

# A multiplicative deformation of the Möbius function for the poset of partitions of a multiset

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ABSTRACT. The Möbius function of a partially ordered set is a very convenient formalism for counting by inclusion-exclusion. An example of central importance is the partition lattice, namely the partial order by refinement on partitions of a set  $\{1, \dots, n\}$ . It seems quite natural to generalize this to partitions of a multiset, i.e. to allow repetition of the letters. However, the Möbius function is not nearly so well-behaved. We introduce a multiplicative deformation, denoted  $\mu'$ , for the Möbius function of the poset of partitions of a multiset and show that it possesses much more elegant formulas than the usual Möbius function in this case.

## 1. Introduction

The partially ordered set (or “poset”) of set partitions of  $\{1, \dots, n\}$  ordered by refinement, denoted  $\Pi_n$ , has a wealth of interesting properties. Among them are lexicographic shellability (a topological condition on an associated simplicial complex called the order complex) and a natural symmetric group action permuting elements in an order-preserving manner which induces an  $S_n$  representation on (top) homology of the order complex. These features lead to formulas for counting by inclusion-exclusion, growing out of the fact that a function called the Möbius function for the partition lattice has the elegant formula  $\mu_{\Pi_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$ . This formula may be explained either topologically by counting descending chains in a lexicographic shelling

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(see [Bj]) or via representation theory (see [St1]), by interpreting this Möbius function as the dimension of a symmetric group representation on top homology of the order complex; up to a sign twist, this representation is an induced linear representation, induced from the cyclic group  $C_n$  up to the symmetric group  $S_n$ , hence having dimension  $n!/n$ .

It is natural to consider more generally the poset of partitions of a multiset  $\{1^{\lambda_1}, \dots, k^{\lambda_k}\}$ , i.e. to allow repetition in values, or in other words to consider the quotient poset  $\Pi_n/S_\lambda$  for  $S_\lambda$  the Young subgroup  $S_{\lambda_1} \times \dots \times S_{\lambda_k}$  of the symmetric group. For example, the poset  $\Pi_7/(S_3 \times S_2 \times S_2)$  consists of the partitions of the multiset  $\{1^3, 2^2, 3^2\}$  into unordered blocks; a sample element would be the partition made up of blocks  $\{1, 1, 3\}$ ,  $\{1, 2\}$ ,  $\{2\}$ , and  $\{3\}$ , which is sometimes represented more compactly as  $1^2, 3|1, 2|2|3$ . We say that an element  $u$  is less than or equal to an element  $v$ , denoted  $u \leq v$ , whenever the multiset partition  $u$  is a refinement of the partition  $v$ , or in other words each block of  $v$  is obtained by merging blocks of  $u$ . For example, if  $u = 1|1$  and  $v = 1^2$ , then  $u \leq v$  because  $v$  is obtained from  $u$  by merging the two blocks of  $u$ . When  $\lambda = (1, 1, \dots, 1)$ , i.e.  $\lambda_1 = \dots = \lambda_k = 1$ , then  $\Pi_n/S_\lambda$  is the usual partition lattice  $\Pi_n$ . On the other hand, the case  $\lambda = (n)$  yields another important poset, namely the poset  $P_n$  of partitions of the positive integer  $n$  into smaller positive integers.

A Möbius function formula for the poset of partitions of any multiset would have a wide array of enumerative applications. A natural first question is whether this poset is shellable (or perhaps even has an interesting group representation on top homology of its order complex), since this could again give a convenient way of calculating the Möbius function. This turns out not to be the case – the example of Ziegler which we include in Section 2 demonstrates that this is not possible. This note therefore suggests a different approach.

Recall that the Möbius function of a finite partially ordered set  $P$  is a function  $\mu_P$  on pairs  $u, v \in P$  with  $u \leq v$ . It is defined recursively as follows:  $\mu_P(u, u) = 1$ ,

$$\mu_P(u, v) = - \sum_{u \leq z < v} \mu_P(u, z)$$

for  $u < v$ , and  $\mu_P(u, v) = 0$  otherwise. One may encode the Möbius function as an upper triangular matrix  $M$  by letting the rows and columns be indexed by the elements of  $P$  with  $M_{u,v} = \mu_P(u, v)$ , using a total order that is consistent with the partial order  $P$  to order the row and column indices. This is the inverse matrix to the “incidence matrix”  $I$  having  $I_{u,v}$  equalling 1 for  $u \leq v$  and 0 otherwise. See [St2].

To see by way of an example how the Möbius function is useful for counting by inclusion-exclusion, consider the problem of counting lattice points in the portion of  $\mathbb{R}^2$  consisting of points  $(x, y)$  with  $|x| \leq q$  and  $|y| \leq q$  such that  $(x, y)$  does not lie on any of the lines  $x = 0, y = 0, x = y$ . A natural approach is to count all lattice points  $(x, y) \in \mathbb{R}^2$  with  $|x| \leq q$  and  $|y| \leq q$ , subtract off those on each of the three lines, then add back 2 copies of the unique lattice point lying on all three lines. Thus, each allowable lattice point is counted once, and each forbidden point is counted 0 times altogether. Notice that the coefficients assigned to the various intersections of lines (i.e. coefficient 1 for the empty intersection, 2 for the triple intersection and -1 for each line by itself) are in fact the Möbius function values  $\mu_P(\mathbb{R}^2, u)$  for the various elements  $u$  in the poset  $P$  of intersections (or “intersection poset”) ordered by containment. The recursive definition of the Möbius function is designed exactly to count allowable points once and all others 0 times. Notice that the partition lattice may be viewed as another instance of an intersection poset, now taking the intersections of hyperplanes  $x_i = x_j$  in  $\mathbb{R}^n$  in which two coordinates are set equal.

A remarkably fruitful method for computing Möbius functions (as well as for finding beautiful formulas for them) grows out of the following topological interpretation for the Möbius function. Given a finite poset  $P$ , its order complex is the simplicial complex  $\Delta(P)$  whose  $i$ -dimensional faces are the chains  $u_0 < \cdots < u_i$  of  $i + 1$  comparable, distinct elements of  $P$ . It turns out that  $\mu_P(u, v)$  equals the reduced Euler characteristic for the order complex of the subposet of  $P$  comprised of exactly those elements  $z$  with  $u < z < v$  (see e.g. [Ro]). Recall that the reduced Euler characteristic of a simplicial complex is  $-1 + f_0 - f_1 + f_2 - \cdots = -1 + \beta_0 - \beta_1 + \beta_2 - \cdots$  where  $f_i$  is the number of  $i$ -dimensional faces in the complex and  $\beta_i$  is the rank of the  $i$ -th homology group of the complex.

Surprisingly many posets arising in combinatorics have a property called shellability (more specifically, EL-shellability) enabling easy calculation of the reduced Euler characteristic (and thus the Möbius function) by showing that the order complex is homotopy equivalent to a wedge of spheres and in fact giving a way to count these spheres by counting the so-called descending chains of the poset. See [Bj] to learn more about this technique. Shellability for a graded poset also implies an algebraic/topological property called the Cohen-Macaulay property. We will soon see that the poset of partitions of a multiset cannot be shellable in general, because it is not always Cohen-Macaulay.

A simplicial complex is Cohen-Macaulay if all its maximal faces have the same dimension and for each face  $\sigma$ , the subcomplex called the link of  $\sigma$  (which may be thought of as a neighborhood of  $\sigma$  in the original complex) also has all its maximal faces of equal dimension  $i$  and its reduced homology concentrated in this same dimension  $i$ . See [Mu], [St2] or [St3] and references therein for further background, including powerful bridges built largely by Stanley and Hochster in the 70's between combinatorics (e.g. counting faces) and commutative algebra (e.g. computing Hilbert series of rings), using the topology of simplicial complexes along the way. See [BH] for partial results and further open questions regarding the topology of the order complex for the poset of partitions of a multiset; in [BH], a relatively new technique called discrete Morse theory is used to analyze the topology of the order complex, since shellability is not an option in this case. Discrete Morse theory translates topological questions into combinatorial questions of finding nice graph matchings, and can be used in situations where other techniques such as shellability are not applicable.

Our main interest here is in the Möbius function, and in variations on it, for the poset of partitions of a multiset. While the special case of the partition lattice has been studied a great deal, little is known about the more general case. The point of this note is to suggest questions about this more general poset, in particular suggesting a variant on the Möbius function which we believe in this case may be much more useful to study than the usual Möbius function; we only scratch the surface here as far as carrying out this study, but we do prove two formulas demonstrating that our variation on the Möbius function is much more well-behaved than the usual Möbius function in these cases, and which seem like a good starting point for further study.

## 2. A cautionary example and a consequent deformation of the Möbius function

Ziegler showed in [Zi, p. 218] that  $\Pi_n/S_n$  is not Cohen-Macaulay for  $n \geq 19$ . The Möbius function for  $\Pi_n/S_\lambda$  also is not well-behaved, seemingly because poset intervals are no longer products of smaller intervals coming from single blocks. For instance,  $\mu_{\Pi_4/S_4}(\hat{0}, 1^2|1^2) \neq [\mu_{\Pi_2/S_2}(\hat{0}, 1^2)]^2$ . We introduce a variant of the Möbius function, denoted  $\mu'$ , which is defined to be multiplicative in the following sense: given a partition into blocks  $B_1, \dots, B_j$ , let  $\mu'(\hat{0}, B_1 | \dots | B_j) = \prod_{i=1}^j \mu'(\hat{0}, B_i)$ . For each block  $B_i$ , let  $\mu'(\hat{0}, B_i) = -\sum_{\hat{0} \leq u < B_i} \mu'(\hat{0}, u)$ , and let  $\mu'(\hat{0}, \hat{0}) = 1$ . In the case of the partition lattice,  $\mu' = \mu$ . The function  $\mu'$  first arose in [He] as the inverse matrix to a weighted version of the incidence

matrix used to count quasisymmetric functions by inclusion-exclusion by collecting them into groups comprising symmetric functions. See [Eh] and references therein for background on quasisymmetric functions and their connections to flag  $f$ -vectors, i.e. to vectors counting “flags” of faces  $F_0 \subset F_1 \subset \cdots \subset F_r$  with each face contained in the next.

Ziegler gave the following example to show that  $\Pi_n/S_n$  is not Cohen-Macaulay for  $n \geq 19$  [Zi, p. 218]. In that paper, he denotes  $\Pi_n/S_n$  by  $P_n$ , because he thinks of it as consisting of the partitions of an integer into smaller integers.

EXAMPLE 2.1 (Ziegler). *The open interval  $(1^6|1^5|1^3|1^2|1^2|1, 1^8|1^7|1^4)$  prevents  $\Pi_{19}/S_{19}$  from being Cohen-Macaulay. Simply note that  $1^8|1^7|1^4$  may be refined to  $1^6|1^5|1^3|1^2|1|1$  in two different ways:*

$$8 = 6 + 2 = 5 + 3$$

$$7 = 5 + 2 = 6 + 1$$

$$4 = 3 + 1 = 2 + 2$$

*Any saturated chain from  $\hat{0}$  to  $1^6|1^5|1^3|1^2|1^2|1$  together with any saturated chain from  $1^8|1^7|1^4$  to  $\hat{1}$  gives a face  $F$  in the order complex  $\Delta(\Pi_{19}/S_{19})$  such that  $lk(F)$  is a disconnected graph, precluding the Cohen-Macaulay property.*

In [Eh], Richard Ehrenborg encoded the flag  $f$ -vector of any finite graded poset with unique minimal and maximal elements as a function called a quasisymmetric function. Quasisymmetric functions recently have been shown to play a key role in the theory of combinatorial Hopf algebras; they also may be regarded as a generalization of symmetric functions (a widely studied family of functions that arise in representation theory as a convenient way of encoding characters). An application to determining the flag  $f$ -vector for graded monoid posets in [He] (encoded as the quasisymmetric function  $F_{\mathcal{P}}$  introduced in [Eh]) led us to examine  $\mu'$  in the first place. We gave an inclusion-exclusion counting formula in [He] where we grouped collections of quasisymmetric functions together into symmetric functions, leading us to use a weighted version of the incidence matrix having  $\mu'$  as its inverse matrix.

Thus,  $\mu'$  is not just an abstraction admitting pleasant formulas, but does arise in counting by inclusion-exclusion, albeit in a way that is not yet well understood. The aim of this note is to take a first step towards understanding to what extent  $\Pi_n/S_\lambda$  shares the nice properties of  $\Pi_n$ . The fact that  $\mu'$  satisfies much nicer formulas than the usual Möbius function suggests that the order complex may not be the best complex to associate to this poset, that rather one should study a complex with

$\mu'$  as its reduced Euler characteristic. It seems that this should be a boolean cell complex, in the sense of [Bj2], where one poset chain may give rise to multiple cells in the complex, with weighted incidence numbers counting these.

### 3. Combinatorial formulas

Next we give formulas for  $\mu'$  for two natural seeming classes of multisets. Let  $n = \sum_{i=1}^j n_i$ . Since  $(-1)^{n-1} \frac{(n-1)!}{\prod_{i=1}^j (n_i)!}$  is not necessarily an integer unless some  $n_i = 1$ , it is too much to ask for the following to hold for all multisets.

**THEOREM 3.1.** *If  $P$  is the poset  $\Pi_n / (S_{n_1} \times \cdots \times S_{n_k} \times S_1)$  of partitions of  $\{1^{n_1}, \dots, k^{n_k}, k+1\}$ , so  $n-1 = n_1 + \cdots + n_k$ , then*

$$\mu'_P(\hat{0}, \hat{1}) = (-1)^{n-1} \frac{(n-1)!}{\prod_{i=1}^k (n_i)!}.$$

**PROOF.** Suppose by induction this is true for rank less than  $n$ . We decompose the set of partitions below the maximal element according to the content of the block containing the distinguished letter  $k+1$ . For any fixed block  $B$  such that  $|B| < n-1$  and  $k+1 \in B$ , note that

$$\sum_{\{u|B \in u\}} \mu'_P(\hat{0}, \bar{u}) = 0$$

where  $\bar{u}$  is obtained from  $u$  by deleting the block  $B$ . Since  $\mu'_P(\hat{0}, u) = \mu'(\hat{0}, B)\mu'(\hat{0}, \bar{u})$ , we have

$$\sum_{\{u|B \in u\}} \mu'(\hat{0}, u) = \mu'(\hat{0}, B) \sum_{\{u|B \in u\}} \mu'(\hat{0}, \bar{u}) = 0.$$

Therefore, we need only sum over all possible blocks  $B$  which contain  $k+1$  and satisfy  $|B| = n-1$ . For each such  $B$ , we sum over partitions

$u$  containing any particular such block  $B$ , yielding

$$\begin{aligned}
 \sum_B \sum_{\{u|B \in u\}} \mu'(\hat{0}, u) &= (-1)^{n-2} \sum_{j=1}^k \frac{(n-2)!}{(n_j-1)! \prod_{i \neq j} (n_i)!} \\
 &= (-1)^{n-2} \sum_{j=1}^k \frac{(n-2)!(n_j)}{\prod_{i=1}^k (n_i)!} \\
 &= (-1)^{n-2} \frac{(n-2)!}{\prod_{i=1}^k (n_i)!} \sum_{j=1}^k n_j \\
 &= (-1)^{n-2} \frac{(n-1)!}{\prod_{i=1}^k (n_i)!}.
 \end{aligned}$$

This implies

$$\mu'(\hat{0}, \hat{1}) = - \sum_B \sum_{\{u|B \in u\}} \mu'(\hat{0}, u) = (-1)^{n-1} \frac{(n-1)!}{\prod_{i=1}^k (n_i)!}.$$

□

**PROPOSITION 3.1.** *Let  $P$  be the poset  $\Pi_n/S_n$  of partitions of  $\{1^n\}$ . Then  $\mu'_P(\hat{0}, \hat{1}) = (-1)^{n-1}$  for  $n$  a power of 2 and  $\mu'_P(\hat{0}, \hat{1}) = 0$  otherwise.*

**PROOF.** By induction,  $\mu'_P(\hat{0}, 1^{n_1} | \dots | 1^{n_k}) = 0$  for  $n_1 + \dots + n_k = n$  and  $k > 1$  unless each  $n_i$  is a power of 2, in which case it equals  $(-1)^{n-k}$ . Hence, if we restrict to partitions where each block has size a power of 2 and give a bijection between such partitions of  $\{1^n\}$  involving an even number of parts and those involving an odd number of parts, this will imply the result, since the partition with a single block of size  $n$  is paired with another partition by this bijection if and only if  $n$  is a power of 2. Such a correspondence is obtained by pairing a partition having a unique largest block with the partition obtained by splitting this block into two equal parts. □

In particular, the formula in Theorem 3.1 suggests the possibility of  $\mu'$  recording the dimension of an induced representation on (top) homology of an associated cell complex, specifically a linear representation induced from the group  $S_{n_1} \times S_{n_2} \times \dots \times S_{n_k}$  to the full symmetric group  $S_{n-1}$ . See [He2] for a generalization of lexicographic shellability applicable to cell complexes having  $\mu'$  as reduced Euler characteristic.

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