Proposition 7.2. *Ordering saturated chains by using any monomial term order to order fibres then lexicographically ordering saturated chains within each fibre yields a content-lex facet order.*

PROOF. Syzygy intervals now must be defined to have weakly increasing labels $a_1,\ldots,a_r$ such that there is a Gröbner basis leading term which divides $a_1\cdots a_r$ and has smallest divisor a_1 and largest divisor a_r. To be a minimal skipped interval, we must also have that neither $a_2\cdots a_r$ nor $a_1\cdots a_{r-1}$ is divisible by a Gröbner basis leading term. Then the proof of Theorem 5.2 applies. □

The remainder of this section is concerned with cancelling pairs of critical cells to obtain a Morse function with no critical cells below dimension $-1+\frac{\deg(\lambda)-1}{d-1}$.

Definition 7.3. An *increasing leading term*, or *ILT*, is a Gröbner basis leading term, with labels arranged in weakly increasing order. We will often use the term ILT to refer to an ILT that labels a specific syzygy interval.

We will use the variables d', d'' to represent the degree of an arbitrary Gröbner basis leading term, so we always will have $d', d'' \leq d$. Denote a syzygy interval with ILT $a_1\cdots a_{d'}$ by $I(a_1,\ldots,a_{d'})$. In contrast to the quadratic Gröbner basis case, now there may be several Gröbner basis leading terms specifying the same syzygy interval. This fact, that a single syzygy interval may have several ILTs beginning and ending with the same pair of labels, causes one substantial new issue to arise: the critical cells resulting from one syzygy interval may comprise several overlapping Boolean algebras, since different ILTs will give rise to different non-essential sets, as in Example 7.4.

Example 7.4. *Consider the syzygy interval labeled* $a_1, \dots, a_4$ *that goes with the Gröbner basis leading terms* $a_1a_2a_4$ *and* $a_1a_3a_4$*. One Boolean algebra of critical cells, based on ILT* $I(a_1, a_2, a_4)$ *consists of labels sequences* $a_1a_2a_3a_4$ *and* $a_3a_1a_2a_4$*, while the Boolean algebra that is associated with* $I(a_1, a_3, a_4)$ *consists of label sequences* $a_1a_2a_3a_4$ *and* $a_2a_1a_3a_4$*. The critical cell labeled* $a_1a_2a_3a_4$ *is shared by the two Boolean algebra.*

Lemma 7.7 deals with such overlap by providing an acyclic matching for any such collection of overlapping Boolean algebras.

Define the non-essential set for $I(a_1, \dots, a_{d'})$, denoted $S(a_1, \dots, a_{d'})$, similarly to the $d = 2$ case, but now it may have two types of members:

(1) individual labels, in exact analogy to the non-essential set members for the $d = 2$ case, i.e. labels which either (a) appear in topologically decreasing positions below $I(a_1, \dots, a_{d'})$ (or above $I(a_1, \dots, a_{d'})$ in the exceptional circumstances discussed in Theorem 5.17) from which they may shift into $I(a_1, \dots, a_{d'})$ via a gradient path without causing $I(a_1, \dots, a_{d'})$ to cease to be a minimal skipped interval, or (b) labels that have thus shifted into the interior of $I(a_1, \dots, a_{d'})$.
(2) collections $\{b_2, \dots, b_{d'}\}$ of labels that appear either immediately above a label b_1 with which they form an ILT, or which collectively appear in the interior of one or more ILTs strictly above such a label b_1, with $I(a_1, \dots, a_{d'})$ serving as the highest of these ILTs. Furthermore, we require there to be a gradient path from the former to the latter which shifts the collection of labels upward into the interior of the various ILTs in order for the collection of labels to belong to $S(a_1, \dots, a_{d'})$.

Follow the conventions of Theorem 5.17 to decide which such individual labels and collections of labels should belong to $S(a_1, \dots, a_{d'})$, not allowing collections of labels that shift downward into $I(a_1, \dots, a_{d'})$ from above. If there is a need for "blocking" as in Theorem 5.17, an individual label will always serve this function rather than a collection of labels. If a label $\lambda \in I(a_1, \dots, a_{d'})$ meets the above requirements to be included individually in $S(a_1, \dots, a_{d'})$, then it is not also included as part of a collection of labels. With these conventions, the proof that this gives a well-defined matching is identical to the proof of Theorem 5.17.

Proof of Theorem 7.1. Proposition 7.2 provides the Morse function that serves as our starting point. We will cancel all critical cells of dimension less than $-1 + \frac{\deg(\lambda)-1}{d-1}$, using a fairly similar, but somewhat more subtle, approach to the $d = 2$ argument. Notice that each critical cell of dimension less than $-1 + \frac{\deg(\lambda)-1}{d-1}$ is contributed by a saturated chain with average minimal skipped interval height greater than $d-1$. But any minimal skipped interval of height greater than $d-1$ is a syzygy interval with more than d labels, so it is consists of an ILT with at least one additional label interspersed. Such extra labels either allow the critical cell to be cancelled similarly to the $d = 2$ case, or in the case of a syzygy interval with multiple ILTs, Lemma 7.7 gives a matching in which all unmatched cells have average interval height at most $d-1$ for the I-intervals related to the syzygy interval. Thus, cells left unmatched must then have another syzygy interval of height greater than $d-1$ at lower ranks. This allows us to repeat the argument until eventually reaching a syzygy interval which causes the cell to be cancelled.

The fact that there is indeed a unique gradient path from a critical cell τ to a critical cell σ for each pair τ, σ to be cancelled follows from Theorem 4.2. When a single label is shifted into a syzygy interval, the gradient path is identical to the one given in the $d = 2$ case. When a collection of labels is shifted upward into a syzygy interval, the gradient path is the one described in the proof of Theorem 4.2. Theorem 4.2 also proves the uniqueness of these gradient paths, whether shifting a single label or a collection of labels. As before, we give a complete acyclic matching on each Boolean algebra of critical cells, as long as it is not part of a collection of overlapping Boolean algebras. Similarly to the $d = 2$ case, we match all cells in this Boolean algebra by including/excluding a single non-essential set member ν from the interior of the syzygy interval. It is convenient to choose ν to be the individual label with highest topologically decreasing position outside the syzygy interval, if there is such a label, and otherwise to choose the collection of labels with highest topologically decreasing position outside the syzygy interval. Lemma 7.7 provides the matching for collections of overlapping Boolean algebras.

With these choices, acyclicity is similar to the $d = 2$ case, since Lemma 7.7 will verify acyclicity of the matching on a single collection of overlapping Boolean algebras resulting from several ILTs on a single syzygy interval. Applying results of [BW], the desired resolution is immediate from this acyclic matching. □

Next we prove Lemma 7.7 in the special case of degree $d = 3$. In this case we deduce a stronger result than for general d, but the proof is also much simpler than in general, but gives the flavor of the upcoming proof for degree d.

Definition 7.5. A pair of ILTs $a_1 \dots a_r$ and $b_1 \dots b_s$ with $a_1 \le b_1, a_r \le b_r$ are *concatenating* if either (1) $a_1 < b_1$ and $a_r = b_i$ for some $1 \le i < s$, or (2) $a_r < b_s$ and $b_1 = a_i$ for some $1 < i \le r$.

Lemma 7.6. *Let I_Λ be a toric ideal with Gröbner basis with leading terms all of degree at most 3. Let $I = I(a_1, \dots, a_{d'})$ be a syzygy interval, given by one or more ILTs, each of which gives rise to a Boolean algebra of critical cells. Then this collection of overlapping Boolean algebras has an acyclic matching which matches all critical cells with average interval height at most 2 for I together with any descents coming from labels shifted out of I.*

PROOF. Order the ILTs $M_1, \dots, M_k$. If some M_i has degree 2, then the Boolean algebra for each M_j is contained in the Boolean algebra for M_i, so we use the complete matching on a single Boolean algebra. Otherwise, we have labels $\lambda_1, \dots, \lambda_k$ such that for $1 \le i \le k$, $M_i = a_1 \lambda_i a_{d'}$, for fixed initial and final labels $a_1, a_{d'}$. Let Δ_i be the set of critical cells in the Boolean algebra $B^{(i)}$ for M_i which are not shared with any earlier Boolean algebra $B^{(i')}$ for $i' < i$. Notice that Δ_i consists of exactly those critical cells in $B^{(i)}$ which have $\lambda_1, \dots, \lambda_{i-1}$ all shifted to topologically decreasing positions outside I. Thus, Δ_i has the structure of a Boolean algebra, resulting from all other labels in the non-essential set for $I(a_1, \lambda_i, a_{d'})$, so this has a complete acyclic matching unless this set is empty. But when the set is empty, then I consists of only the three labels $a_1, \lambda, a_{d'}$, as well as labels essential to concatenating ILTs, so matching is not necessary. In the case of concatenating ILTs, the average interval height is still at most 2. $\square$

The situation gets much more complex when labels other than a_1 and $a_{d'}$ may divide more than one of the Gröbner basis leading terms specifying ILTs on the syzygy interval.

Lemma 7.7. *Suppose that a single expanding interval I has multiple ILTs. Then the resulting collection of overlapping Boolean algebras has an acyclic matching such that all unmatched cells have minimal skipped interval average height at most $d - 1$.*

PROOF. Choose a total order $M_1, \dots, M_k$ on the ILTs for $I(a_1, \dots, a_{d'})$. Thus, each M_i is a Gröbner basis leading term with smallest divisor a_1, largest divisor $a_{d'}$ and with divisors of intermediate value, all of which appear as labels that can shift in/out of the syzygy interval. Thus, any shifting of labels in/out of the syzygy interval still gives a syzygy interval as long as at least one of these ILTs appears entirely within the syzygy interval. Each ILT $I(M_i)$ has its own non-essential set, denoted

$S(M_i)$, giving rise to its own Boolean algebra of critical cells. Denote by Δ_r the collection of critical cells in the Boolean algebra given by M_r which are not shared with any of the earlier Boolean algebras given by $M_1, \dots, M_{r-1}$.

We will provide an acyclic matching on each such Δ_r. Note that Δ_r consists of those subsets of $S(M_r)$ which shift enough labels to outside the syzygy interval $I(M_r)$ so that the label sequence on $I(M_r)$ is not divisible by any of the monomials $M_1, \dots, M_{r-1}$. If there is any label in $S(M_r)$ that does not divide any of the monomials $M_1, \dots, M_{r-1}$, then we obtain a complete acyclic matching on Δ_r by including/excluding one such label in $I(M_r)$. Next we consider the case where each member of $S(M_r)$ does divide some earlier M_i.

Fix an ordering $\lambda_1, \dots \lambda_l$ on the elements of $S(M_r)$. It will be convenient in the next section if we order them from highest to lowest topologically decreasing position outside $I(M_r)$. Now apply the following matching procedure to each critical cell in Δ_r:

(1) match the cell by including/excluding λ_1 from $I(M_r)$ unless shifting λ_1 to inside $I(M_r)$ yields a cell in an earlier Δ_i,
(2) if the cell is not yet matched, then match by including/excluding λ_2 from $I(M_r)$, unless this yields a matching partner which was already matched at the first step or which belongs to an earlier Δ_i
(3) continue inductively, matching the cell by including/excluding λ_i from $I(M_r)$ if the cell was not already matched based on any of the labels $\lambda_1, \dots, \lambda_{i-1}$ and the partner cell based on shifting λ_i also does not belong to an earlier Δ_i and is not already matched based on any earlier label $\lambda_{i'}$ with $i' < i$.

Notice that a cell cannot be matched based on the label λ_i if either (a) $\lambda_i \notin I(M_r)$ and shifting λ_i into $I(M_r)$ gives a cell in an earlier Boolean algebra, or (b) shifting λ_i in or out of $I(M_r)$ gives a cell previously matched. Thus, any unmatched critical cell that has exactly the labels $M = \{\mu_1, \dots, \mu_k\}$ shifted to outside $I(M_r)$ will have the property that each such μ_i is necessary outside $I(M_r)$ either to avoid overlap with an earlier Boolean algebra or in order for some $\nu \in I(M_r)$ coming earlier than μ_i also not to allow matching. That is, in the latter case there must be some $\mu_{i'} < \nu < \mu_i$ with $\mu_{i'} \notin I(M_r)$, such that $\mu_{i'}$ "covers" multiple ILTs (see Definition 7.8), some of which could also be covered by ν, and the rest of which are also covered by μ_i.

Definition 7.8. A label μ *covers* an earlier ILT M_j if μ divides the Gröbner basis leading term specifying M_j.

Assign to each $\mu_i \in M$ either an ILT $M_{i'}$ which it exclusively covers, or an ILT $M_{i''}$ that it would exclusively cover if the earliest forbidden $\nu \in I(M_r)$ were shifted to outside $I(M_r)$, or which it would exclusively cover after some number of iterations of this reasoning, i.e. an ILT which makes it impossible to shift μ_i from outside $I(M_r)$ to inside $I(M_r)$ as a matching step. Call this ILT which is assigned to μ_i the *indexing ILT* of μ_i.

If we can show that every label in $S(M_r)$ belongs to one of the k indexing ILTs, this will imply $|T| \leq k \cdot (d-2)$, as desired. Suppose some $\lambda \in S(M_r)$ is not in any of the indexing ILTs, and choose the label λ of this form which comes earliest in our ordering on labels in $S(M_r)$. First note that $\lambda \in I(M_r)$, since otherwise λ would belong to its own indexing ILT. We will show next that the cell with λ shifted to outside $I(M_r)$ is not matched based on a label of higher precedence than λ. Since shifting λ to outside $I(M_r)$ also cannot give a cell belonging to an earlier Δ_i, we will be able to conclude that the critical cell will be matched based on λ. Thus, any unmatched cell will satisfy $|T| \leq k \cdot (d-2)$.

Now we prove the claim that the cell may be matched by shifting λ to outside $I(M_r)$. When λ is shifted to outside $I(M_r)$, each μ_i of higher precedence which appears outside $I(M_r)$ cannot be shifted to inside $I(M_r)$ as a matching step, by virtue of its indexing ILT, since λ cannot cover this indexing ILT. Likewise any λ_i of higher precedence which appears within $I(M_r)$ in the critical cell cannot be shifted to outside $I(M_r)$ without rendering some $\mu_{i'}$ unnecessary for covering its indexing ILT, since otherwise we would have matched based on the smallest λ_i which did not have this property; in particular, this means that λ_i must belong to the indexing ILT for $\mu_{i'}$ in the critical cell. Shifting λ to outside $I(M_r)$ does not change this relationship, so the cell with λ shifted to outside $I(M_r)$ also cannot match by shifting λ_i. Thus, λ is the first label allowing matching for both cells, so both are indeed matched by shifting λ.

In the case where all elements of $S(M_r)$ are individual labels that shift to topologically decreasing positions outside $I(M_r)$, this yields the following upper bound on average interval height for this portion of the interval system, using the fact that total height is one less than the total number of labels involved, and that k is non-negative:

$$\frac{\text{total height}}{\text{no. of intervals}} = \frac{d-1+|T|}{k+1} \leq$$

$$\frac{d-1+k\cdot(d-2)}{k+1} = (d-2) + \frac{1}{k+1} \leq d-1.$$

Let us now handle the more general case, where some non-essential set members are collections of labels. All labels belonging to such collections will contribute individually to the bound $|T| \leq k\cdot(d-2)$ when the labels appear within $I(M_r)$, because each label contributes individually to monomial degree. When such a collection of labels appears outside the ILT, it would increase the number of intervals k by one, but would increase the total height by as much as $d-1$, seemingly invalidating the above computation of average interval height. However, the highest label in the newly created ILT must also form a descent with the label immediately above it, and we may use this descent rather than the new ILT in order to compute the above bound, since the descent will not also be counted in a similar computation for any other syzygy interval. We may safely ignore the newly created ILT in the bound computation, since it also has height at most $d-1$.

Acyclicity will follow from the Cluster Lemma of [Jo] (see Lemma 2.6), using the filtration of subcomplexes $B_1 \subseteq B_1 \cup B_2 \subseteq \cdots \subseteq B_1 \cup \cdots \cup B_k$ where $B_1 \cup \cdots \cup B_i$ is the union of Boolean algebras given by $M_1, \dots, M_i$. All we need to do is show that the matching on each $B_i \setminus (B_1 \cup \cdots \cup B_{i-1})$ is acyclic. But if there were a cycle, let μ_i be the highest precedence label to be inserted as a matching step in the cycle. This would necessitate a downward step in the cycle shifting μ_i back into the interior of $I(M_r)$, but this would be preceded and followed by matching steps inserting labels of lower precedence than μ_i. This contradicts our greedy matching procedure, because it would instead make the downward edge a matching edge inserting μ_i, since this has higher precedence than the matching step of either endpoint. Thus, there are no cycles. □

8. Rationality of Morse number generating function

In this section we describe a finite state automaton that generates exactly the language of label sequences for surviving critical cells, in the case of a Gröbner basis of degree d. The existence of such a generating function again implies the language is regular, and hence that the generating function for Morse numbers is a rational function which gives upper bounds on the terms in the Poincare'-Betti series. In contrast, for $d \geq 3$ the Poincare'-Betti series is not always rational. The generating function for Morse numbers does come close enough to the Poincare'-Betti series to achieve the vanishing of Betti numbers described by Theorem 7.1. Due to the similarity of the finite state automaton to the one given in the quadratic Gröbner basis case, less detail is provided here than in Section 6.

The states in the automaton keep track of the set of previously encountered ILTs and individual labels, in their order of most recent appearance. Reading label sequences from top to bottom, the following are the legal transitions from one state to another.

(1) a single label λ that is larger than its predecessor, i.e. which forms a descent with the label above it. For the transition to be to a final state, we require the further property that λ is separated from each previously encountered ILT $I(a_1,\ldots,a_{d'})$ which satisfies $a_1 < \lambda < a_{d'}$ and $\lambda a_2\cdots a_{d'}, \lambda a_1\cdots a_{d'-1} \notin \mathrm{in}(I_\Lambda)$. by a label with which λ does not commute.
(2) a single label which forms an ILT together with its predecessor, exactly as in the $d = 2$ case
(3) a collection $\{a_1,\ldots,a_{d'-1}\}$ of labels with $d' > 2$, which together with the most recently encountered label $a_{d'}$ form an ILT $I(a_1,\ldots,a_{d'})$ such that (a) the labels $\{a_2,\ldots,a_{d'}\}$ cannot all simultaneously shift upward into the interior of higher ILTs to yield a critical cell which does not have any of the labels $\{a_2,\ldots,a_{d'}\}$ individually as members of any non-essential set, and (b) no label above $I(a_1,\ldots,a_{d'})$ may shift downward into $I(a_1,\ldots,a_{d'})$ by a gradient path to yield a critical cell. Such a transition leads to a final state.
(4) a collection of labels that collectively complete an ILT, with an allowable collection of interspersed labels. Allowable collections are those that arise as a result of concatenating ILTs, as described below, and those which may be within the ILT in a surviving critical cell when there are multiple ILTs on the same syzygy interval. In this case the transition is to a non-final state, and we will justify below that there are only a finite number of these transitions.

The point is to use non-final states for label sequences for critical cells that are cancelled, if the concatenation of additional labels may yield a critical cell that is not cancelled. to be cancelled. The "concatenating" ILTs mentioned in the fourth type of transition come from situations such as the following example.

Example 8.1. *Consider a label sequence abcde where abd and cde are each Gröbner basis leading terms. The label c cannot be shifted out of the ILT $I(a,b,d)$ to yield a critical cell, because the ascend (d,e) would no longer be part of a minimal skipped interval, so $c \notin S(a,b,d)$ for the label sequence abcde, though it would belong to $S(a,b,d)$ in the label sequence abcd.*

Specifically, a pair of ILTs $M_1 = a_1 \cdots a_r$ and $M_2 = b_1 \cdots b_s$ are *concatenating* if either (a) a_r divides M_2 with $a_r \neq b_s$ and $a_1 < b_1$, or (b) b_1 divides M_1 with $b_1 \neq a_1$ and $a_r < b_s$.

The fourth type of transition also accommodates the matching procedure of Lemma 7.7.

Proposition 8.2. *The automaton has finitely many states and transitions.*

PROOF. There is a finite list of possible ILTs, even when we consider all possible label interspersions that could still allow the cell not to be cancelled, i.e. from concatenating ILTs and from multiple ILTs on a single syzygy interval. This follows from the fact that the semi-group ring is finitely generated, and that each Gröbner basis leading term has finite degree, so labels occurring in the interior of an ILT with multiplicity greater than the Gröbner basis degree will always allow critical cell cancellation. The transitions out of a state are limited by the finite list of labels. □

Proposition 8.3. *Word length equals critical cell dimension shifted by two.*

PROOF. Any label sequence which has more I-intervals than J-intervals will be cancelled, unless there are two concatenating ILTs such that their concatenation contains another ILT, causing three or more overlapping I-intervals in which one is discarded, in such a way that no labels may be shifted from the interior of any of these ILTs without making the cell non-critical. But in the case of this type of concatenation, where two ILTs share labels and cover a third ILT, this means we can use just the labels in these two ILTs for labelling the transitions in the finite state automaton, so we get the correct word length. □

Using the observations and propositions above, it is not hard to generalize the automaton from the $d = 2$ case to obtain:

Theorem 8.4. *The surviving critical cells are labeled by the words of a regular language, with word length measuring cell dimension, shifted by two. Thus, the generating function for Morse numbers is a rational generating function which is determined by the given finite state automaton.*

9. Some remarks and open questions

Remark 9.1. When a variable does not appear in any syzygies, then it may be "factored out" before starting our analysis, similar to the situation with computing Tor groups directly. Specifically, if some z_i does not appear in any generators of the toric ideal I_Λ for $k[z_1, \dots, z_n]/I_\Lambda$, then the partial order Λ is the product of an infinite chain together with the poset of monomials ordered by divisibility in $k[z_1, \dots, \hat{z}_i, \dots, z_n]/I_\Lambda$. Thus, any finite interval is the product of a finite chain together with a monoid poset interval $(0, \lambda)$ for the ring $k[z_1, \dots, \hat{z}_i, \dots, z_n]/I_\Lambda$; the order complex of such an interval is the suspension of the join of the order complexes for the two terms in the product, so the suspension of the join of a simplex (i.e. the order complex of a chain) with the order complex $\Delta(0, \lambda)$.

Remark 9.2. We sometimes have gradient paths which reverse a decreasing sequence of labels of length $d > 2$ to produce an ILT, in which case the permutation on labels is not 321-avoiding. We have not matched and cancelled any such pairs of critical cells. Theorem 2.14 shows there are at most two gradient paths between a pair of critical cells related by such a reversal for lexicographic discrete Morse functions; the proof of Theorem 2.14 generalizes to those facet orders which satisfy the crossing condition, so in particular to content-lex facet orders.

Theorem 9.3. *Suppose I_Λ has degree at most three. Then for each critical cell τ in our complex Δ^M of critical cells after cancellation, $\partial(\tau)$ is a linear combination of critical cells of content strictly earlier than τ.*

PROOF. Suppose τ, σ, $\dim(\tau) = \dim(\sigma)+$, are surviving critical cells with equal content and there is a gradient path from τ to σ. Then τ, σ each have no syzygy intervals with non-empty non-essential set. Any gradient path from τ to σ must sort labels, but in such a way that σ still has no syzygy intervals with non-empty non-essential set. This can only be accomplished by reversing three or more descending labels to form a new ILT. This ILT must come from a Gröbner basis leading term of degree exactly three, since pairs of labels comprising degree 2 leading terms cannot be swapped without passing to an earlier content class. Lemma 7.6 ensures that the three or more labels must occur in a single string of descending labels within τ, to avoid τ being cancelled by virtue of a syzygy interval with non-empty non-essential set.

The ILT to be created cannot come from a Gröbner basis leading term of degree greater than three, both because of the assumptions of our theorem, and also because this would decrease critical cell dimension by more than one, implying σ could not be in the image of the boundary map applied to τ. Theorem 2.14 shows there are at most two gradient paths reversing three labels, resulting from the Coxeter relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ being applied at the conclusion of a reduced expression. But one may easily check that one will indeed get two gradient paths when we reverse three labels as required for σ in the boundary of τ, and furthermore, that these will be oriented so that the two ways in which σ is incident to τ will cancel. Thus, σ will appear with coefficient 0 in the boundary of τ. $\square$

Remark 9.4. *The following example shows that the Morse function bound on which Tor groups vanish is sharp. Consider*

$$k[\Lambda] = k[z_1, \dots, z_{2d}]/(z_1 \cdots z_d - z_{d+1} \cdots z_{2d}),$$

or equivalently,

$$k[z_1 z_2, z_3 z_4, \dots, z_{2d-1} z_{2d}, z_1 z_{d+1}, z_2 z_{d+2}, \dots, z_d z_{2d}].$$

This clearly has a Gröbner basis of degree d and none of lower degree. The interval $(1, z_1 \cdots z_d)$ in Λ is disconnected.

Question 9.5. *Is there a nice description of the gradient paths between surviving critical cells? This would be needed for a completely explicit description of the boundary maps in our resolution, since these are sums over such gradient paths.*

Question 9.6. *Is it possible to improve our discrete Morse function into one that would provide a combinatorial proof of the following theorem? If an affine semi-group ring is standard graded, and its toric ideal of syzygies has a Gröbner basis of degree d, then its $(d-1)$-st Veronese is Koszul.*

Example 9.7. There is only one circumstance in which critical cells could skip more than $2d-3$ consecutive elements of a saturated chain, and this only may happen in the $d > 3$ case. Namely, if there are distinct Gröbner basis leading terms with the same initial and final labels, this may result in overlapping Boolean algebras of critical cells, with cells with large syzygy intervals not necessarily cancelled.

Question 9.8. *In [HRW],* Tor *groups related to quotients of affine semi-group rings by monomial ideals are translated to homology of certain relative complexes $\Delta(\lambda, A)$, where λ specifies a monoid poset interval and A is a graphic subspace arrangement. Does our Morse function translate to this setting to provide useful new information?*

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