CHAINS OF MODULAR ELEMENTS AND LATTICE CONNECTIVITY

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ABSTRACT. We show that the order complex of any finite lattice with a chain $\hat{0} < m_1 < \cdots < m_r < \hat{1}$ of modular elements is at least $(r-2)$-connected.

In [St], Stanley shows that a finite lattice L with a maximal chain consisting entirely of modular elements is supersolvable and therefore graded and EL-shellable in the sense of [Bj1]. Hence, the order complex $\Delta(L \setminus \{\hat{0}, \hat{1}\})$ has the homotopy type of a wedge of top dimensional spheres. In particular, if L has a maximal chain $\hat{0} < m_1 < \cdots < m_r < \hat{1}$ with each m_i modular then its order complex $\Delta(L \setminus \{\hat{0}, \hat{1}\})$ is at least $(r-2)$-connected. We generalize this connectivity lower bound to chains of modular elements that are not necessarily maximal.

Knowledge about the topological structure of the order complex of a given lattice can be of use, for example, in the study of subspace arrangements and in the study of free resolutions. See for instance [GM], [ZZ], [Bj2], and [GPW]. If instead of determining the homology entirely, one merely is able to prove a connectivity lower bound, this already may provide useful information. For example, connectivity lower bounds for LCM lattices directly translate to upper bounds on the regularity of a monomial ideal; a connectivity lower bound for monoid posets gives a bound on the rate for resolving the residue field over an associated toric ring, i.e. the coordinate ring of an associated toric variety. Terminology as well as specific results in these directions may be found, for instance, in [MS], [HRW], [HW], [Pe] and [PRS].

We assume that the reader is familiar with the basic notions from topological combinatorics. All relevant definitions can be found in [Bj3]. All lattices we consider here will be finite with minimum element $\hat{0}$ and maximum element $\hat{1}$. One of many equivalent definitions of modularity in a lattice L is that $m \in L$ is modular if for each $x \in L$

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there is an isomorphism

$$[m, x \vee m] \cong [x \wedge m, x]$$

given by sending each $z \in [m, x \vee m]$ to $z \wedge x$ (cf. [Sc]). Note that this definition is dual to that given by Stanley in [St], that is, $m \in L$ is modular according to our definition if and only if m is modular, according to Stanley's definition, in the lattice L^* defined by $x \leq_{L^*} y$ if $y \leq_L x$. This discrepancy in definitions is harmless, because the order complex of any poset P is the same as that of P^*.

There are also instances in the literature in which a weaker notion of modularity is used, where x is said to be modular if $(x \vee y) \wedge z = (x \wedge z) \vee y$ for all $y \leq z$; usually, but not always, this is called left-modularity. This is not the same as the notion of modularity which we use, and which is used in [Sc] and [St].

Here is our main result, as mentioned above.

Theorem 1. *Let L be a finite lattice with a chain $\hat{0} < m_1 < \cdots < m_r < \hat{1}$ of modular elements. Then $\Delta(L \setminus \{\hat{0}, \hat{1}\})$ is at least $(r-2)$-connected.*

Proof. We proceed by induction on r, the base case $r = 0$ being trivial. Now suppose L has a chain $\hat{0} < \cdots < m_r < \hat{1}$ of modular elements with $r > 0$. Let $C(m_r)$ be the set of complements to m_r in L.

The elements of $C(m_r)$ form an antichain in L. Indeed, assume that $x \leq y$ are both complements to any modular element $m \in L$. Then $m \vee x = m \vee y = \hat{1}$ and $m \wedge x = m \wedge y = \hat{0}$. Since m is modular, $[x \wedge m, x] \cong [m, x \vee m]$ and $[y \wedge m, y] \cong [m, y \vee m]$. Thus

$$[\hat{0}, x] \cong [m, \hat{1}] \cong [\hat{0}, y],$$

which is impossible unless $x = y$, since L is finite.

The Homotopy Complementation Formula of Björner and Walker (cf. [BW]) now yields:

$$\Delta(L \setminus \{\hat{0}, \hat{1}\}) \simeq \bigvee_{a \in C(m_r)} \Sigma\left(\Delta(\hat{0}, a) * \Delta(a, \hat{1})\right),$$

where $\bigvee$ denotes wedge, Σ denotes suspension and $*$ denotes join. We will show below that for each $a \in C(m_r)$, the lattice $(a, \hat{1})$ has a chain of $r - 1$ distinct modular elements, namely $a \vee m_1, a \vee m_2, \ldots, a \vee m_{r-1}$. It follows from our inductive hypothesis that $\Delta(a, \hat{1})$ is $(r-3)$-connected. Since $\Delta(\hat{0}, a)$ is (-2)-connected, the join $\Delta(\hat{0}, a) * \Delta(a, \hat{1})$ is $(r-3)$-connected and its suspension is $(r-2)$-connected. A wedge of complexes, each of which is $(r-2)$-connected, is itself $(r-2)$-connected.

It remains to verify the claim that $a \vee m_1, a \vee m_2, \ldots, a \vee m_{r-1}$ is a chain of $r - 1$ distinct modular elements in $(a, \hat{1})$. It is well known (see for example [Sc, Theorem 2.1.6]) that if m is modular in L and $x \in L$, then $m \vee x$ is modular in the interval $[x, \hat{1}]$. Set $m_0 = \hat{0}$. For $0 \leq i < j \leq r$ we have

$$\hat{0} \leq a \wedge m_i \leq a \wedge m_j \leq a \wedge m_r = \hat{0},$$

so $a \wedge m_i = a \wedge m_j = \hat{0}$. Thus

$$[m_i, a \vee m_i] \cong [0, a] \cong [m_j, a \vee m_j].$$

Since L is finite and $m_i < m_j$, we cannot have $a \vee m_i = a \vee m_j$.

□

Next is an example of a subspace arrangement with nonshellable intersection lattice for which our connectivity bound is sharp. There are many similar examples, as well as numerous examples from group theory, in which context normal subgroups are modular in the lattice of subgroups of a finite group.

Example 2. Begin with the braid arrangement in $\mathbb{R}^n$ (for $n > 2$) generated by hyperplanes $x_i = x_j$ for $1 \leq i < j \leq n$. Now add an additional variable x_{n+1} and replace the hyperplane $x_{n-1} = x_n$ by the codimension two subspace $x_{n-1} = x_n = x_{n+1}$. The dual lattice L^* of the intersection lattice L for the resulting subspace arrangement has a chain

$$\hat{0} < (x_1 = \cdots = x_{n-1}) < \cdots < (x_1 = x_2 = x_3) < (x_1 = x_2) < \hat{1}$$

of modular elements which is not contained in any larger chain of modular elements. Indeed, the only way to enlarge the given chain is to add the subspace $S : x_1 = \cdots = x_n$. Let T be the subspace $x_{n-1} = x_n = x_{n+1}$. One may check that $S \wedge T$ is the subspace $x_1 = \cdots = x_{n+1}$ while $S \vee T = \hat{1}$, implying $[S, S \vee T]$ is the intersection lattice for the graphic arrangement given by $K_n \setminus \{e_{n-1,n}\}$ while $[S \wedge T, T]$ is the intersection lattice for the graphic arrangement given by K_{n-1}. These last two intersection lattices are not isomorphic, so S is not modular. One can show that $\Delta(L \setminus \{\hat{0}, \hat{1}\}) \simeq \Sigma\Delta(\Pi_{n-1} \setminus \{\hat{0}, \hat{1}\})$, and thus $\Delta(L \setminus \{\hat{0}, \hat{1}\})$ is $(n-4)$-connected but not $(n-3)$-connected.

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