

# ON OPTIMIZING DISCRETE MORSE FUNCTIONS

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ABSTRACT. In 1998, Forman introduced discrete Morse theory as a tool for studying CW complexes by producing smaller, simpler-to-understand complexes of critical cells with the same homotopy types as the original complexes. This paper addresses two questions: (1) under what conditions may several gradient paths in a discrete Morse function simultaneously be reversed to cancel several pairs of critical cells, to further collapse the complex, and (2) which gradient paths are individually reversible in lexicographic discrete Morse functions on poset order complexes. The latter follows from a correspondence between gradient paths and lexicographically first reduced expressions for permutations. As an application, a new partial order on the symmetric group recently introduced by Remmel is proven to be Cohen-Macaulay.

## 1. INTRODUCTION

Forman introduced discrete Morse theory in [10] as a tool for studying the homotopy type and homology groups of finite CW-complexes. In joint work with Eric Babson, we introduced a way of constructing “lexicographic discrete Morse functions” for the order complex of any finite poset with  $\hat{0}$  and  $\hat{1}$  in [2]. Lexicographic discrete Morse functions have relatively few critical cells: when one builds the complex by sequentially attaching facets using a lexicographic order, each facet introduces at most one new critical cell, while facet attachments leaving the homotopy type unchanged do not contribute any new critical cells. The most natural of poset edge-labellings and chain-labellings seem to yield lexicographic discrete Morse functions in which one may easily give a systematic description of critical cells. However, these are usually not minimal Morse functions.

The purpose of the present paper is to provide tools for “optimizing” discrete Morse functions by cancelling pairs of critical cells, both in general and specifically in lexicographic discrete Morse functions. This paper develops and justifies machinery that has recently been applied to examples in [12], [13], and as provided in Section 9. We do not address the very interesting question of how to turn an arbitrary discrete Morse function into an optimal one, but rather provide tools that seem to work well at turning very natural discrete Morse functions on complexes that arise in practice, especially on poset order complexes, into ones with smaller Morse numbers (and often into optimal Morse functions). These tools have led to connectivity lower bounds (in [13]) and to proofs that posets are Cohen-Macaulay, without needing to find a shelling (in [12]). Using our machinery typically seems to require somewhat lengthy proofs, but this is because there are several things one needs to check, each of which is often straightforward.

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We introduce and justify two new tools for cancelling critical cells in discrete Morse functions, in Sections 3 and 7, respectively:

- (1) A criterion for reversing several gradient paths, each of which is individually reversible, to simultaneously cancel several pairs of critical cells.
- (2) A result showing that a pair of critical cells  $\tau, \sigma$  in a lexicographic discrete Morse function may be cancelled whenever there is a certain type of gradient path from  $\tau$  to  $\sigma$ , by virtue of a correspondence between gradient paths and certain types of reduced expressions for permutations

Section 7 also introduces the notion of a least-content-increasing labelling, a class of labellings for which critical cell cancellation seems to be particularly manageable. Section 4 very briefly shows how filtrations of simplicial complexes may help in constructing discrete Morse functions. Section 5 reviews the notion of lexicographic discrete Morse function, in preparation for later sections. It is quite helpful in cancelling critical cells to view these as coming from a filtration. Section 6 gives a new characterization for lexicographically first reduced expressions for permutations, as needed to prove (2) above.

Finally, Sections 8 and 9 apply the above results to prove that two posets are homotopically Cohen-Macaulay: the poset  $PD(1^n, q)$  of collections of independent lines in a finite vector space, and the poset  $\Pi_{S_n}$  of partitions of  $\{1, \dots, n\}$  into cycles. The former is the face poset of a matroid complex, and so is well-known to be shellable (see [5]), but is included to illustrate on a familiar example a strategy for constructing and optimizing a discrete Morse function that has also worked on much more complex examples. The latter poset was recently defined by Remmel as a new partial order on elements of the symmetric group. See [12] and [13] for much more difficult applications of the machinery provided in this paper.

## 2. BACKGROUND

Recall that any permutation may be expressed as a product of adjacent transpositions. Denote by  $s_i$  the adjacent transposition swapping  $i$  and  $i+1$ . An **inversion** in a permutation  $\pi$  is a pair  $(i, j)$  with  $i < j, \pi(i) > \pi(j)$ . A **reduced expression** for  $\pi$  is a minimal expression for  $\pi$  as a product of adjacent transpositions. Any two reduced expressions for the same permutation are connected by a series of braid relations, i.e. relations  $s_i \circ s_j = s_j \circ s_i$  for  $|j-i| > 1$  and  $s_i \circ s_{i+1} \circ s_i = s_{i+1} \circ s_i \circ s_{i+1}$ . See [Hum] or [Ga] for more about reduced expressions for permutations. In our setting, permutations in  $S_n$  act by permuting positions of labels in a label sequence of length  $n$ , i.e.  $s_i$  swaps the  $i$ -th and  $(i+1)$ -st labels.

**2.1. Discrete Morse theory.** Forman introduced discrete Morse theory for CW complexes in [10], then Chari provided a combinatorial reformulation for regular cell complexes in [7]. Most discrete Morse functions in the literature are in terms of this combinatorial reformulation, in which one constructs an “acyclic matching” on the face poset of the complex.

Let  $\sigma^{(d)}$  denote a cell of dimension  $d$ . Forman defines a function  $f$  which assigns real values to the cells in a regular cell complex (as defined next) to be a **discrete Morse function** if for each cell  $\sigma^{(d)}$ , the sets  $\{\tau^{(d-1)} \subseteq \sigma^{(d)} \mid f(\tau^{(d-1)}) \geq f(\sigma^{(d)})\}$  and  $\{\tau^{(d+1)} \supseteq \sigma^{(d)} \mid f(\tau^{(d+1)}) \leq f(\sigma^{(d)})\}$  each have cardinality at most one. Making these requirements for all cells implies that for each cell  $\sigma$ , at most one of the two cardinalities is nonzero. When both cardinalities are 0,  $\sigma$  is called a **critical**

**cell.** For non-regular CW-complexes, Forman makes the additional requirement  $f(\sigma) < f(\tau)$  for each  $\sigma$  which is a non-regular face of  $\tau$ .

Recall that any CW complex  $M$  is equipped with characteristic maps  $h_\tau : B \rightarrow M$  sending a closed ball  $B$  of dimension  $p + 1$  to  $\tau^{(p+1)}$ . A face  $\sigma^{(p)}$  is a **regular face** of  $\tau$  if the restriction of  $h$  to  $h^{-1}(\sigma)$  is a homeomorphism and the closure  $\overline{h^{-1}(\sigma)}$  is a closed  $p$ -ball. If all face incidences in a CW complex are regular, then the complex is a **regular CW complex**. All simplicial complexes and boolean cell complexes are regular (see [4], [21]). On the other hand, the minimal CW complexes for a 2-sphere and the real projective plane have non-regular faces.

Figure 1 gives an example of a discrete Morse function on a 1-sphere resulting from a height function. Critical cells record changes in topological structure as a complex is built by sequentially inserting cells in the order specified by the Morse function; non-critical cells may be eliminated by elementary collapses without changing the homotopy type. The non-critical cells come in pairs  $\sigma^{(p)} \subseteq \tau^{(p+1)}$  which prevent each other from being critical by satisfying  $f(\sigma) \geq f(\tau)$ . Elementary collapses eliminating these pairs are possible because  $\sigma$  is a free face of  $\tau$  in a certain partial complex (see Section 3 for details).

Any discrete Morse function on a regular cell complex gives rise to a matching on its face poset, by the aforementioned pairing on non-critical cells. Recall that the **face poset** of a regular cell complex is the partial order on cells with  $\sigma < \tau$  for each  $\sigma$  in the boundary of  $\bar{\tau}$ .

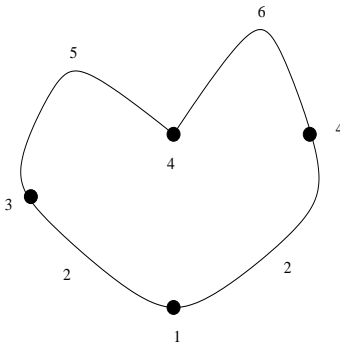


FIGURE 1. A discrete Morse function

**Definition 2.1** (Chari). *A matching on (the Hasse diagram of) the face poset of a regular cell complex is **acyclic** if the directed graph obtained by directing matching edges upward and all other edges downward has no directed cycles. Recall that the Hasse diagram of a poset is the graph whose vertices are poset elements and whose edges are covering relations, i.e. comparabilities  $u < w$  such that there is no intermediate element  $v$  satisfying  $u < v < w$ .*

Observe that the face poset matching resulting from a discrete Morse function is always acyclic, because the edges are oriented in the direction in which  $f$  decreases. Conversely, many different (but in some sense equivalent) discrete Morse functions may be constructed from any face poset acyclic matching.

**Remark 2.2.** *Given an acyclic matching on a facet poset with  $n$  elements, the corresponding discrete Morse functions are the points in a cone in  $\mathbb{R}^n$  bounded by hyperplanes  $x_i = x_j$  coming from pairs of comparable poset elements  $v_i < v_j$ . The*

acyclic matching determines which side of each hyperplane contributes to the cone, and its acyclicity ensures that the cone is non-empty.

See Figure 2 for the acyclic matching corresponding to the discrete Morse function of Figure 1.

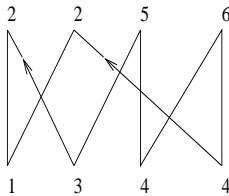


FIGURE 2. An acyclic matching

Let  $m_i$  be the number of critical cells of dimension  $i$  in a discrete Morse function on  $\Delta$ , and let  $b_i$  denote the Betti number recording the rank of  $H_i(\Delta)$ . By convention, we include the empty set in our face poset, so as to get a reduced version of discrete Morse theory, and we let  $\tilde{m}_i, \tilde{b}_i$  denote the reduced Morse numbers and reduced Betti numbers, respectively. Forman showed in [10] that  $\Delta$  collapses onto a CW-complex  $\Delta^M$  which has  $m_i$  cells of dimension  $i$  for each  $i$ , such that  $\Delta^M$  is homotopy equivalent to  $\Delta$ . The existence of such a complex  $\Delta^M$  implies that the following results from traditional Morse theory, the first two of which are called the Morse inequalities, carry over to discrete Morse theory. Critical cells of dimension  $i$  will play the role of critical points of index  $i$ .

- (1)  $\tilde{m}_j \geq \tilde{b}_j$  for  $-1 \leq j \leq \dim(\Delta)$
- (2)  $\sum_{i=0}^{j+1} (-1)^i \tilde{m}_{j-i} \geq \sum_{i=0}^{j+1} (-1)^i \tilde{b}_{j-i}$  for  $0 \leq j \leq \dim(\Delta)$ , with equality achieved when  $j = \dim(\Delta)$
- (3) If  $\tilde{m}_i = 0$  for all  $i \geq 0$ , then  $\Delta$  is collapsible.
- (4) If  $\tilde{m}_i = 0$  for all  $i \neq j$  for some fixed  $j$ , then  $\Delta$  is homotopy equivalent to a wedge of  $j$ -spheres.

**Remark 2.3.** Another consequence of the homotopy equivalence of  $\Delta$  to such a complex  $\Delta^M$  is that any  $\Delta$  with  $\tilde{m}_i(\Delta) = 0$  for all  $i < j$  is  $(j-1)$ -connected.

**Question 2.1.** Is there a notion of rank-selected lexicographic discrete Morse functions for graded posets?

Figure 1 gives an example of a discrete Morse function with  $b_0 = b_1 = 1$  and  $m_0 = m_1 = 2$ . Letting  $f(\emptyset) = 1.5$  turns this into a reduced discrete Morse function with  $\tilde{b}_0 = 0, \tilde{m}_0 = \tilde{b}_1 = 1$  and  $\tilde{m}_1 = 2$ . The Morse numbers are larger than the Betti numbers because there is a critical cell of dimension 0 that is labelled 4 which locally looks as though it is creating a new connected component as the complex is built from bottom to top and there is a critical cell of dimension one that is labelled 5 which locally appears to be closing off a 1-cycle, but these two critical cells actually cancel each other's effect.

Recall that the **order complex** of a finite poset  $P$  with  $\hat{0}$  and  $\hat{1}$  is the simplicial complex, denoted  $\Delta(P)$ , whose  $i$ -faces are chains  $\hat{0} < v_0 < \dots < v_i < \hat{1}$  of comparable poset elements. Since  $\tilde{\chi}(\Delta(P)) = \mu_P(\hat{0}, \hat{1})$  (cf. [19]), a discrete Morse function on  $\Delta(P)$  gives a Möbius function expression  $\mu_P(\hat{0}, \hat{1}) = \sum_{i=-1}^{\dim(\Delta(P))} (-1)^i \tilde{m}_i$ , one of the original motivations of [2]. A poset is **homotopically Cohen-Macaulay** if

each interval is homotopy-equivalent to a wedge of spheres of top dimension. This implies Cohen-Macaulayness of  $\Delta(P)$  over any field, and for  $\dim(\Delta(P)) > 1$  that  $\Delta(P)$  is simply-connected.

**Remark 2.4.** A shelling (see [6] for background) immediately implies the existence of a discrete Morse function whose (reduced) critical cells are all top-dimensional (cf. [2], [7]). However, discrete Morse functions may also give information about simplicial complexes and CW-complexes that are far from shellable.

Forman shows in [10] that whenever a discrete Morse function has two critical cells  $\sigma^{(p)}$  and  $\tau^{(p+1)}$  such that there is a unique gradient path from  $\tau$  to  $\sigma$  (i.e. a path upon which  $f$  decreases at each step), then one obtains a new acyclic matching in which  $\sigma$  and  $\tau$  are no longer critical by reversing this gradient path, analogously to in traditional Morse theory. We call this process **critical cell cancellation**. Uniqueness of the gradient path from  $\tau$  to  $\sigma$  implies that reversing it does not introduce any directed cycles. We still get a matching because vertices along the path are matched with others along the path, and reversal redistributes which pairs are matched so as to incorporate the endpoints into the matching. This reversal process for instance straightens the 1-sphere in Figure 1 into one in which two of the critical cells have been eliminated. On the other hand, the minimal Morse numbers for the dunce cap are strictly larger than its Betti numbers, reflecting the fact that it is contractible but not collapsible.

Section 7 provides machinery for checking uniqueness of a gradient path in a lexicographic discrete Morse function. In practice, we often need to reverse several gradient paths simultaneously; section 3 introduces a notion of face poset for  $\Delta^M$ , denoted  $P^M$ , and uses this to provide a criterion for checking whether several pairs of critical cells may be cancelled simultaneously. Chari's construction does not apply directly because it is not clear a priori what are the face incidences in  $\Delta^M$ , or how to define a face poset for a non-regular CW-complex: for instance, one face may be incident to another in multiple ways (e.g. the real projective plane, viewed as a CW-complex with one 0-cell, one 1-cell and one 2-cell), or a cell may differ by more than one in dimension from maximal cells in its boundary (e.g. a 2-sphere, realized as a CW-complex with a 0-cell and a 2-cell).

### 3. CRITICAL CELL CANCELLATION VIA MULTI-GRAPH FACE POSETS

Forman showed in [10] that a discrete Morse function gives a way of collapsing a regular cell complex  $\Delta$  onto a CW complex  $\Delta^M$  of critical cells such that  $\Delta \simeq \Delta^M$ . Theorem 3.4 follows from an analysis of the relationship between face incidences in one such  $\Delta^M$  and gradient paths in  $\Delta$ . First we recall from [17, Theorem 3.2] one very explicit way of constructing such a complex  $\Delta^M$ , and then make some observations about  $\Delta^M$ .

**Constructing  $\Delta^M$ :** Begin with the empty complex and sequentially attach cells in a way that illustrates how  $\Delta$  collapses onto  $\Delta^M$ , as follows. At each step, attach a single cell not yet in the complex, all of whose faces have already been attached. Choose any such cell  $c_1$ , and if it is critical then glue it to the complex. Otherwise,  $c_1$  is matched with a cell  $c'_1$ , and by virtue of our insertion procedure,  $c_1$  will be in the boundary of  $c'_1$ . If  $c_1$  is the only cell in the boundary of  $c'_1$  that has not yet been inserted, then we may attach the cells  $c_1, c'_1$  to the complex both at once without changing the homotopy type. This is because  $c_1$  is a free face of  $c'_1$ , enabling

an elementary collapse to eliminate  $c_1, c'_1$  as we construct  $\Delta^M$ . Kozlov used the acyclicity of the matching to show that there is an attachment order for cells which allows each matched pair of non-critical cells to be inserted both at once with one a free face of the other, and hence collapsed away.

**Remark 3.1.** *A critical cell  $\sigma^{(p)}$  is incident to a critical cell  $\tau^{(p+1)}$  in  $\Delta^M$  if there is a gradient path from  $\tau$  to  $\sigma$  in the discrete Morse function on  $\Delta$ . Each such gradient path gives a distinct way in which  $\sigma$  is incident to  $\tau$  in  $\Delta^M$ , implying that  $\sigma$  is not a regular face of  $\tau$  unless there is a unique gradient path from  $\tau$  to  $\sigma$ .*

An irregular face occurs in  $\Delta^M$ , for example, if two different  $p$ -faces of  $\tau^{(p+1)}$  in  $\Delta$  get identified with the same face  $\sigma$  through a series of collapses leading from  $\Delta$  to  $\Delta^M$ . Alternatively, if there is a directed path in  $F(\Delta)$  from a critical cell  $\tau^{(p+1)}$  to a critical cell  $\sigma^{(q)}$  for  $q < p$ , and this path does not pass through any critical cells of intermediate dimension, then this also yields a face  $\sigma$  which is an irregular face of  $\tau$  in  $\Delta^M$ . With these observations in mind, we define the **multi-graph face poset**, denoted  $P^M$ , for the complex  $\Delta^M$  of critical cells as follows:

- (1) The vertices in  $P^M$  are the cells in  $\Delta^M$ , or equivalently the critical cells in the discrete Morse function  $M$  on  $\Delta$ .
- (2) There is one edge between a pair of cells  $\sigma^{(p)}, \tau^{(p+1)}$  of consecutive dimension for each gradient path from  $\tau$  to  $\sigma$ .

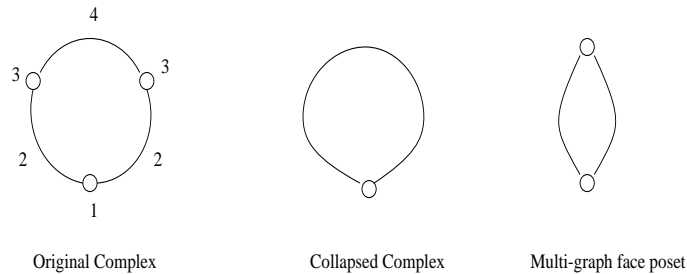


FIGURE 3. A “face poset”  $P^M$  for  $\Delta^M$

Figure 3 gives an example of a discrete Morse function  $M$  on a regular cell complex  $\Delta$ , together with the non-regular CW complex  $\Delta^M$  onto which  $\Delta$  collapses and its multi-graph face poset  $P^M$ . Recall that the discrete Morse function in Figure 1 was optimizable by reversing a single gradient path. Figure 4 shows its multi-graph face poset, with critical cells labelled by their Morse function values. Note that any one of the four gradient paths between pairs of critical cells may be reversed to obtain an optimal Morse function.

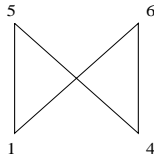


FIGURE 4. A multi-graph face poset

For  $f$  to be a discrete Morse function, Forman required  $f(\sigma) < f(\tau)$  whenever  $\sigma$  is a non-regular face of  $\tau$ . Notice that any discrete Morse function on  $\Delta^M$  gives rise

to an acyclic matching on  $P^M$ , by again orienting edges in the direction in which  $f$  decreases; we can never match faces  $\sigma^{(p)}, \tau^{(p+1)}$  which are incident in multiple ways or which differ in dimension by more than one. Next we verify the converse, that any acyclic matching on  $P^M$  implies the existence of a discrete Morse function on  $\Delta^M$  whose critical cells are the unmatched elements of  $P^M$ .

**Proposition 3.2.** *Let  $M$  be a discrete Morse function such that there is a unique gradient path  $\gamma_i$  from critical cell  $\tau_i$  of dimension  $d_i$  to critical cell  $\sigma_i$  of dimension  $d_i - 1$  for  $1 \leq i \leq r$ . If there are no permutations  $\pi \in S_r$  other than the identity such that there is a gradient path from  $\tau_i$  to  $\sigma_{\pi(i)}$  for  $1 \leq i \leq r$ , then reversing the gradient paths  $\gamma_i$  will not create any directed cycles.*

PROOF. Let us initially assume that the gradient paths are non-overlapping. When  $r = 2$ , we can reverse both gradient paths  $\gamma_1, \gamma_2$  without creating any directed cycles unless there is a cycle involving the reversals of both  $\gamma_1$  and  $\gamma_2$ . The existence of a cycle involving  $\gamma_1^{rev}$  and  $\gamma_2^{rev}$  would imply that  $\gamma_i$  has elements  $v_{i,1}, v_{i,2}$  for  $i = 1, 2$  such that the original digraph has paths from  $v_{1,1}$  to  $v_{2,2}$ , from  $v_{2,1}$  to  $v_{1,2}$ , and from  $v_{i,1}$  to  $v_{i,2}$  for  $i = 1, 2$ . Thus, we obtain directed paths  $\tau_1 \rightarrow v_{1,1} \rightarrow v_{2,2} \rightarrow \sigma_2$  and  $\tau_2 \rightarrow v_{2,1} \rightarrow v_{1,2} \rightarrow \sigma_1$  in the original directed graph. It is straightforward to extend this to larger  $r$ .

Now suppose some edge is shared by two different gradient paths  $\gamma_i$  and  $\gamma_j$ . Then the first and last edges shared by  $\gamma_i, \gamma_j$  must be oriented upward, so that their endpoints are each only in a single matching edge. We get gradient paths from  $\tau_i$  to  $\sigma_j$  and  $\tau_j$  to  $\sigma_i$  by switching gradient paths at the shared edge, implying the existence of a transposition  $\pi$  of the type that is forbidden.  $\square$

**Remark 3.3.** An easy way to ensure that a collection of gradient paths  $\gamma_1, \dots, \gamma_r$  which are individually reversible will also be simultaneously reversible is for the discrete Morse function  $f$  to satisfy  $f(\tau_i) < f(\sigma_j)$  for each pair  $i < j$ . This was the approach taken in [2].

Subsequent applications (in [12], [13] and later sections of this paper) seem to require a more careful analysis, as provided in Theorem 3.4.

**Theorem 3.4.** *Any acyclic matching on  $P^M$  gives rise to a discrete Morse function for  $\Delta$  in which the critical cells of  $\Delta$  are the unmatched elements of  $P^M$ .*

PROOF. Let  $M$  be the acyclic matching on  $F(\Delta)$  that gave rise to  $\Delta^M$  and  $P^M$ . We must show that simultaneously reversing all the gradient paths in  $F(\Delta)$  which correspond to edges in an acyclic matching for  $P^M$  gives a new acyclic matching on  $F(\Delta)$ . There are three things to check, the third of which was already done in Proposition 3.2.

- (1) that reversing this set of gradient paths is a well-defined operation, i.e. there are no edges in the Hasse diagram for  $F(\Delta)$  that must be reversed in one gradient path and cannot be reversed in another.
- (2) that reversing these gradient paths still gives a matching.
- (3) that reversing all of these gradient paths does not create any directed cycles.

To prove 1, note that any shared gradient path segments must be of the form  $UD \dots DU$  so as to come from a matching, where  $U$  and  $D$  denote upward and downward-oriented edges respectively. If one gradient path  $\gamma_1$  needs this segment reversed, while another gradient path  $\gamma_2$  containing the segment is not reversed, we

reverse the shared segment without reversing all of  $\gamma_2$ , to nonetheless get a matching. Acyclicity is preserved by similar reasoning to that used in Proposition 3.2.

The second requirement follows from the fact that vertices along a gradient path are matched with others along that gradient path. The only potential difficulty would be if two gradient paths shared a segment  $UD \dots DU$  and we reversed both paths, but then the matching for  $P^M$  would have a directed cycle.  $\square$

Section 7 will provide a way of determining when a gradient path between a single pair of critical cells in a lexicographic discrete Morse function is reversible. This result and Theorem 3.4 appear to be most useful when applied jointly.

#### 4. DISCRETE MORSE FUNCTIONS ON FILTRATIONS

This section shows how a filtration may simplify the task of constructing an acyclic matching by splitting it into smaller, typically much more manageable pieces. We independently discovered Lemma 4.1, but it also appears as the ‘‘Cluster Lemma’’ in [16], and the idea has been used widely (e.g. see [1], [14], [17], and [20]). The argument below also applies to non-regular CW complexes obtained from regular ones by a series of collapses, using the generalized notion of acyclic matching from Section 3.

**Lemma 4.1.** *Let  $\Delta$  be a regular CW complex which decomposes into collections  $\Delta_\sigma$  of cells, indexed by the elements  $\sigma$  in a partial order  $P$  which has a unique minimal element  $\hat{0} = \Delta_0$ . Furthermore, assume that this decomposition is as follows:*

- (1) *Each cell belongs to exactly one  $\Delta_\sigma$ .*
- (2) *For each  $\sigma \in P$ ,  $\cup_{\tau \leq \sigma} \Delta_\tau$  is a subcomplex of  $\Delta$ .*

*For each  $\sigma \in P$ , let  $M_\sigma$  be an acyclic matching on the subposet of  $F(\Delta)$  consisting of the cells in  $\Delta_\sigma$ . Then  $\cup_{\sigma \in P} M_\sigma$  is an acyclic matching on  $F(\Delta)$ .*

PROOF. Let  $D$  be the directed graph obtained by orienting matching edges upward and all other edges in  $F(\Delta)$  downward. By design,  $D$  does not include any upward-oriented edges between cells in different components. Suppose there is a downward-oriented edge from a cell in  $\Delta_\tau$  to a cell in  $\Delta_\sigma$  for  $\sigma \neq \tau$ . This implies  $\sigma \leq \tau$ , because  $\cup_{\rho \leq \tau} \Delta_\rho$  is a subcomplex of  $\Delta$ . If  $\sigma \neq \tau$ , then  $\sigma < \tau$ , which means it will be impossible for the directed path ever to return to  $\Delta_\tau$ .  $\square$

The lexicographic discrete Morse functions of [2] to be discussed shortly may be viewed as coming from a filtration  $\Delta_1 \subseteq \dots \subseteq \Delta_k$  with  $\Delta_j \setminus \Delta_{j-1} = F_j \setminus (\cup_{i < j} F_i)$ , where  $F_1, \dots, F_k$  is a lexicographic order on the facets of an order complex.

#### 5. REVIEW OF LEXICOGRAPHIC DISCRETE MORSE FUNCTIONS

This section briefly reviews lexicographic discrete Morse functions for poset order complexes from [2], in preparation for later sections. We also briefly indicate how lexicographic discrete Morse functions yield an easy new proof that every interval in the weak order for  $S_n$  is a ball or a sphere of specified dimension.

Any lexicographic order on the saturated chains of any finite poset  $P$  with minimal and maximal elements  $\hat{0}$  and  $\hat{1}$  gives rise to a discrete Morse function on its order complex with a relatively small number of critical cells as follows. Let  $\lambda$  be a labelling of Hasse diagram edges (or more generally, a chain-labelling) with integers such that  $\lambda(u, v) \neq \lambda(u, w)$  for  $v \neq w$ . We obtain a total order  $F_1, \dots, F_r$  on facets of  $\Delta(P)$  by lexicographically ordering the label sequences on saturated chains.



**Remark 5.1.** *Often we will refer to ranks of elements in a maximal chain. We do not assume posets are graded, but rather allow the rank of an element to depend on the choice of maximal chain within which it is considered.*

A key observation for the [2] construction was that each of the maximal faces in  $F_j \cap (\cup_{i < j} F_i)$  has rank set of the form,  $1, \dots, i, j, \dots, n$ , i.e. it omits a single interval  $i + 1, \dots, j - 1$  of consecutive ranks, by virtue of our use of a lexicographic order. Call the rank interval  $[i + 1, j - 1]$  a **minimal skipped interval** of  $F_j$  with **support**  $i + 1, \dots, j - 1$  and **height**  $j - i - 1$ . Call the collection of minimal skipped intervals for  $F_j$  the **interval system** of  $F_j$ .

Notice that the faces in  $F_j \setminus (\cup_{i < j} F_i)$ , i.e. belonging to  $F_j$  but not to any earlier facets, are the faces of  $F_j$  that include at least one rank from each of the minimal skipped intervals of  $F_j$ . For each  $j$ , [2] constructs an acyclic matching on the set of faces in  $F_j \setminus \cup_{i < j} F_i$ . It is immediate from Proposition 4.1 (and was verified by other means in [2]) that the union of these matchings is acyclic on  $\Delta(P)$ .

Each  $F_j \setminus (\cup_{i < j} F_i)$  has a single critical cell if the homotopy type of  $\Delta(P)$  changes with the attachment of  $F_j$ , and otherwise  $F_j \setminus (\cup_{i < j} F_i)$  contains no critical cells. (See [3] for a related notion of shellability, called a weak shelling.) We say that  $F_j$  **contributes** a critical cell when  $F_j \setminus (\cup_{i < j} F_i)$  has a critical cell.  $F_j$  contributes a critical cell if and only if the interval system of  $F_j$  covers all ranks in  $F_j$  after the truncation procedure described shortly; the dimension of the critical cell is then one less than the number of intervals in the truncated interval system.

For convenience, order the minimal skipped intervals  $I_1, \dots, I_m$  of  $F_j$  so that their lowest rank elements sequentially increase in rank.

**Description of critical cells in a lexicographic discrete Morse function:**

- if the minimal skipped intervals of  $F_j$  (namely the  $I$ -intervals) do not collectively have support covering all the ranks in  $F_j$ , then  $F_j$  does not contribute a critical cell
- if the  $I$ -intervals of  $F_j$  have disjoint support covering all ranks in  $F_j$ , then the critical cell in  $F_j \setminus (\cup_{i < j} F_i)$  consists of the lowest rank from each of the minimal skipped intervals
- if there is overlap in the minimal skipped intervals of  $F_j$ , but they cover all ranks in  $F_j$ , then iterate the following procedure to obtain a potential critical cell:
  - (1) include the lowest rank from  $I_1$  in the critical cell
  - (2) truncate all the remaining minimal skipped intervals by chopping off any ranks that they share with  $I_1$
  - (3) discard  $I_1$  and any skipped intervals that now strictly contain other intervals in our collection
  - (4) re-index the remaining truncated minimal skipped intervals to begin with a newly chosen  $I_1$
  - (5) repeat until there are no more minimal skipped intervals

**Definition 5.2.** *The truncated, minimal intervals obtained by the above procedure are called the  $J$ -intervals of  $F_j$ , and are nonoverlapping. If the  $J$ -intervals do not cover all ranks, then  $F_j$  does not contribute a critical cell.*

**Remark 5.3.** *Since each maximal chain contributes at most one critical cell, we refer interchangeably to critical cells and to the maximal chains contributing them.*

For instance, the label sequence of a critical cell will refer to the label sequence for the maximal chain which contributes the critical cell.

To construct a lexicographic discrete Morse function and compute its Morse numbers requires a labelling in which one may understand its minimal skipped intervals, so as to determine which saturated chains contribute critical cells and what are their dimensions. To cancel critical cells by gradient path reversal, one also needs to understand the matching construction of [2] enough to recognize gradient paths between critical cells and check their uniqueness.

**Description of acyclic matching on  $F_j \setminus \cup_{i < j} F_i$ :**

- If the  $I$ -intervals leave some rank uncovered, then there is a cone point in  $F_j \cap (\cup_{i < j} F_i)$ , so match by including/excluding the lowest rank such cone point.
- Otherwise, match any cell that differs from the potential critical cell on some  $J$  interval based on the lowest rank  $J$ -interval where it differs from  $F_j$ 's potential critical cell. Specifically, match by including/excluding the element of lowest rank in this interval.
- If the  $I$ -intervals cover all ranks but the  $J$ -intervals leave some rank uncovered, match all remaining cells by including excluding the element at the lowest such uncovered rank.

**Remark 5.4.** If all the  $I$ -intervals in a lexicographic discrete Morse function on a poset  $P$  have height at most  $d - 1$ , then each interval  $(x, y)$  in  $P$  is at least  $(-1 + \frac{\text{rk}(y) - \text{rk}(x) - 1}{d - 1})$ -connected.

**Example 5.5.** Consider the ring  $k[ab, a^2, c, d, e, b^2] \cong k[x_1, \dots, x_6]/(x_2x_6 - x_1^2)$  and the partial order on (equivalence classes of) monomials by divisibility. Label  $m_1 \prec m_2$  by the quotient  $m_2/m_1$ . Order the labels  $x_i < x_j$  for each  $i < j$ . Consider the interval  $(1, x_2x_3x_4x_5x_6)$ , and its saturated chain  $F_j = 1 \prec x_2 \prec x_2x_3 \prec x_2x_3x_6 \prec x_2x_3x_5x_6 \prec x_2x_3x_4x_5x_6$ , depicted in Figure 5. This chain is labelled  $x_2x_3x_6x_5x_4$ , or more compactly 23654 (recording indices).

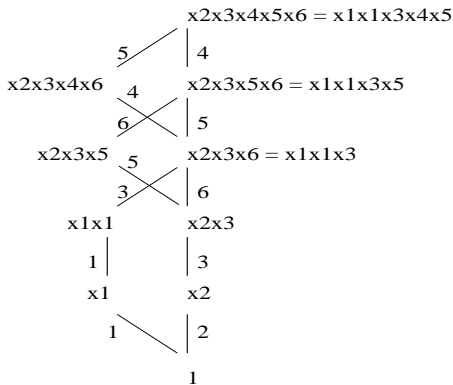


FIGURE 5. A saturated chain and its minimal skipped intervals

$F_j$  has minimal skipped intervals  $I_1, I_2, I_3$ , with support  $\{1, 2\}$ ,  $\{3\}$ , and  $\{4\}$ , respectively.  $I_1$  results from the syzygy  $(x_2x_6 - x_1^2)x_3 = 0$ , causing a lexicographically earlier facet  $F_i$  labelled 11354. The intervals  $I_2$  and  $I_3$  come from

descents. The intervals  $I_1, I_2, I_3$  are non-overlapping and cover  $F_j$ , so  $F_j$  contributes a critical cell  $\text{Crit}(F_j)$  with rank set  $\{1, 3, 4\}$ .  $I_1, I_2$ , and  $I_3$ . Thus,  $\text{Crit}(F_j) = x_2 < x_2x_3x_6 < x_2x_3x_6x_5$ .

**5.1. Example: the weak order for the symmetric group.** In [9], Edelman and Walker study the poset of regions in a hyperplane arrangement with a chosen base region, partially ordered by inclusion of the set of hyperplanes separating a region from the base region  $B$ . They show that each interval is a ball or a sphere of a proscribed dimension. The weak order for any Coxeter group may be viewed as the poset of regions for a Coxeter arrangement.

**Remark 5.6.** *The weak order for the symmetric group has a lexicographic discrete Morse function yielding an elementary proof of the result from [9] in this case.*

The idea is to label each covering relation by the simple reflection being applied. The minimal skipped intervals in the resulting lexicographic discrete Morse function are labelled by reduced expressions of the following forms:

- (1)  $s_j \circ s_i$  for  $j > i + 1$ , since  $s_i \circ s_j$  is lexicographically earlier
- (2)  $s_{j+1} \circ s_j \circ s_{j-1} \circ \cdots \circ s_{i+1} \circ s_i \circ s_{j+1}$  for some  $i \leq j$ , since  $s_j \circ s_{j+1} \circ s_j \circ s_{j-1} \circ \cdots \circ s_{i+1} \circ s_i$  is lexicographically earlier

Within any interval it is straightforward to cancel all but at most critical cell, using Theorems 3.4 and 7.5, and to see that the dimension of any surviving critical cell agrees with the result of [9].

**Question 5.1.** *May lexicographic discrete Morse functions be used to deduce more general results about posets of regions or be applied to the (open) question of whether intervals in higher Bruhat orders are homotopy equivalent to balls or spheres?*

## 6. LEXICOGRAPHICALLY FIRST REDUCED EXPRESSIONS FOR PERMUTATIONS

This section gives a new characterization for lexicographically first reduced expressions for permutations, in the sense of [8]. This provides a very natural explanation for [8, Theorem 2.5], which characterized the “type” of any lexicographically first reduced expression, i.e. the possible vectors  $(m_1, \dots, m_{n-1})$  where  $m_i$  counts the number of appearances of the simple reflection  $s_i$  in a lexicographically first reduced expression. Our main interest in Theorem 6.1 below is that it will help us in Section 7 to show that certain gradient paths in lexicographic discrete Morse functions are unique and hence reversible.

Recall that a reduced expression for a permutation  $\pi$  is the **lexicographically first reduced expression** for  $\pi$  if its reduced word precedes all other reduced words for reduced expressions for  $\pi$  in lexicographic order, i.e. when we say  $w_1 < w_2$  if  $w_1$  has a smaller letter at the first place that the two words differ.

**Theorem 6.1.** *Every permutation has a unique reduced expression  $s_{i_1} \circ \cdots \circ s_{i_k}$  with all of the following properties:*

- (1)  $s_i$  is never immediately followed by  $s_j$  for  $j > i + 1$
- (2) There are no reduced expressions in the commutation class for  $s_{i_1} \circ \cdots \circ s_{i_k}$  which have  $s_{i+1} \circ s_i \circ s_{i+1}$  appearing consecutively.

The proof is somewhat similar to a proof in [11] for the fact that any two reduced expressions for a fixed permutation are connected by a series of braid relations.

PROOF. Existence follows from the fact that the reversal of the lexicographically first reduced expression for the inverse permutation must take this form. To prove uniqueness, first check that any allowable reduced expression  $\gamma$  has exactly one copy of  $s_u$  for  $u$  the maximal index appearing in  $\gamma$ . Otherwise there would be a copy of  $s_{u-1}$  between each pair of copies of  $s_u$ ; avoiding  $s_u \circ s_{u-1} \circ s_u$  in the commutation class would necessitate another  $s_{u-1}$  between the same two copies of  $s_u$ , forcing an intermediate  $s_{u-2}$ , and this continues indefinitely, contradicting the finiteness of  $\gamma$ .

Similarly for each  $t \leq u$ , each pair of  $s_{t-1}$ 's in  $\gamma$  is separated by an  $s_t$ . Moreover, each  $s_{t-1}$  which is followed by an  $s_t$  must immediately precede an  $s_t$ . If  $\gamma$  includes  $s_t$  and of  $s_{t-d}$  for  $d > 1$ , but no copies of  $s_{t-d'}$  for  $1 \leq d' < d$ , then there is at most one copy of  $s_{t-d}$  and it must appear after the last copy of  $s_t$ . Thus, the only flexibility is in which allowable adjacent transpositions do appear, but varying this will change the permutation itself. Thus, there is at most one allowable reduced expression for any fixed permutation.  $\square$

**Corollary 6.2.** *A reduced expression  $\omega = s_{i_1} \circ \dots \circ s_{i_k}$  is the lexicographically first reduced expression for a permutation  $\pi$  if and only if all of the following hold:*

- (1)  $\omega$  never has  $s_j$  immediately followed by  $s_i$  for  $i < j - 1$
- (2)  $\omega$  never has consecutive simple reflections  $s_{i+1} \circ s_i \circ s_{i+1}$
- (3)  $\omega$  is not in the same commutation class as any reduced expressions containing consecutive simple reflections  $s_{i+1} \circ s_i \circ s_{i+1}$

PROOF. There is a unique reduced expression for  $\pi$  satisfying these conditions, by a nearly identical proof to the one used for Theorem 6.1. These conditions are necessary, because otherwise a lexicographically earlier reduced expression could be obtained by applying a Coxeter relation.  $\square$

In the language of [8], the third condition says that  $\omega$  is not  $C_1$ -equivalent to any reduced expressions containing consecutive simple reflections  $s_{i+1} \circ s_i \circ s_{i+1}$ .

**Remark 6.3.** *Theorem 2.5 of [8] showed that a vector  $(m_1, \dots, m_{n-1})$  is the type vector of a lexicographically first reduced expression if and only if  $0 \leq m_{n-1} \leq 1$  and  $0 \leq m_i \leq m_{i+1} + 1$  for each  $1 \leq i \leq n - 2$ . This also follows immediately from the arguments used in the proof of Theorem 6.1 above.*

## 7. GRADIENT PATHS IN LEXICOGRAPHIC DISCRETE MORSE FUNCTIONS

This section uses properties of reduced expressions for permutations to give conditions under which a gradient path between a pair of critical cells in a lexicographic discrete Morse function is unique, and hence may be reversed to cancel the pair of critical cells. While the results in this section are quite technical, they have recently been useful in applications (see [12] and [13], as well as later sections of this paper). We also believe the proofs in this section may be generalizable, for instance to deal with nonsaturated chain segments, at least in special cases related to least-content-labellings, as defined shortly.

**Definition 7.1.** *A poset edge labelling is **least-increasing** if every interval has a (weakly) increasing chain as its lexicographically smallest saturated chain. It is **least-content-increasing** if in addition the lexicographically smallest chain is at least as small as the increasing rearrangement of every other label sequence on the interval. Within a least-increasing labelling, any increasing chain that is not lexicographically smallest on its interval is called a **delinquent chain**.*

Later we will show that least-content-increasing labellings satisfy the fairly technical conditions of Theorem 7.5 below. Notice that the least-increasing condition is less restrictive than an EL-labelling, since intervals may have several increasing chains. See Sections 8 and 9 as well as [13] for examples of least-content-increasing labellings. Least-content-increasing labellings are particularly well-suited to cancelling pairs of critical cells whose label sequences have equal content, i.e. equal unordered multiset of labels, using the following observation:

**Remark 7.2.** *If  $\tau, \sigma$  are critical cells whose label sequences have equal content in a lexicographic discrete Morse function coming from a least-content-increasing labelling, then any gradient path from  $\tau$  to  $\sigma$  must preserve content at each step and also may never introduce inversions. Thus, gradient path steps must sort labels whenever the label sequence changes at all.*

Any gradient path from a critical cell  $\tau^{(p+1)}$  to a critical cell  $\sigma^{(p)}$  must alternate between ranks  $p+1$  and  $p$ , because it cannot contain two consecutive upward steps or end with an upward step. Denote the downward step in a gradient path which deletes the  $i$ -th element from a chain by  $d_i$ . Denote by  $u_r$  the unique upward step from a non-critical cell to its matching partner, if the newly inserted chain element is the  $r$ -th element in the chain. We will focus on gradient paths resulting from labellings such that each  $d_i$  deletes from a chain a descent and then  $u_i$  replaces this by a lexicographically earlier ascent in which the two labels have been swapped. Thus, we may view the pair of steps  $d_i \circ u_i$  as an adjacent transposition  $s_i = (i, i+1)$  swapping the labels in positions  $i, i+1$  on the lexicographically earliest saturated chain containing the order complex face.

**Definition 7.3.** *An adjacent transposition  $s_i$  acts effectively on a chain  $C$  that includes elements  $v_{i-1}, v_i, v_{i+1}$  of ranks  $i-1, i, i+1$  and has a descent at rank  $i$  if  $s_i$  sends  $C$  to a chain in which  $v_i$  is replaced by  $v'_i$  such that  $\lambda(v_{i-1}, v_i) = \lambda(v'_i, v_{i+1})$  and  $\lambda(v_{i-1}, v'_i) = \lambda(v_i, v_{i+1})$ . A reduced expression  $\omega$  acts effectively on  $C$  if each of its adjacent transpositions acts effectively in turn.*

Let  $\lambda(u, v)$  denote the sequence of edge labels on the lexicographically earliest saturated chain from  $u$  to  $v$ . Then  $s_i$  acts effectively on a (not necessarily saturated) chain  $C = \hat{0} < v_1 < \cdots < v_{i-1} < v_i < v_{i+1} < \cdots < v_{r-1} < \hat{1}$  if  $s_i(C)$  replaces  $v_i$  by some  $v'_i$  such that  $rk(v'_i) = rk(v_{i-1}) + 1$  and the concatenated label sequence  $\lambda(v_{i-1}, v'_i) \circ \lambda(v'_i, v_{i+1})$  is obtained from  $\lambda(v_{i-1}, v_i) \circ \lambda(v_i, v_{i+1})$  by sorting labels into increasing order.

**Definition 7.4.** *A poset edge-labelling (or chain-labelling) has the ordered zigzag property if  $\lambda(u_i, v_i) < \lambda(u_i, v_{i+1})$  for  $1 \leq i \leq k$  implies that  $\lambda(u', v_1) < \lambda(u', v_{k+1})$  for every  $u'$  which satisfies both  $u' < v_1$  and  $u' < v_{k+1}$ .*

Any product of chains has the ordered zigzag property when each covering relations is labelled by the coordinate being increased. To apply Theorem 7.5, it will suffice to have the ordered zigzag property on a subposet which includes the two critical cells  $\sigma, \tau$  to be cancelled as well as all cells with intermediate Morse function values. See [13] for such an application.

In the next theorem, we will use reduced expressions for permutations to show that certain gradient paths are reversible. By convention, we apply adjacent transpositions from left to right. Let  $\tau^{(p+1)}$  be a critical cell which includes ranks

$i_0, i, i+1, \dots, j-1, j$ . Denote by  $u_\tau, v_\tau$  the elements at ranks  $i_0, j$ , respectively in  $\tau$ , and say that these are the  $k$ -th and  $l$ -th elements of  $\tau$ , respectively. Let  $\sigma^{(p)}$  be a critical cell that agrees with  $\tau$  except between ranks  $i_0$  and  $j$ . Theorem 7.5 will show that when a certain type of gradient path from  $\tau$  to  $\sigma$  exists, then it is the only gradient path from  $\tau$  to  $\sigma$ , and hence may be reversed to cancel  $\tau$  and  $\sigma$ .

**Theorem 7.5.** *Let  $M$  be a lexicographic discrete Morse function satisfying the ordered zigzag property, with critical cells  $\tau, \sigma$  as above. Suppose for each  $k < r < l$ ,  $s_r$  acts effectively on any chain  $C$  that (1) coincides with  $\tau$  except strictly between ranks  $i_0$  and  $j$ , (2) has a descent at rank  $r$ , and (3) includes ranks  $r \pm 1$ . Let  $\omega$  be a reduced expression that acts effectively on  $\tau^{(p+1)}$  in such a way that  $\omega$  followed by some  $d_r$  for  $k < r < l$  yields a critical cell  $\sigma^{(p)}$  that agrees with  $\tau$  except between ranks  $i_0$  and  $j$ . Then the gradient path from  $\tau$  to  $\sigma$  is unique unless it ends with steps  $s_{r+1} \circ s_r \circ d_{r+1}$  or differs by commutation relations from one ending this way, in which case there are exactly two gradient paths from  $\tau$  to  $\sigma$ .*

PROOF. Let  $\omega^* = \omega \circ s_i$ , where  $i$  is the rank of the final downward step  $d_i$  in the given gradient path from  $\tau$  to  $\sigma$ . We break the proof into the following three parts:

- (1) If  $\omega^*$  involves only  $s_{k+1}, \dots, s_{l-1}$ , then  $\omega^*$  does not contain any consecutive adjacent transpositions  $s_t \circ s_u$  for  $u > t+1$ . It also avoids consecutive adjacent transpositions  $s_{t+1} \circ s_t \circ s_{t+1}$ , except when the second  $s_{t+1}$  occurs as the final downward step. Furthermore,  $\omega^*$  is not equivalent up to commutation relations to any reduced expression involving  $s_{t+1} \circ s_t \circ s_{t+1}$  except with the second  $s_{t+1}$  as the final downward step.
- (2) Let  $\pi$  be the permutation which has  $\omega^*$  as one of its reduced expressions. Every gradient path  $\gamma$  from  $\tau$  to  $\sigma$  corresponds to a reduced expression for  $\pi$ . That is,  $\gamma$  has the form  $d_{i_1} \circ u_{i_1} \circ \dots \circ d_{i_q}$  for some indices  $i_1, \dots, i_q$  such that  $s_{i_1} \circ \dots \circ s_{i_q}$  is a reduced expression for  $\pi$ .
- (3) Every permutation has a unique reduced expression that avoids consecutive transpositions  $s_i \circ s_j$  for  $j > i+1$  and whose commutation class avoids  $s_{i+1} \circ s_i \circ s_{i+1}$ . In addition, every permutation has at most one reduced expression that avoids consecutive adjacent transpositions  $s_i \circ s_j$  for  $j > i+1$  and whose commutation class avoids  $s_{i+1} \circ s_i \circ s_{i+1}$  except as the final three steps. Thus, the gradient path  $\omega$  from  $\tau$  to  $\sigma$  is unique, except in the designated case where there are two gradient paths.

We begin with (1). The series of steps  $d_t \circ u_t \circ d_u$  for  $u > t+1$  yields a chain lacking rank  $u$ , and having an ascent at rank  $t$ . Thus, it is the top of an ‘‘up’’ edge in  $F(\Delta(P))$ ; it is matched with the lower-dimensional face in which the cone point coming from the ascent at rank  $t$  is deleted. Thus, the gradient path cannot continue. Similarly, the steps  $d_{t+1} \circ u_{t+1} \circ d_t \circ u_t \circ d_{t+1}$  yield a chain with rank  $t+1$  omitted, having an ascent at rank  $t$ , precluding continuation of the gradient path. A reduced expression in the same commutation class as one containing  $s_{t+1} \circ s_t \circ s_{t+1}$  also gets stuck after the second  $d_{t+1}$ , by the same reasoning, unless this is the final step.

When  $\omega^*$  is equivalent up to commutation to a reduced expression which ends with steps  $s_{t+1} \circ s_t \circ s_{t+1}$ , we obtain the second gradient path as follows. Apply the braid relation to obtain  $s_t \circ s_{t+1} \circ s_t$  then consider the unique reduced expression obtained from this by commutation relations, which avoids ever having  $s_i$  followed by  $s_j$  for  $j > i+1$ . This gradient path will end with  $d_t \circ u_t$ . This does not give

the desired critical cell because the wrong ranks are covered on  $J$ -intervals, but we may apply  $d_{t+1}$  to obtain the desired critical cell.

Now we turn to (2). Notice that  $\tau$  and  $\sigma$  must agree up through rank  $i_0$ , since  $\omega$  leaves these ranks unchanged. Thus, every gradient path from  $\tau$  to  $\sigma$  must also leave ranks  $1, \dots, i_0$  fixed in order for the discrete Morse function  $f$  to be non-increasing along the gradient path from  $\tau$  to  $\sigma$ . The ordered zigzag property ensures that every gradient path from  $\tau$  to  $\sigma$  also must leave ranks  $j$  and higher untouched, so only uses  $d_r, u_r$  for  $k < r < l$ . Each such  $d_r$  must be followed by the upward step  $u_r$ , because applying  $d_r$  to a chain  $C$  ensures that  $d_r(C)$  has an ascent (and hence a cone point in  $d_r(C) \setminus (\cup_{F_i < \text{lex } d_r(C)} F_i)$ ) on the newly uncovered interval. There must be no lower cone points, or we would be stuck at the top of an up edge. Each pair of steps  $d_r \circ u_r$  for  $r > j + 1$  swaps two consecutive chain labels if the labels are out of order prior to  $d_r$ , so that a series of effective steps  $d_{i_1} \circ u_{i_1} \circ \dots \circ d_{i_k} \circ u_{i_k}$  eliminates exactly the same inversions that  $s_{i_1} \circ \dots \circ s_{i_k}$  would. Thus, every gradient path corresponds to an expression for  $\pi$  as a product of adjacent transpositions.

What remains to show is that non-reduced expressions do not give rise to gradient paths. Such an expression will eventually apply some adjacent transposition  $s_r$  at a rank  $r$  where the labels  $\lambda_1, \lambda_2$  are increasing. The step  $d_r$  would not go in the direction the Morse function decreases, because the edge would be a matching edge if  $\lambda_1, \lambda_2$  is the lowest increasing pair of consecutive labels in the chain obtained by deleting rank  $r$ ; otherwise, the subsequent upward step would not be a matching step, because the chain would instead match by inserting or deleting a lower rank cone point. Either way, the gradient path cannot be completed.

The first statement in Part (3) was proven as Theorem 6.1. The second statement is proven using similar ideas. The only variation needed is that once we may have two copies of  $s_i$  after the final  $s_{i+1}$ , but then we must have one copy of  $s_{i-1}$  between the copies of  $s_i$ . The rest of the analysis proceeds just as in Theorem 6.1.  $\square$

**Remark 7.6.** *Theorem 7.5 also applies to critical cells  $\tau, \sigma$  where  $\tau$  includes ranks  $i, i + 1, \dots, j - 1, j, k$  for  $k > j + 1$ ,  $\sigma$  includes ranks  $i, i + 1, \dots, j - 1, k$ , and  $\tau, \sigma$  agree outside of the interval from rank  $i$  to  $k$ .*

**Corollary 7.7.** *If the conditions of the previous theorem are met, and in addition  $\tau, \sigma$  have label sequences differing by a 321-avoiding permutation, then the gradient path from  $\tau$  to  $\sigma$  is unique.*

PROOF. A reduced expression for a 321-avoiding permutation cannot have consecutive reflections  $s_{i+1}s_i s_{i+1}$  or  $s_i s_{i+1} s_i$ . This eliminates the possibility of two choices for the conclusion of the gradient path.  $\square$

**Remark 7.8.** *It would be nice if Theorem 7.5 could be extended to deal with gradient paths on nonsaturated chain segments, at least for least-content-increasing labellings. This is related to the question of describing incidences in the complex  $\Delta^M$  of critical cells for a lexicographic discrete Morse function  $M$ .*

In [13], the preceding theorem is applied to monoid posets to provide connectivity lower bounds in terms of Gröbner basis degree. Monoid posets are not Cohen-Macaulay in general, so something other than shelling was needed.

**Example 7.9.** Recall the poset from Example 5.5. Consider the saturated chains  $F_j$  and  $F_i$ , labelled 26543 and 23654, respectively.  $F_j$  has minimal skipped intervals

with rank sets  $\{1\}, \{2\}, \{3\}, \{4\}$ . The first interval results from a syzygy, and the others come from descents. The facet  $F_i$  was already discussed in Example 5.5.  $F_j$  and  $F_i$  contribute critical cells  $\tau = x_2 < x_2x_6 < x_2x_6x_5 < x_2x_6x_5x_4$  and  $\sigma = x_2 < x_2x_3x_6 < x_2x_3x_6x_5$ , respectively. There is a gradient path  $d_4 \circ u_4 \circ d_3 \circ u_3 \circ d_2$  from  $\tau$  to  $\sigma$ , shown in Figure 6.

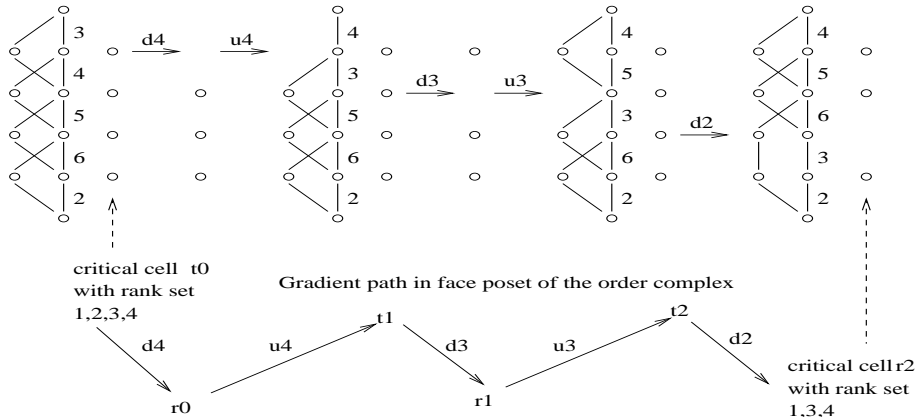


FIGURE 6. A gradient path between two critical cells

It is not hard to check the conditions of Theorem 7.5, using the fact that for any elements  $u_1 < u_2 < u_3$  with descending labels  $\lambda(u_1, u_2) > \lambda(u_2, u_3)$  from the label set  $\{x_3, \dots, x_6\}$ , there exists  $u_1 < z < u_3$  with  $\lambda(u_1, z) = \lambda(u_2, u_3)$  and  $\lambda(z, u_3) = \lambda(u_1, u_2)$ . Theorem 7.5 ensures that the gradient path constructed above is the only gradient path from  $\tau$  to  $\sigma$ . The point is that  $x_3$  is **non-essential** to the syzygy in  $\sigma$ , allowing it to be shifted to its location in  $\tau$  above the syzygy interval. Theorem 7.5 ensures that there is a unique way for a gradient path to shift the label  $x_3$  downward from its position in  $\tau$  to inside the syzygy interval in  $\sigma$ .

Next is an example with two gradient paths from  $\tau^{(p+1)}$  to  $\sigma^{(p)}$ , where we examine orientation to determine that  $\tau^{(p+1)}$  has 0 boundary due to cancellation.

**Example 7.10.** Consider the semigroup ring  $k[x_1, \dots, x_7]/(x_4x_5x_6 - x_1x_2x_3) \sim k[ab, cd, ef, ad, be, cf, g]$  and the partial order on its monomials by divisibility. As before, label poset edges with the indices of the variables being multiplied. Let us examine the interval  $(1, x_4x_5x_6x_7)$ . It has a critical 1-cell  $\sigma$  in the saturated chain labelled 7, 4, 5, 6 and it has critical 2-cells  $\tau_1, \tau_2$  in the saturated chains labelled 7, 3, 2, 1 and 7, 6, 5, 4, respectively. The order complex has a 2-sphere coming from  $\tau_1$ , and then attaching  $\sigma$  gives a wedge of a 2-sphere with a 1-sphere. When we next attach  $\tau_2$ , notice that there are two gradient paths from  $\tau_2$  to  $\sigma$ , and  $\sigma$  is incident to  $\tau$  in two different ways (coming from two distinct series of collapses). To decide whether attaching  $\tau_2$  creates a 2-sphere or a copy of  $\mathbb{R}P_2$ , we need to examine relationships between orientations (see [10] for more about orientation). The orientations are such that  $\tau_2$  has 0 boundary, due to cancellation, and we deduce that  $\Delta(1, x_4x_5x_6x_7)$  consists of a pair of 2-spheres, joined at two distinct points, and has Betti numbers  $\tilde{b}_0 = 0, \tilde{b}_1 = 1, \tilde{b}_2 = 2$ .

**Proposition 7.11.** *If an edge labelling is least-content-increasing, then Theorem 7.5 holds for  $\tau, \sigma$  of equal content without needing to assume the ordered zigzag property or that  $\omega$  acts effectively.*



PROOF. Since the initial and final saturated chains have the same label content, and content is non-increasing, it must be preserved at each step. This together with the fact that the labelling is least-increasing implies that each gradient path downward step which changes the label sequence must sort labels on the interval where the chain element was deleted. Hence,  $\omega$  acts effectively.

Assuming  $\tau$  and  $\sigma$  agree above rank  $s$  means in particular both have the same set of labels in the same order above rank  $s$ . Any gradient path step involving ranks above  $s$  must sort labels from this fixed order above rank  $s$ , precluding completion of the gradient path to  $\sigma$ , since gradient path steps can never create new inversions above rank  $s$ . Thus, we are assured that all ranks above  $s$  are left untouched by all gradient paths from  $\tau$  to  $\sigma$ .  $\square$

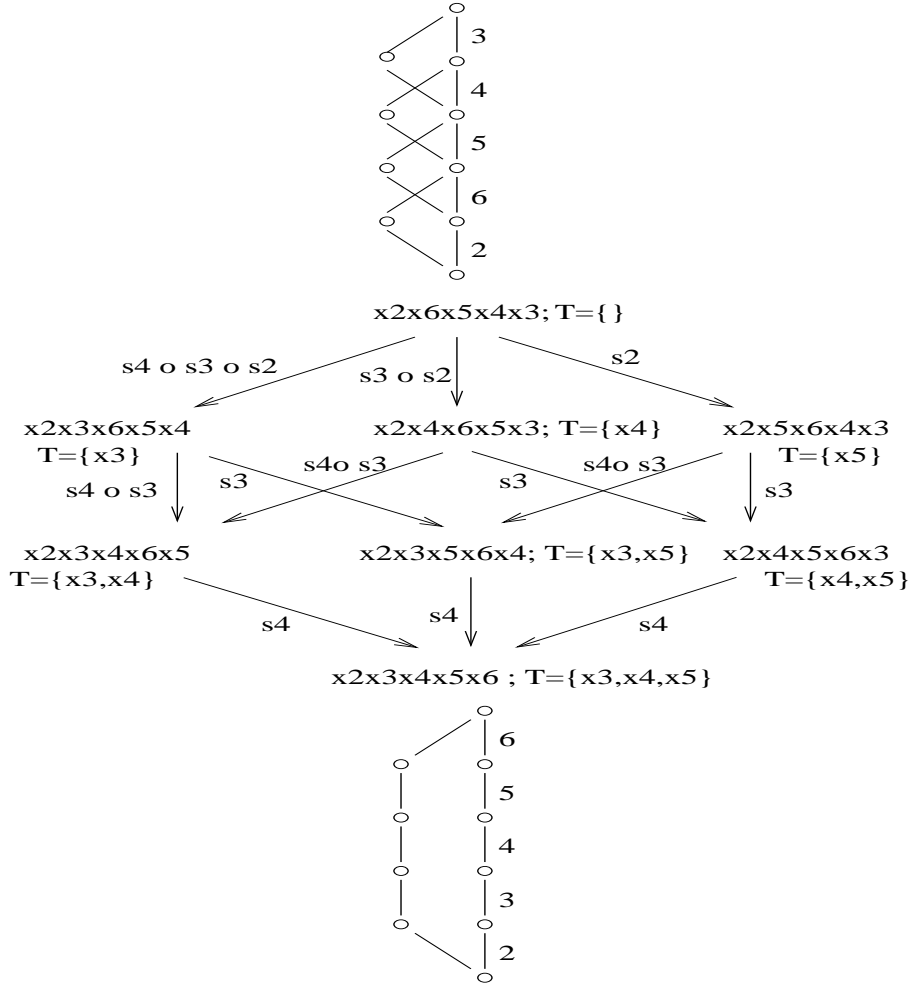


FIGURE 7. A Boolean algebra of critical cells in  $P^M$

Recall from Example 5.5 the saturated chain  $F_j$  labelled  $x_2x_3x_6x_5x_4$ , contributing the critical cell  $\sigma = x_2 < x_2x_3x_6 < x_2x_3x_6x_5$ . In Example 7.9, we used reduced expressions for permutations (and Theorem 7.5) to show there is a unique gradient path from  $\tau = x_2 < x_2x_6 < x_2x_5x_6 < x_2x_4x_5x_6$  to  $\sigma$ , given by the reduced

expression  $s_4 \circ s_3 \circ s_2$ . Now we use Theorem 3.4 to reverse several gradient paths simultaneously by collecting critical cells into a Boolean algebra within  $P^M$ .

**Example 7.12.** Each  $T \subseteq S = \{x_3, x_4, x_5\}$  corresponds to a critical cell  $\text{Crit}(T)$  belonging to facet  $F(T)$  as follows.  $F(T)$  has label sequence  $x_2 x_T x_6 x_{(S \setminus T)^{\text{rev}}}$ , where  $x_T$  lists members of  $T$  in increasing order, and  $x_{(S \setminus T)^{\text{rev}}}$  lists members of  $S \setminus T$  in decreasing order.  $F(T)$  has one minimal skipped interval labelled  $x_2 x_T x_6$  coming from the syzygy  $x_2 x_6 = x_1^2$ . Each rank above this interval is covered by a minimal skipped interval of height one coming from a descent. Theorem 7.5 implies that the set of critical cells  $\{\text{Crit}(T) | T \subseteq S\}$  sits inside the multi-graph face poset  $P^M$  as a Boolean algebra, depicted in Figure 7. Its covering relations are  $\text{Crit}(T \cup \{x_i\}) \prec \text{Crit}(T)$  for each  $T \subseteq S$  and each  $x_i \in S \setminus T$ . Matching each  $T \setminus \{x_3\}$  with  $T \cup \{x_3\}$  gives a complete acyclic matching on the Boolean algebra.

## 8. THE COHEN-MACAULAY PROPERTY FOR THE POSET $PD(1^n, q)$ OF PARTIAL DECOMPOSITIONS OF AN $n$ -DIMENSIONAL VECTOR SPACE OVER $F_q$ INTO 1-DIMENSIONAL SUBSPACES

Let  $V$  be a finite vector space, i.e. an  $n$  dimensional vector space over a finite field  $\mathbb{F}_q$ . Let  $PD(1^n, q)$  denote the poset of partial decompositions of  $V$  into 1-dimensional subspaces. That is, the elements of  $PD(1^n, q)$  are collections  $l_1, \dots, l_r$  of linearly independent lines in  $V$ , with a maximal element  $\hat{1}$  adjoined.

**Definition 8.1.** A set of lines is **independent** if they are spanned by vectors that are linearly independent.

$PD(1^n, q)$  has covering relations  $u \prec v$  if  $v$  is obtained from  $u$  by adding another linearly independent line to the collection.  $PD(1^n, q)$  is the poset of independent sets for the matroid whose ground set consists of the lines in  $V$ , i.e. the points in the projective space  $\mathbb{P}V$ . Thus,  $PD(1^n, q)$  is shellable, by virtue of its order complex being the barycentric subdivision of a matroid complex (see [5]). However, Cohen-Macaulayness may also easily be proven using discrete Morse theory in a way that is indicative of how our machinery also seems to apply to much more complex posets.

We will construct a discrete Morse function in terms of an ordering for the ground set of lines in  $V$ , then use gradient path reversal to optimize it into one which has only top-dimensional critical cells. The surviving critical cells will be indexed by matroid bases with internal activity 0, allowing us to recover the fact that a matroid complex has the homotopy type of a wedge of spheres of top dimension, where the number of spheres is the number of matroid bases with internal activity 0. Though our proof is for a specific matroid complex, the same argument will work without modification for any isthmus-free matroid.

**8.1. Construction of lexicographic discrete Morse function.** First we need an edge-labelling, then we will study the resulting lexicographic discrete Morse function. Following matroid theory, we call the set of all lines in  $V$  our **ground set**, and choose an ordering  $\omega$  on the ground set. Label each covering relation  $\{l_1, \dots, l_r\} \prec \{l_1, \dots, l_{r+1}\}$  with the line  $l_{r+1}$  being added to the set of linearly independent lines, and use the order  $\omega$  on the ground set to order our labels. Label covering relations  $\{l_1, \dots, l_n\} \prec \hat{1}$  with a fixed label that is set to be larger than all lines in  $PD(1^n, q)$ . The resulting lexicographic order on the label sequences on saturated chains gives rise to the lexicographic discrete Morse function we will use.

**Proposition 8.2.** *This labelling is least-content-increasing.*

PROOF. Any descent comes from a consecutive pair of lines  $l_j, l_i$  with the larger one (in terms of the order  $\omega$ ) inserted first. This insertion order may be reversed to obtain an earlier saturated chain on the interval which has the same content.  $\square$

**Definition 8.3.** *Any independent set of lines that spans  $V$  is a **basis**. An element  $l$  in a basis  $B$  is **internally active** in  $B$  if  $l$  cannot be replaced by a smaller element of  $F$  (with respect to  $\omega$ ) to obtain an alternate basis for  $F$ . In other words,  $l$  is internally active when there are no smaller elements whose unique expression in terms of  $B$  has nonzero coefficient for  $l$ . The **internal activity** of a basis is the number of internally active elements in it. (See [5] for more detail.)*

Our first task is to characterize minimal skipped intervals.

**Definition 8.4.** *Let  $N$  be a saturated chain  $\hat{0} \prec x_1 \prec \cdots \prec x_k \prec x \prec y_1 \prec \cdots \prec y_l \prec \hat{1}$  with label set  $B$ . Suppose that  $N$  has strictly increasing labels from  $x$  to  $\hat{1}$ . Then  $(x, \hat{1})$  is called a **top interval** of  $N$  if the following conditions are both met:*

- (1) *The label for  $x \prec y_1$  is not internally active in  $B$ .*
- (2) *The label on each  $y_i \prec y_{i+1}$  is internally active in  $B$ .*

**Proposition 8.5.** *The lexicographic discrete Morse function given by our ground set labelling/ordering has exactly the following minimal skipped intervals: (1) label sequence descents, and (2) top intervals.*

PROOF. An interval may only have one increasing chain unless the interval is of the form  $(x, \hat{1})$ , since our labelling restricted to  $(x, y)$  for  $y \neq \hat{1}$  is the standard EL-labelling on a Boolean algebra. Thus, all minimal skipped intervals come from either descents or intervals  $(x, \hat{1})$ , since our labelling is least-content-increasing.

For a skipped interval  $(x, \hat{1})$  to be a minimal skipped interval in a saturated chain  $M = \hat{0} \prec m_1 \prec m_2 \prec \cdots \prec x \prec y \prec \cdots \prec \hat{1}$ , there must be an earlier saturated chain  $M'$  that agrees with  $M$  through  $x$  but has covering relation  $x \prec y'$  with strictly earlier label  $w'$  than the label  $w$  on  $x \prec y$ . Let  $w_1, \dots, w_i$  be the labels on covering relations in the saturated chain  $M$  below  $x$ . Then  $w'$  is not in the span of the vectors  $w_1, \dots, w_i$ , since  $w_1, \dots, w_i, w'$  extends to a basis. But for the skipped interval  $(x, \hat{1})$  to be minimal,  $w'$  must be in  $\langle w_1, \dots, w_i, w \rangle$ , because otherwise the expression for  $w'$  in terms of  $B$  would involve some later label, causing that label to not be internally active. This would mean there would be a smaller minimal skipped interval beginning later in that saturated chain, a contradiction. Thus, the expression for  $w'$  in terms of  $B$  must involve  $w$ , which means  $w$  may be replaced by  $w'$  to obtain a new basis, implying  $w$  is not internally active. If any later label were internally active, that would contradict the minimality of our minimal skipped interval from  $x$  to  $\hat{1}$ , so we have shown that any minimal skipped interval not coming from a descent must be a top interval.  $\square$

**Corollary 8.6.** *A saturated chain contributes a critical cell if and only if it has label sequence  $b_m b_{m-1} \cdots b_1 a c_1 \cdots c_p$  with  $b_m > b_{m-1} > \cdots > b_1 > a$ ,  $a < c_1 < \cdots < c_p$ , in which  $c_1, \dots, c_p$  are all internally active but  $a$  is not internally active.*

PROOF. A saturated chain contributes a critical cell if and only if its minimal skipped intervals cover all ranks, so the saturated chain must have descending labels up until a minimal skipped interval  $(x, \hat{1})$ . However, top intervals are the only minimal skipped intervals of the form  $(x, \hat{1})$ .  $\square$

**8.2. Cancellation of critical cells.** This section uses a notion called the non-essential set of a saturated chain to collect critical cells into Boolean algebras within  $P^M$ . We will use complete acyclic matchings on these Boolean algebras, whenever the non-essential set is nonempty, to construct an acyclic matching on  $P^M$ . The key will be to choose the non-essential set in such a way that any subset  $S$  of the non-essential set will index a critical cell in such a way that the critical cells given by subsets of the non-essential set comprise a Boolean algebra within  $P^M$ . It is important to define the non-essential set in a way that does not depend on the order in which labels appear on a saturated chain, to ensure that all saturated chains indexed by subsets of the non-essential set will have the same non-essential set.

**Definition 8.7.** *The non-essential set of a saturated chain  $N$ , denoted  $NE(N)$ , is the set of internally active labels on  $N$ .*

Let  $a < b_1 < b_2 < \cdots < b_n$  be the labels on  $N$ . Let  $b_S$  denote the label sequence  $b_{i_1}b_{i_2}\cdots b_{i_r}$  for  $S = \{i_1, \dots, i_r\} \subseteq [n]$  with  $i_1 < \cdots < i_r$ . Let  $b_S^{rev}$  denote the label sequence  $b_{i_r}b_{i_{r-1}}\cdots b_{i_1}$ . Let  $S^C$  denote  $[n] \setminus S$ . In this notation, each saturated chain that contributes a critical cell is labelled  $b_{(S^C)^{rev}}ab_S$  for some set of labels and some choice of  $S$ .

**Remark 8.8.** *If  $N$  contributes a critical cell that is not top-dimensional, then its top interval includes one or more internally active labels, implying  $NE(N) \neq \emptyset$ .*

**Proposition 8.9.** *If the saturated chain  $N$  labelled  $b_{(S^C)^{rev}}ab_S$  contributes a critical cell, then so does the saturated chain labelled  $b_{(T^C)^{rev}}ab_T$  for any  $T \subseteq NE(N)$ .*

PROOF. This follows from our characterization of minimal skipped intervals along with the fact that permuting the order in which 1-spaces are created in a saturated chain does not affect whether they are internally active in the basis  $B$ .  $\square$

We will refer to the critical cell contributed by the saturated chain  $b_{(T^C)^{rev}}ab_T$  as being indexed by  $T$ , and denote this as  $Crit(T)$ . The next lemma will be applied both to  $PD(1^n, q)$  and to the poset discussed in Section 9.

**Lemma 8.10.** *Let  $\lambda$  be a least-content-increasing labelling on a finite poset with  $\hat{0}$  and  $\hat{1}$ . Suppose  $\lambda$  restricted to  $(\hat{0}, x)$  for each  $x \neq \hat{1}$  is a CL-labelling. Let  $N$  be a saturated chain contributing a critical cell. Let  $S$  be the set of labels on  $N$ , excluding the minimal label, denoted  $a$ , appearing on  $N$ . Let  $T$  be a subset of  $S$  such that there is a saturated chain labelled  $(U^C)^{rev}aU$  which contributes a critical cell  $Crit(U)$  for every  $U \subseteq T$ . If there is a gradient path from  $Crit(U)$  to  $Crit(U) \cup \{i\}$  for each  $U \subseteq T$ ,  $i \in T \setminus U$ , then the critical cells indexed by subsets of  $T$  form a Boolean algebra within  $P^M$ .*

PROOF. Construct a gradient path from  $Crit(U)$  to  $Crit(U \cup \{i\})$  for any  $U \subseteq T$ ,  $i \in T \setminus U$  by shifting the label  $i$  upward from its position in the descending part of a saturated chain to the interior of the top interval. That is, downward steps in the gradient path delete the chain element immediately above  $i$ , then upward steps replace the newly uncovered descent by the ascent which has the pair of labels sorted. To deduce uniqueness of each such gradient path, we use the version of Theorem 7.5 given in Remark 7.6, since the labelling on  $PD(1^n, q)$  is a least-content-increasing labelling. The fact that there are no other gradient paths between critical

cells of consecutive dimension follows from the fact that there cannot be a gradient path from  $Crit(U_1)$  to  $Crit(U_2)$  if the saturated chain contributing the latter has any inversions not present in the former; if there is any  $j \in U_1 \setminus U_2$ , then  $Crit(U_2)$  will have an inversion between  $j$  and the minimal label on the saturated chain, and this inversion will be missing from  $Crit(U_1)$ . Thus, we are done.  $\square$

**Corollary 8.11.** *The critical cells indexed by subsets of  $NE(N)$  form a Boolean algebra within  $P^M$ .*

PROOF. Lemma 8.10 applies because  $PD(1^n, q) - \hat{1}$  is a simplicial poset with each interval getting the standard EL-labelling for a Boolean algebra.  $\square$

**Proposition 8.12.** *The union of the complete acyclic matchings on these Boolean algebras is an acyclic matching on  $P^M$ .*

PROOF. The point is to preclude directed cycles involving multiple Boolean algebras, by showing that once we pass from one Boolean algebra to another, we may never return to the original Boolean algebra. The idea is to partially order the Boolean algebras resulting from various label contents and notice that any gradient path which passes from one Boolean algebra to another must be a step downward in our partial order on Boolean algebras, making it impossible for a cycle to return to the original Boolean algebra.

The partial order is on choices of basis for  $V$ , where by convention each basis vector must have a leading 1. One basis  $B_1$  will be smaller than another basis  $B_2$  in our partial order if  $B_1$  is obtained from  $B_2$  by replacing a single vector by a lexicographically smaller vector. Notice that a downward step in the face poset on the order complex  $\Delta(PD(1^n, q))$  deletes a single chain element, either preserving the earliest coatom possible in an extension to a saturated chain, or else causing this coatom to decrease in our partial order on bases.  $\square$

**Theorem 8.13.**  *$PD(1^n, q)$  is homotopically Cohen-Macaulay. It has the homotopy type of a wedge of spheres of top dimension, with the number of spheres equalling the number of bases for  $\mathbb{F}_q^n$  with internal activity 0.*

PROOF. It suffices to show that each interval has critical cells only of top dimension. This is done by cancelling critical cells in the lexicographic discrete Morse function resulting from our edge-labelling. Theorem 3.4 ensures that we can cancel the critical cells belonging to our union of complete acyclic matchings on Boolean algebras, since this is an acyclic matching on  $P^M$ . Remark 8.8 ensures that every critical cell that is not top-dimensional belongs to such a Boolean algebra  $B_n$  for  $n \geq 1$ , and hence to a Boolean algebra in which all cells are indeed matched and cancelled. Thus, the interval  $(\hat{0}, \hat{1})$  is homotopy equivalent to a wedge of spheres of top dimension. Note that the surviving critical cells are labelled by the decreasing chains with empty non-essential set, i.e. where none of the labels are internally active. There is one such saturated chain for each basis with internal activity 0, yielding the homotopy type result.

The same labelling and argument works for intervals  $(x, \hat{1})$ . Let  $x = l_1, \dots, l_r$ , and note that the labels given by bases for  $l_1, \dots, l_r$  cannot belong to the non-essential set of a saturated chain on this interval. Intervals  $(\hat{0}, y)$  for  $y \neq \hat{1}$  are Boolean algebras, and hence are shellable, so Cohen-Macaulayness follows.  $\square$

9. THE COHEN-MACAULAY PROPERTY FOR THE POSET  $\Pi_{S_n}$  OF PARTITIONS INTO PERMUTATION CYCLES

Jeff Remmel recently defined the following permutation-analogue of the partition lattice (personal communication). Let  $\Pi_{S_n}$  be the poset of permutations in  $S_n$ , partially ordered as follows, with a maximal element  $\hat{1}$  adjoined. A permutation  $\sigma$  covers a permutation  $\tau$  if  $\tau$  may be obtained from  $\sigma$  by splitting a single cycle of  $\sigma$  (written in cycle notation) into two smaller cycles by the following procedure: let  $a_1, \dots, a_r$  be the elements of the cycle in  $\sigma$  listed in such a way that  $\sigma(a_i) = a_{i+1}$  for each  $i$ . Choose some  $S \subseteq [r]$ , and let  $\tau$  have cycles  $(a_{i_1}, \dots, a_{i_k})$  with  $S = \{i_1, \dots, i_k\}$  and  $(a_{j_1}, \dots, a_{j_{r-k}})$  with  $[r] \setminus S = \{j_1, \dots, j_{r-k}\}$ . In other words,  $\sigma$  is obtained from  $\tau$  by shuffling together two cycles of  $\tau$  to obtain one larger cycle, with any cyclic shift of each cycle allowed prior to shuffling. (Shuffling takes place on two labelled circles, in which shuffling amounts to some interspersing of labels.)

**Remark 9.1.** *The number of poset elements at corank  $k$  is the signless Stirling number of the first kind  $c(n, k)$ . The coatoms are the  $(n-1)!$  possible permutations which are  $n$ -cycles.*

Remmel previously showed that all intervals  $(x, y)$  for  $y \neq \hat{1}$  are isomorphic to intervals in the partition lattice, and hence are EL-shellable (personal communication). In particular, intervals  $(\hat{0}, y)$  for  $y$  a coatom have  $(n-1)!$  decreasing chains, so have Möbius function  $\mu(\hat{0}, y) = (-1)^{n-1}(n-1)!$ . We will show that  $\Pi_{S_n}$  is Cohen-Macaulay, but it remains open whether or not it is shellable.

**9.1. Construction of lexicographic discrete Morse function.** Label covering relations with ordered pairs  $(i, \pi)$  where  $i$  is an integer and  $\pi$  is a permutation in  $S_{[2, n]}$ , obtained as follows. Each covering relation  $u < v$  for  $v \neq \hat{1}$  merges two cycles  $C_1, C_2$  by shuffling them. Let  $i = \max(\min C_1, \min C_2)$  and let  $\pi$  be the lexicographically smallest permutation obtainable from  $v$  by shuffling the cycles of  $v$  (in the sense described above).

Order integers  $i$  linearly and order permutations  $\pi$  by the lexicographic order on their expressions in one-line notation. The integer label takes precedence over the permutation for ordering labels. Label all covering relations  $u < \hat{1}$  with some fixed label that is chosen to be larger than all other labels being used.

**Remark 9.2.** *Ascents and descents are determined by the first coordinate, since just like in the partition lattice saturated chains have the distinct integers  $2, \dots, n$  as the first coordinates of their labels.*

**Proposition 9.3.** *The labelling is least-content-increasing.*

PROOF. This is immediate for intervals  $(x, y)$  with  $y \neq \hat{1}$ , since then we have an interval in the partition lattice, and the first coordinate gives an EL-labelling in which all label sequences have equal content. If  $y = \hat{1}$ , then the saturated chains with smallest content are those whose coatom is the lexicographically smallest permutation  $\pi$  obtainable by shuffling together the cycles in  $x$ . Each such saturated chain has label set  $\{(i, \pi) | 2 \leq i \leq n\}$ , i.e. with second coordinate fixed as  $\pi$ . The lexicographically smallest saturated chain on our interval is the unique increasing chain with this content, implying the result.  $\square$

Next we characterize the minimal skipped intervals. Recall that a saturated chain contributes a critical cell if and only if its interval system covers all ranks, so

we also characterize which saturated chains contribute critical cells. By convention, list the elements of a cycle in the order in which they eventually appear in a coatom of the saturated chain being considered.

**Definition 9.4.** Let  $N$  be a saturated chain  $\hat{0} \prec x_1 \prec \cdots \prec x_k \prec x \prec y_1 \prec \cdots \prec y_l \prec \hat{1}$  with increasing labels from  $x$  to  $\hat{1}$ . A **top interval** in  $N$  is an interval  $(x, \hat{1})$  such that (1) there is a coatom above  $x$  that is smaller than  $y_1$ , and (2)  $y_1$  is the smallest coatom above  $y_1$ .

**Proposition 9.5.** A saturated chain  $N$  has two types of minimal skipped intervals, descents and top intervals.

PROOF. Since the labelling is least-content-increasing, each descent gives a minimal skipped interval of height one, and all other minimal skipped intervals come from delinquent chains. However, our labelling restricted to any interval  $(x, y)$  for  $y \neq \hat{1}$  is essentially the standard EL-labelling for the partition lattice, so delinquent chains are only possible on intervals  $(x, \hat{1})$ , and then must take the form above.  $\square$

Thus, any saturated chain contributing a critical cell must have descending labels immediately followed by a top interval.

**9.2. Cancelling critical cells.** We will again use non-essential sets to collect critical cells into Boolean algebras, much as we did in our analysis of  $PD(1^n, q)$ . The non-essential set of a saturated chain will depend only on its coatom permutation  $\sigma$  (specifically on the inversions in  $\sigma$ ) and on the following tree structure associated to any saturated chain.

**Definition 9.6.** A covering relation merging cycles  $C_1, C_2$  gives rise to a **merge step** denoted  $e_{i,j}$ , where  $i = \min C_1$  and  $j = \min C_2$ . Associate a labelled, rooted tree on vertex set  $\{1, \dots, n\}$  to a saturated chain by letting 1 be the root and including each edge  $e(i, j)$  where  $e_{i,j}$  is a merge step in the saturated chain. Notice that labels are decreasing along tree paths to the root, and there are  $(n-1)!$  such trees.

The non-essential set of a saturated chain will be a certain subset of its merge steps. It will develop that these two pieces of data associated to a saturated chain, namely its co-atom and its tree structure, will remain constant on certain Boolean algebras of critical cells, ensuring that all elements of such a Boolean algebra have the same non-essential set. To decide which merge steps belong to the non-essential set, we will need a notion of forced and unforced inversions in the co-atom permutation. The non-essential set of a saturated chain will consist of those merge steps which shuffle two cycles in the lexicographically smallest possible way.

Associate to each co-atom  $\sigma$  the permutation  $o(\sigma)$  obtained by viewing the unique cycle in  $\sigma$  as a permutation in one-line notation;  $\sigma$  is cyclically shifted so that 1 appears in the leftmost position, implying 1 is a fixed point of the permutation  $o(\sigma)$ .

**Definition 9.7.** A merge step  $e_{m_1, m_2}$  in a saturated chain with co-atom  $\sigma$  is said to have an **unforced** inversion if  $o(\sigma)$  has an inversion  $(i, j)$  with  $i, j$  belonging to the two distinct cycles,  $C_1, C_2$ , respectively, to be merged by  $e_{m_1, m_2}$ , but  $C_1$  has no inversions  $(i, k)$  with  $i < j < k$ . Any other inversions between elements of  $C_1, C_2$  are said to be **forced**.

The following lemma will help us to define non-essential sets as consisting of merge steps with no unforced inversions.

**Lemma 9.8.** *An interval  $(x, \hat{1})$  in a saturated chain  $N$  is a top interval if and only if it consists of merge steps  $(1, i_1), (1, i_2), \dots, (1, i_l)$  with  $i_1 < i_2 < \dots < i_l$  such that  $e_{(1, i_1)}$  has at least one unforced inversion, but all subsequent merge steps  $e_{(1, i_j)}$  have only forced inversions.*

PROOF. Let  $\underline{a} = (a_1, \dots, a_k)$  and  $\underline{b} = (b_1, \dots, b_l)$  be two permutation cycles, cyclically shifted to have their smallest elements listed first. Now view the sequences  $a_1, \dots, a_k$  and  $b_1, \dots, b_l$  as two tapes to be read from left to right. To merge the sequences in the lexicographically smallest possible way, we proceed as follows:

- (1) Compare the current positions on the  $\underline{a}$  and  $\underline{b}$  tapes at each step.
- (2) Choose the smaller of the two values currently being viewed.
- (3) Append this number to the shuffled sequence being constructed.
- (4) Move forward one position on the tape from which we chose a number.

From this viewpoint, it is clear that we cannot add  $b_j > a_i$  to the merged sequence before putting  $a_i$  into the merged sequence unless there is some  $a_{i'} > b_j$  appearing earlier than  $a_i$  on the  $\underline{a}$ -tape, since we will not choose  $b_j$  until the  $\underline{a}$ -tape is at a value  $a_{i'}$  that is larger than  $b_j$ . Thus, for  $\underline{a}$  and  $\underline{b}$  to be merged in the lexicographically smallest possible way, all inversions between  $\underline{a}$  and  $\underline{b}$  must be forced inversions.

In the other direction, if  $\underline{a}$  and  $\underline{b}$  are merged in a way that is not lexicographically smallest possible, consider the first place where the merged sequence differs from the lexicographically smallest choice. That is, consider the first step where we compare an element  $a_i$  on the  $\underline{a}$  tape to an element  $b_j$  on the  $\underline{b}$  tape and choose the larger of the two. Without loss of generality, say  $a_i < b_j$ . For this to be the first deviation from the lexicographically smallest choice, we are assured that there are no  $(a_i, c)$  inversions on the  $\underline{a}$  tape for  $a_i < b_j < c$ , because once such a  $c$  is reached on the tape, we are assured that one tape or the other will have some  $d \geq c$  as its current value until either we reach the end of one tape or the other, or we deviate from the lexicographically smallest choice of how to merge the two tapes. Thus, we cannot get to the state of having  $a_i, b_j$ , which are both smaller than  $c$ , as the current states of the two tapes, without deviating from the lexicographically smallest merged sequence before this, a contradiction.  $\square$

We have already associated a tree structure  $T(N)$  to each saturated chain  $N$ . To define the non-essential set of  $N$ , we will also need a set partition on  $[2, n]$ . We obtain this partition, denoted  $\Pi(N)$ , from  $T(N)$  by deleting all tree edges of the form  $(1, i)$ , then taking the graph components as the partition blocks.

**Definition 9.9.** *The non-essential set of  $N$  consists of those tree edges  $(1, i)$  such that the component of the partition  $\Pi(N)$  containing  $i$  does not have any unforced inversions with other components of  $\Pi(N)$ .*

**Proposition 9.10.** *Each merge step  $(1, i) \in NE(N)$  may appear either in the top interval or in a unique position below the top interval which causes the labels below the top interval to be decreasing. Shifting the merge step  $(1, i)$  from one of these two positions to the other does not change the tree structure.*

PROOF. The first assertion follows from Lemma 9.8. To see that tree structure is preserved, we show that the block minimums are preserved, since tree edges connect block minimums at merge steps. Shifting  $(1, i)$  upward to within the top interval or downward to the decreasing segment of labels can only change block minimums



for blocks whose minimum element is  $i$ . However, any merge steps  $(i, j)$  for  $j > i$  must occur below the lower of the two positions where  $(1, i)$  may appear.  $\square$

**Corollary 9.11.** *Let  $N$  be a saturated chain which contributes a critical cell. Then the labels in the interior of the top-interval of  $N$  all belong to  $NE(N)$ . In addition, any subset of  $NE(N)$  may appear in the interior of the top interval.*

PROOF. This follows immediately from Proposition 9.10.  $\square$

**Corollary 9.12.** *Critical cells are arranged into Boolean algebras  $B_m$  in  $P^M$ , based on their non-essential sets, with  $m$  being the size of the non-essential set.*

PROOF. This is immediate from Lemma 8.10 of the previous section.  $\square$

**Remark 9.13.** *Corollaries 9.11 and 9.12 allow us to collect all non-top-dimensional critical cells, along with some top-dimensional ones, into Boolean algebras  $B_m$  with  $m \geq 1$ , since all non-top-dimensional critical cells have nonempty non-essential set.*

Theorem 3.4 shows that any acyclic matching on  $P^M$  gives a collection of pairs of critical cells that may be cancelled simultaneously. Each Boolean algebra  $B_m$  for  $m \geq 1$  has a complete acyclic matching. The next result implies that the union of these matchings is an acyclic matching on  $P^M$ .

**Proposition 9.14.** *There are no directed cycles visiting multiple Boolean algebras.*

PROOF. Similarly to with  $PD(1^n, q)$ , we partially order Boolean algebras of critical cells and show that gradient paths may only proceed from later Boolean algebras to strictly earlier ones in this partial order. Any gradient path from one Boolean algebra to another one either alters the coatom in a way that makes it lexicographically smaller or alters the tree structure by turning a pair of edges  $(i, j), (j, k)$  into the pair  $(i, j), (i, k)$ . Thus, we partially order coatoms lexicographically and order tree structures using covering relations  $(i, j), (i, k) \prec (i, j), (j, k)$ . Upward steps are impossible in both partial orders, precluding return to the original Boolean algebra, i.e. precluding a cycle involving distinct Boolean algebras.  $\square$

**Theorem 9.15.**  $\Pi_{S_n}$  *is a homotopically Cohen-Macaulay poset.*

PROOF. The optimized discrete Morse function has only top-dimensional critical cells, ensuring the interval  $(\hat{0}, \hat{1})$  is homotopy-equivalent to a wedge of spheres of top dimension. The proof extends to arbitrary intervals as follows. Remmel observed that all intervals  $(x, y)$  with  $y \neq \hat{1}$  are shellable. Our discrete Morse function easily generalizes to intervals  $(x, \hat{1})$  by excluding from the non-essential set of a saturated chain restricted to  $(x, \hat{1})$  any labels appearing below  $x$ . We may still define forced and unforced inversions in terms of co-atom permutations.  $\square$

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